Classification and Phylogenetic Analysis of African Ternary Rhythm Timelines

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Abstract

A combinatorial classification and a phylogenetic analysis of the ten 12/8 time, seven-stroke *bell* rhythm timelines in African and Afro-American music are presented. New methods for rhythm classification are proposed based on measures of *rhythmic oddity* and *off-beatness*. These combinatorial classifications reveal several new uniqueness properties of the *Bembé* bell pattern that may explain its widespread popularity and preference among the other patterns in this class. A new distance measure called the *swap*-distance is introduced to measure the non-similarity of two rhythms that have the same number of strokes. A swap in a sequence of notes and rests of equal duration is the location interchange of a note and a rest that are adjacent in the sequence. The swap distance between two rhythms is defined as the minimum number of swaps required to transform one rhythm to the other. A phylogenetic analysis using Splits Graphs with the swap distance shows that each of the ten bell patterns can be derived from one of two "canonical" patterns with at most four swap operations, or from one with at most five swap operations. Furthermore, the phylogenetic analysis suggests that for these ten bell patterns there are no "ancestral" rhythms not contained in this set.

1 Introduction

Consider the clock depicted in Figure 1, and assume the clock runs so fast that it makes a full revolution in about two seconds. Now set the clock ticking starting at "noon" (12 O'clock) and let it keep running for ever. Finally let it strike a bell on the hours of twelve, two, four, five, seven, nine, and eleven, for a total of seven strikes per clock cycle, with the first strike of the cycle at twelve. These times are marked with a bell in Figure 1.

The resulting pattern rings out the predominant African rhythm time-line that has travelled to America and beyond, and has become the most well known of all the (12/8)-time bell patterns. It is known internationally mostly by its Cuban name, the *Bembé*, a name given to a Cuban feast celebrated with drums to entertain the *orishas* (divinities) [33]. In the following, simple mathematical arguments will be given that may explain why the *Bembé* has taken center stage among the 12/8 time bell patterns.

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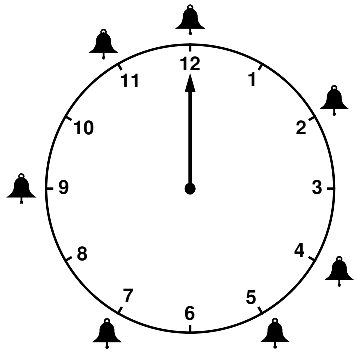

Figure 1: A clock that strikes a bell seven times in one cycle.

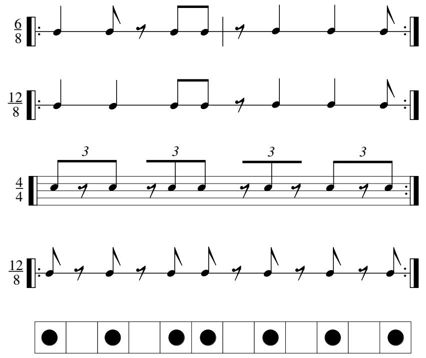

Figure 2: Five ways of representing the *Bembé* bell pattern.

Figure 2 shows five ways in which the *Bembé* bell pattern is usually notated. The third row shows that although the *Bembé* is a *ternary* rhythm normally notated in 6/8 or 12/8 meter, some authors describe it in 4/4 time with *triplets* [27]. The fourth row depicts the rhythm with the smallest convenient notes and rests. The final row shows the Bembé in the *Box Notation Method* developed by Philip Harland at the University of California in Los Angeles in 1962. If we connect the tail to the head of this last diagram and draw it in the form of a circle in clockwise direction, with the head at 12 O'clock, we obtain the clock representation in Figure 1, where the squares in Figure 2 filled with black dots correspond to the positions of the bells in Figure 1. The box notation method is convenient for simple-to-notate rhythms like bell patterns, especially for the mathematical analysis and comparison of rhythms. A commonly used convenient variant of box notation used in text documents is simply to use the letter "x" to denote the strike of the bell or note onset, and the period symbol "." to denote interval units between the note onsets. For example, the *Bembé* pattern of Figure 2 then becomes [x . x . x x . x . x . x].

In [39] a mathematical analysis of the six principal 4/4 time *clave* and bell patterns used in African and Afro-American music was presented. Here we offer a similar study of the ten 12/8 time African rhythm bell patterns (time-lines).

There exist hundreds of timeline patterns for bells, claves and woodblocks traditionally used in music throughout Africa and America, and more recently in world music. In this study however, we are concerned only with the seven-note 12/8 time rhythms. The total number of such possible

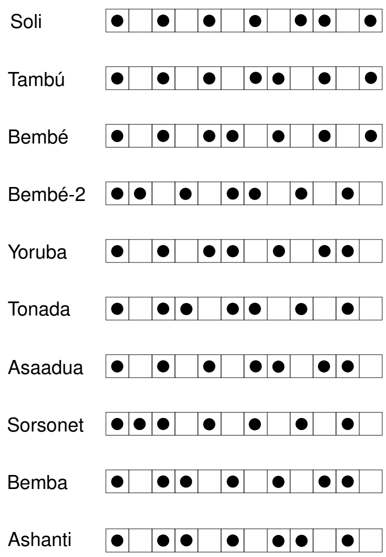

Figure 3: The ten 12/8 time bell patterns in box notation.

rhythms is $12!/(7!)(5!) = 792$ if we do not make restrictions on how small or large the gaps between the notes may be. This is a large number of patterns. However, most of these are not good enough to serve as effective time-line patterns for powerful percussive dance music. We can reduce this large number to an interesting small subset by restricting the maximum size of the inter-onset intervals. For this purpose a useful way to represent these timelines is as an *interval-vector* [35], i.e, a sequence of intervals between the note onsets. The interval-vector for the *Bembé* is (2212221). In [39] a classification of rhythms was proposed based on permutations of the elements of the interval vectors. If one rhythm may be obtained from another by such a permutation then the two rhythms are said to belong to the same *interval combinatorial class*. One may ask how many interval permutations exist of the *Bembé* pattern (2212221). Note that these are *multisets* since repetitions of the elements are permitted [21]. We have seven objects (intervals) of two different types: two of class one and five of class two. Therefore the total number of different permutations of (2212221) is $(7!)/(2!)(5!) = 21$. In the following discussion a method for enumerating the twenty-one rhythms will be outlined.

Although in this severely restricted class there are only twenty-one members, all the traditional music (that I am aware of) appears to use only ten of these, and none of the other 771 seven-note patterns. The ten commonly used rhythms are known by many names in different countries, and there is as yet no international consensus on common terminology for all of them. For the purpose of this study I will call them: *Soli*, *Tambú*, *Bembé*, *Bembé-2*, *Yoruba*, *Tonada*, *Asaadua*, *Sorsonet*, *Bemba*, and *Ashanti*. Figure 3 depicts all ten rhythms in box notation.

A useful geometric representation for such cyclic rhythms is obtained by starting with the clock idea of Figure 1 and connecting consecutive note locations with edges to form a convex polygon. Such a representation not only enhances visualization for classification, but lends itself more readily to geometrical analysis. It has been used by Becker [6] to analyse Javanese Gamelan music, by McLachlan [28] to analyze rhythmic structures from Indonesia and Africa using group theory and Gestalt psychology, by London [26] to study meter representation in general, and most recently, it has been successfully used in a geometrical analysis of the six principal 4/4 time five-stroke clave and bell patterns used in African and Afro-American music [39].

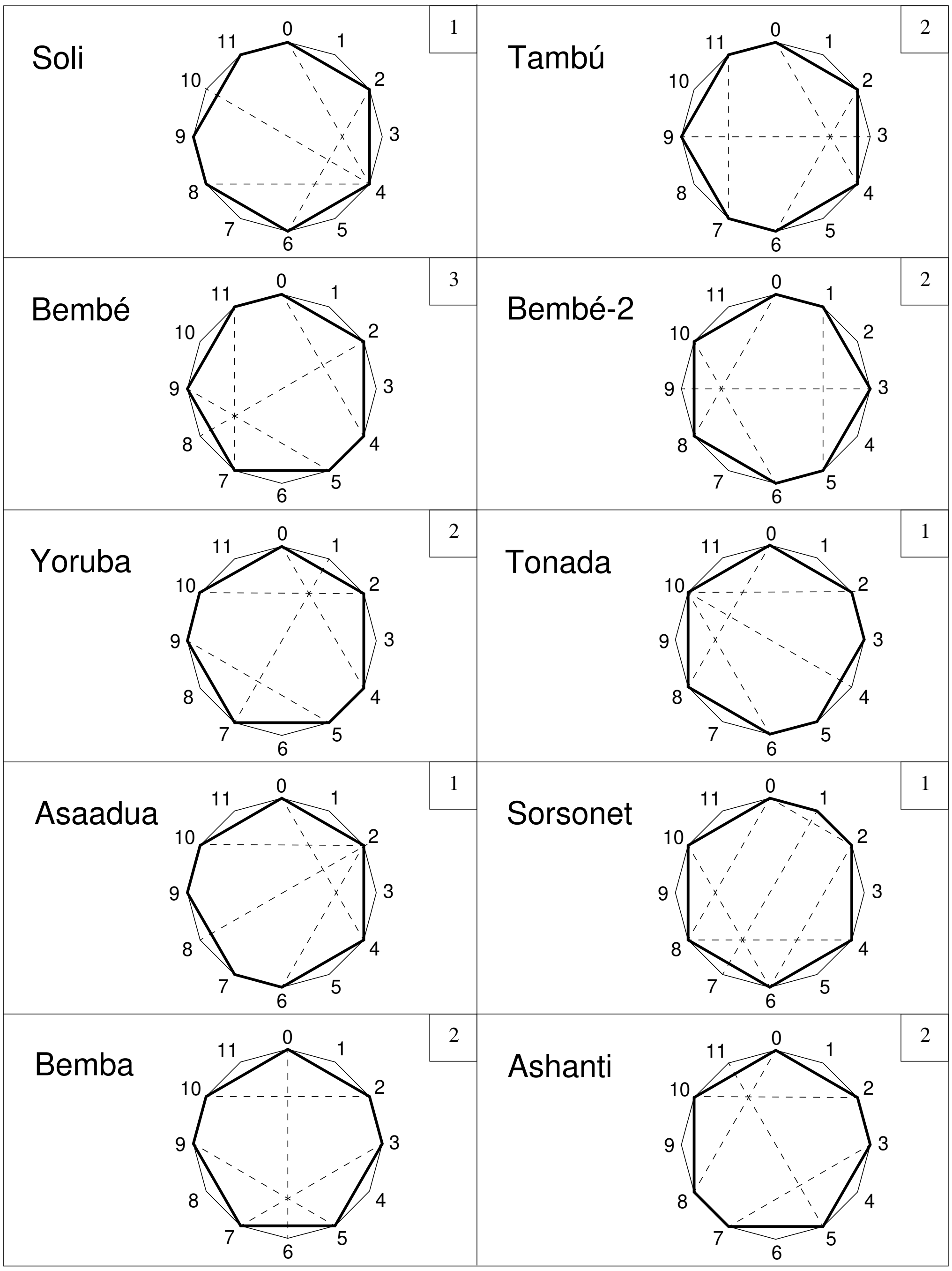

Figure 4: The ten 12/8 time bell patterns represented as convex polygons.

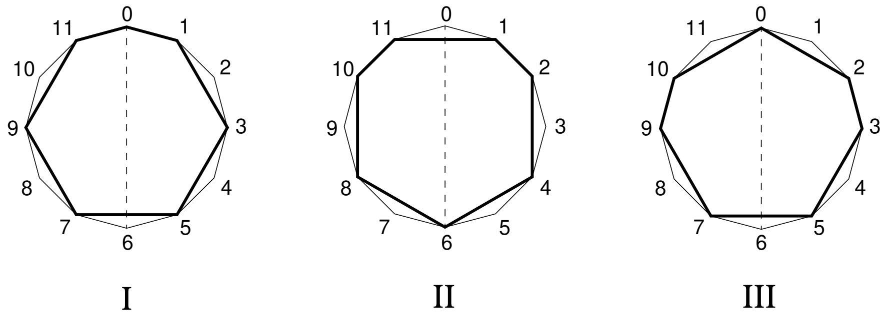

Figure 5: The three cannonical *necklace* patterns that generate all the rhythms.

The ten bell patterns of Figure 3 are represented as convex polygons in Figure 4, where the dashed lines indicate either the base of an isoceles triangle (indicating two equal consecutive time intervals) or an axis of mirror symmetry. Note that, unlike the 4/4 time clave and bell patterns studied in [39], where the presence and number of axes of mirror symmetry helped to distinguish between the different rhythms, here all ten bell patterns contain precisely one axis of symmetry. The 4/4 time and 12/8 time patterns also have very different interval combinatorial classifications. Whereas the six 4/4 time patterns fall into four different interval combinatorial classes [39], all ten 12/8 time patterns fall into one and the same combinatorial class. One may conclude that 12/8 time African rhythm bell patterns are more symmetrical and uniform than their 4/4 time counterparts. A *k-ary necklace* is an equivalence class of k-ary strings under rotation. In this paper we are concerned with *binary* necklaces: the beads come in two colors, note onset interval and rest interval. A necklace is said to be of *fixed density* if the number of beads of one color is fixed [38]. Here we are concerned with binary necklaces of twelve beads with density seven. In the interval combinatorial class determined by the *Bembé* pattern there are only three distinct *necklace* patterns [24]. These three cannonical *necklaces* are shown in Figure 5 with their axes of mirror symmetry in the vertical position and with the two short intervals in the upper semi circle. Cannonical pattern number I generates the *Sorsonet* pattern. Cannonical pattern number II generates the *Soli*, *Tonada*, and *Asaadua* patterns. Cannonical pattern number III generates the *Bembé*, *Bembé-2*, *Tambú*, *Yoruba*, *Bemba*, and *Ashanti* patterns. The number (I, II, III) assigned to the cannonical configuration corresponds to the minimum distance (in terms of the number of intervals) that separates the two short intervals. It is clear that bell patterns with a larger separation (longer sequence of rests) between the two short intervals are prefered. The cannonical pattern number III has the largest possible separation between the two short intervals and adds another special characteristic to the *Bembé* pattern which falls in this class. One may wonder if it is at all useful to play the *Bembé* pattern by starting at a different position in the cycle. However, African rhythm is polyrhythmic and usually at least one drum in the ensemble is playing a 4/4 time rhythm alongside. In this context changing the starting time of the pattern makes it sound quite different. Additional special properties of the *Bembé* will be discussed in the following.

2 The Ten 12/8 Time Bell Patterns

2.1 Soli

The *Rhythm Catalog* of Larry Morris on the World Wide Web [30] contains a rhythm called *Soli* from West Africa with the bell pattern given by [x . x . x . x . x x . x]. The *Soli* actually has three bell patterns including the seven-stroke pattern included in this study, which is the pattern most frequently used in the *Soli* rhythm [18].

2.2 Tambú

The *Tambú*, given by [x . x . x . x x . x . x], is found in several places in the Caribbean, including Curaçao, where it goes by this name [37]. Originally this rhythm was played with only two instruments: a drum and a metallophone called the *heru*. Note that the word Tambú sounds like *tambor*, the Spanish word for drum, and heru sounds like *hierro*, the Spanish word for iron. This bell pattern is also common in West Africa and Haiti [35]. In Haiti it is used for the *Yanvalou*, *Zepaule*, *Camberto*, and *Mahi* rhythms [30]. This rhythm is sometimes called the *long* African bell pattern [16].

2.3 Bembé

This rhythm, denoted by [x . x . x x . x . x . x], is probably the most (internationally) well known of all the African ternary timelines. Indeed, the master drummer Desmond K. Tai has called it the Standard Pattern [23]. In West Africa it is found under various names among the Ewe and Yoruba peoples [35]. In Ghana it is played in the *Agbekor* dance rhythm found along the southern coast of Ghana [10], as well as in the *Bintin* rhythm [30]. It is also the bell pattern of the *Agbadza* rhythm for a recreational dance of the *Ewe* people of eastern Ghana and Togo (see chapter 22 of Collins [12]). Among the Ewe people of Ghana there is a unique rhythm, for five bells only, called the *Gamamla* [25]. The standard pattern is one of the five Gamamla bell patterns played on the *Gankogui* (a double bell), with the first note played on the low pitch bell, and the other six on the high pitch bell. The same is done in the *Sogba* and *Sogo* rythms of the Ewe people. It is played in the *Zebola* rhythm of the *Mongo* people of Congo, and in the *Tiriba* and *Liberté* rhythms of Guinea [18]. This bell pattern is equally widespread in America. In Cuba it is the principal bell pattern played on the *guataca* or hoe blade in the *Bata* rhythms. For example, it is used in the *Columbia de La Habana*, the *Bembé*, the *Chango*, the *Eleggua*, the *Imbaloke*, and the *Palo*. The word *palo* in Spanish means stick and refers also to the sugar cane. The Palo rhythm was played during the cutting of sugar cane in Cuba. The pattern is also used in the *Guiro*, a Cuban folkloric rhythm [25]. In Haiti it is called the *Ibo* [30]. In Brazil it goes by the name of *Behavento* [30]. This rhythm is sometimes called the *short* African bell pattern [16]. In this study I shall refer to it by its most popular international name: the *Bembé*.

2.4 Bembé-2

In Cuba sometimes the *Bembé* rhythm contains two bell patterns; the pattern described in the preceeding, played on the *guataca* or hoe-blade, and the pattern [x x . x . x x . x . x .], played on a low pitch bell [30]. For this reason this secondary bell pattern will be referred to here as *Bembé-2*. This pattern is also a hand-clapping pattern used in Ghana [35] and Tanzania [40].

2.5 Yoruba

The bell pattern [x . x . x x . x . x x .] is widely used in sacred music among the Yoruba people of West Africa [35]. Bettermann [7] calls this rhythm the *Omele*. It is also used in Cuba with the *Columbia* rhythm [36]. The name *Yoruba* will be used for this bell pattern.

2.6 Tonada

The *Tonada* in Cuba is a type of song that illustrates clearly the fusion between the singing style of Andalucia in Spain, and the African rhythms of Cuba. In Andalucia the Tonada is a style of song with voice only. In Cuba the Tonada alternates between segments with only voice and segments with percussion and guitar. The structure is a call-and-response style with the solitary voice calling and the percussion responding. The bell pattern used in the Tonada is [x . x x . x x . x . x .] [36]. In the Caribbean it also appears in Martinique where it is used in the *Bélé* (bel-air) rhythm that accompanies a music and dance originating in the time of slavery [17]. In Africa this pattern is used in Ghana by the Ashanti people [35] and by the *Akan* people in *Adowa* music [31], [32]. It is also used in the *Mandiani* rhythm of Guinea [18].

2.7 Asaadua

The *Asaadua*, expressed as [x . x . x . x x . x x .], is used in processional music of the *Akan* people of central western Ghana [20]. It is played on a *dawuro* bell (also called *atoke* in other parts), a hollow boat-shaped iron bell with a piercing hi-pitched tone that cuts through a score of loud drums. It is also used in the *Kakilambe* and *Sokou* rhythms of Guinea [18].

2.8 Sorsonet

The *Sorsonet* bell pattern given by [x x x . x . x . x . x .] is used by the *Baga* people of Guinea [18] and does not appear to be widely used.

2.9 Bemba

The bell pattern denoted by [x . x x . x . x . x x .] is played with a rhythm found in Northern Zimbabwe called the *Bemba* [35] (not to be confused with the *Bembé* from Cuba). In Cuba it is the bell pattern of the *Sarabanda* rhythm associated with the Palo Monte cult [40].

2.10 Ashanti

The pattern [x . x x . x . x x . x .] is used by the *Ashanti* people of Ghana in several rhythms [35]. It is used in the *Dunumba* rhythm of Guinea [18], and by the *Akan* people of Ghana [31] as a juvenile song rhythm. It is also a pattern used by the *Bemba* people of Northern Zimbabwe, where it is either a hand-clapping pattern, or played by chinking pairs of axe-blades together [23].

3 Why is the Bembé Bell Pattern so Special?

3.1 Isomorphism with the Diatonic Scale

The reader familiar with the piano and the *diatonic* scale may have noticed the similarity between this pitch pattern and the time pattern of the *Bembé* (see Figure 6). If we associate the notes and rests of the *Bembé* time pattern with the white and black keys, respectively, of the diatonic scale

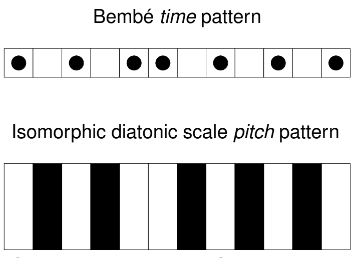

Figure 6: Illustrating the isomorphism between the Bembé time pattern and the diatonic scale pitch pattern found on the piano.

pitch pattern (for the *even tempered* piano) then we obtain an exact isomorphism between the two. Remarkably, Pressing [35] has discovered exact isomorphisms between almost all rhythm timelines and pitch patterns (scales) found in world music. Since the seven white keys of the diatonic scale (C, D, E, F, G, A, B) are so fundamental and important in Western music theory, it is not surprising that the study of this pattern has received a great deal of attention. Several mathematical properties have been suggested as testimony to the specialness of this pattern.

For example, the diatonic scale is generated by the so-called *circle-of-fifths* [8] in which we mark every fifth tone until seven tones are marked and then select all marked tones. If we apply this method instead in the rhythmic domain to generate seven onsets we obtain the canonical necklace pattern III of Figure 5 of which the *Bembé* is a member.

Duncan [14] describes a percolation algorithm that starts with the pattern [x x x x x x x . . . . .] and moves the x's to the right with the goal of producing a less dense pattern. When the algorithm stops the final pattern obtained is the diatonic scale in the pitch domain or the *Bembé* pattern in the time domain. Several group-theoretic [5] and combinatoric [24] properties of the diatonic scale have also been discovered. In the following we add several additional simple characterizing properties of the *Bembé* pattern to this growing list.

3.2 Complementarity

Besides the 7-note 12/8 time bell patterns there are many other 12/8 time patterns with 5,6 and 8 notes. For example, a Cuban 8-note pattern is [x x . x . x x . x x x .] [34]. Note that this pattern may be obtained by inserting a note in between the last two notes of the *Bembé-2* pattern. A 6-note pattern played on the *dawuro* bell in the *Adenkum* rhythm of the *Akan* people of Ghana is the pattern [x . x . x . . x . x x .]. This pattern may be obtained by removing the fourth note in either the *Asaadua* or the *Yoruba*. However, of all the bell patterns that do not contain seven notes, the 5-note patterns are the most popular. Furthermore, amongst these, the two most distinguished are the *Fume-Fume* and the *Columbia* denoted, respectively, by [x . x . x . . x . x . .] and [x . x . . x . x . x .], and depicted as polygons in Figure 7. One of the *Ewe* dances called *Abuteni* uses the *Fume-Fume* pattern [23]. It is also used in the *kple* music of the *Ga* people [32], [1] and in the *Congo* rhythm of Cuba [36]. Bettermann refers to this pattern as the *Inyimbo* rhythm [7]. Indeed, the *Fume-Fume* is described by Jones [23] as "the African signature". Note its structural resemblance to the 4/4 time clave *Son*; both start with an evenly divided, 3-note "call", and a two-note "response". The clave *Son* pattern may equally well be described as the signature of Afro-Cuban-Latin music. The *Columbia* pattern is also used in the *Abakuá* rhythm of Cuba. The *Fume-Fume* and *Columbia* patterns are closely related to each other. Although the *Columbia* can be seen as a rotation of the *Fume-Fume* by five time units in the clockwise direction, the latter can

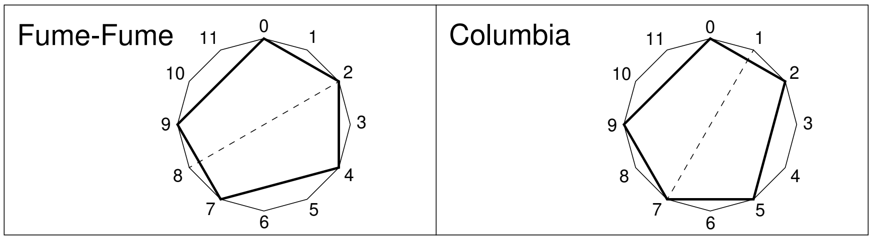

Figure 7: The two principal 5-note 12/8 time bell patterns.

also be obtained from the former much more easily by merely advancing the third note one time unit. These patterns are also special because they have a combinatoric property called the *rhythmic oddity property* discovered by Simha Arom [4]. A rhythm has the rhythmic oddity property if no two onsets partition the entire interval into two subintervals of equal length. Note that none of the ten seven-stroke bell patterns in Figure 3 have this property. For characterizations of this property and algorithms for enumerating rhythms with this property see [9].

It has already been pointed out in the literature that the *complement* or *dual* of the *Bembé* pattern, i.e., the spaces between the notes (corresponding to the black keys on the piano) is the 5-note *Columbia* pattern (disregarding the fact that it starts on an upbeat). Thus when the *Bembé* is played, the silences "play" the *Columbia* pattern which is rotationally equivalent to the *Fume-Fume* pattern. The psychological importance of this phenomenon, called *complementarity* by Pressing [35], has also been documented.

Given that the *Fume-Fume* and *Columbia* are the most important 5-note patterns it would seem desirable that the preferred 7-note patterns should contain one or both of these 5-note patterns as subsets. A quick inspection reveals that the *Fume-Fume* is contained in only three patterns: the *Bembé*, *Yoruba* and *Asaadua*. On the other hand, the *Columbia* is contained in only the *Bembé*, *Yoruba* and *Bemba* patterns. Hence only the *Bembé* and *Yoruba* patterns contain both the *Fume-Fume* and *Columbia* patterns. Furthermore, only the *Bembé* has the additional property that it contains the *Fume-Fume*, or "African signature," as its *dual* pattern (the pattern of silent notes).

3.3 Maximally Even Sets

Clough and Duthett [11] defined the notion of *maximally even sets* with respect to scales represented on a circle. Block and Douthet went further by proposing a measure of *evenness* [8]. Their measure simply adds all the circle-chord lengths determined by pairs of notes in the scale. These definitions may be readily applied to rhythms. It turns out that within the set of rhythms consisting of seven onsets in a bar of twelve units, the *Bembé* pattern, which corresponds to the diatonic scale in Figure 5 III, is a maximally even set. Interestingly, the two configurations that rank just below the *Bembé* pattern, using the Block-Douthett evenness measure, are the ascending melodic minor scale, corresponding to Figure 5 II, and the whole-tone plus one scale, corresponding to Figure 5 I. Therefore the cannonical necklace patterns in Figure 5 correspond to the three most even rhythms.

3.4 Measuring Rhythmic Oddity

Recall that Simha Arom [4] defines a rhythm as having the *rhythmic oddity* property if no two onsets partition the entire interval into two subintervals (bi-partition) of equal length. We have

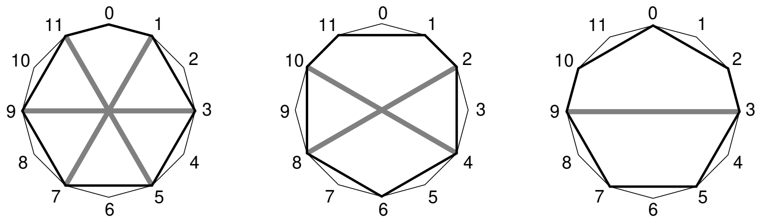

Figure 8: The three necklace patterns showing the number of equal bi-partitions in each.

also seen that none of the ten seven-stroke bell patterns has this property. However, we may define a measure of the amount of rhythmic oddity of a rhythm by the *number* of bi-partitions of equal length that it admits. The fewer equal bi-partitions a rhythm admits, the more rhythmic oddity it posesses. Figure 8 shows the three necklace patterns with the number of equal bi-partitions contained in each. We see that the rhythms belonging to the *Sorsonet* wheel contain three equal bi-partitions, the rhythms belonging to the *Tonada* wheel contain two equal bi-partitions and the rhythms belonging to the *Bembé* wheel contain only one equal bi-partition. Thus the *Bembé* bell pattern belongs to the class of rhythms that have maximum rhythmic oddity.

As an aside, it it worth mentioning that the rhythmic oddity property is related to the *right-angle* property proposed in [39] to classify 4/4 time *clave* patterns. A rhythm has the right-angle property if its rhythm polygon has a vertex (onset) with an interior angle of ninety degrees. Circa 600 B.C. the Greek mathematician Thales of Miletus proved that an angle inscribed in a circle, such that its base is a diameter of the circle, is a right angle [15]. It turns out that the converse of Thales' Theorem is also true: if an angle inscribed in a circle is a right angle then its base must be a diameter of the circle. Therefore, by the converse of Thales' Theorem it follows that if a rhythm has the right-angle property it admits an equal length bi-partition, and hence lacks the oddity property. However, the converse is not necesarily true, as the ten bell patterns studied here demonstrate.

3.5 Measuring Off-Beatness

We conclude this section with one more example of the special properties that characterize the *Bembé* bell pattern. For this I introduce a measure of the *off-beatness* of a rhythm. A twelve-unit interval may be evenly divided (with no remainders) by *four* numbers greater than one and less than twelve. These are the numbers six, four, three and two. Dividing the twelve unit circle by these numbers yields a bi-angle, triangle, square, and hexagon, respectively, as depicted in Figure 9. African music usually contains some drum or other percussion instrument that will play at least one or a portion of these patterns. In polyrhythmic music these four patterns form the possible underlying even pulses. Two of the patterns (bi-angle and square) are binary pulses and two (triangle and hexagon) ternary pulses. Therefore notes played on other positions are off-beat in a strong sense. There are four positions not used by these four even pulse patterns. They are positions 1, 5, 7, and 11. A rhythm that contains an onset in at least one of these four positions will be said to contain the *off-beat* property. A measure of the *off-beatness* of a rhythm is therefore the number of onsets it contains in these four positions. This number is indicated in the upper right-hand corner of each box in Figure 4. The highest value of *off-beatness* is three and only the *Bembé* realizes this value. This property, perhaps more than any other, may help to explain why the *Bembé* enjoys such widespread popularity.

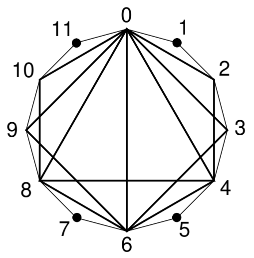

Figure 9: The four off-beat positions not obtainable when dividing twelve by six, four, three or two.

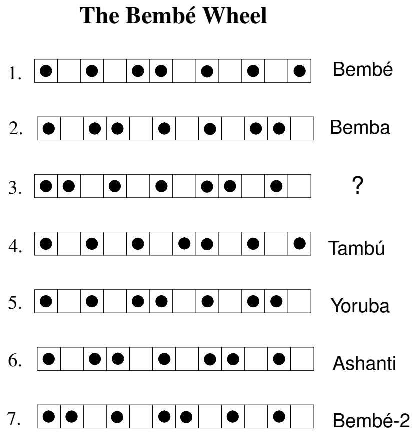

Figure 10: The Bembé wheel and the six known rhythms it generates.

4 Rhythm Wheels

4.1 The Bembé Wheel

Some authors and teachers have noticed that if the *Bembé* bell pattern is played by starting on the fourth onset one obtains the *Tambú* bell pattern [16]. Others have gone further by suggesting pedagogical exercises in which one practices by starting the *Bembé* with each of its seven onsets acting as the first downbeat. In fact Gary Harding of Seattle has given this rhythm generation method the name *Bembé Wheel*. Several Internet sites are devoted to the *Bembé* Wheel which is shown in Figure 10. However, it has not been realized in the popular literature that this simple technique actually generates other traditional rhythms. The second bell pattern in Figure 10 is obtained by starting the *Bembé* on the second onset: the *Bemba* pattern. The third pattern is obtained by starting the *Bembé* on the third onset, and so on. As Figure 10 shows, this generation method yields seven patterns of which six are used in traditional African music. As for the third pattern [x x . x . x . x x . x .], I have not been able to find it anywhere. Note that the seven bell patterns of the *Bembé* Wheel are obtained by appropriate rotations of the cannonical necklace pattern number III in Figure 5.

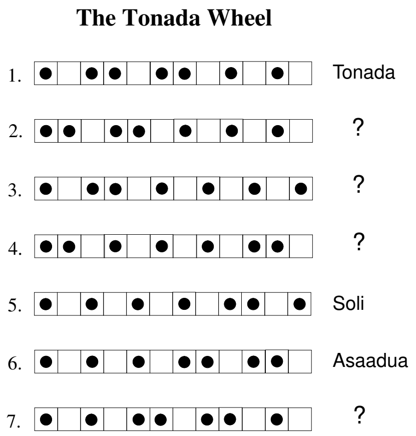

Figure 11: The *Tonada* wheel and the three known rhythms it generates.

4.2 The *Tonada* Wheel

Clearly, just as cannonical necklace pattern number III can be made into a wheel, one can make a wheel out of the two other cannonical patterns. The wheel generated from the cannonical necklace pattern number II will be called the *Tonada* wheel and is depicted in Figure 11. The *Tonada* bell pattern generates only two other rhythmic patterns that I have been able to find. One is the *Asaadua* bell pattern (number six) and the other (number five) is the *Soli*. The *Soli* pattern is the same as the *Al-ramal* drum rhythm from Arab and Persian music dating back to books on rhythm written by SafI-al-DIn in the thirteenth century [41]. Interestingly, SafI-al-DIn depicted the rhythms as a circular "pie" chart divided into equal slices of pie. Each slice corresponded to a time unit, and the slices corresponding to the onsets of the notes were shaded black.

Although we are not concerned with *melodic* rhythms in this paper, African bell patterns have had a great influence on them in America. For example, the Colombian *Bambuco* musical style of the Santander region near Venezuela uses a melodic rhythm of the form [x x . x x . x . x . x .] (see Varney [40]). This is pattern number two of the *Tonada* wheel and can be obtained from the *Asaadua* bell pattern by interchanging the order of the first and second halfs, [x . x . x .] and [x x . x x .], respectively.

4.3 The *Sorsonet* Wheel

The wheel generated from cannonical necklace pattern number I will be called the *Sorsonet* wheel and is depicted in Figure 12. The *Sorsonet* wheel generates only one other pattern that I encountered in the literature. This is pattern number four, a Persian rhythm called *kitaab al-adwaar* that also dates back to SafI-al-DIn [41].

Comparing the three wheels *Sorsonet*, *Tonada*, and *Bembé* corresponding to the three cannonical necklace patterns, one notices that only two patterns are used from necklace I, three from necklace II, and six from necklace III. This suggests a clear direction of preference towards the patterns that have the two short intervals situated as far from each other as possible (or as evenly dispersed as possible) within the cycle.

The reader may have noticed that each of the three wheels generates exactly seven rhythms with this rotation method, for a total of twenty-one, which coincides with the number of different permutations of the interval vector (2212221). This is no coincidence. Indeed, all the permutations may

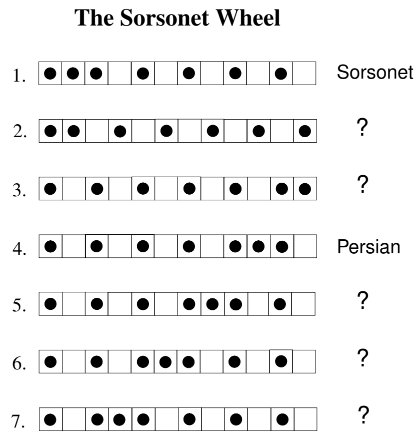

Figure 12: The *Sorsonet* wheel and the two known rhythms it generates.

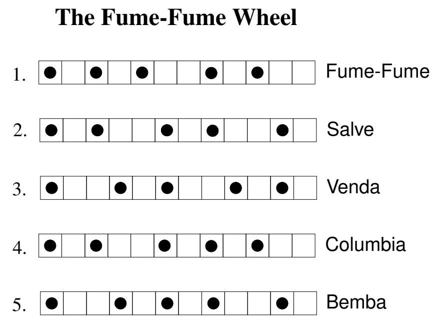

Figure 13: The *Fume-Fume* wheel and the five rhythms it generates.

be enumerated in this way by first generating the necklaces with a variety of existing algorithms [38] and then generating the wheel for each necklace.

4.4 The *Fume-Fume* Wheel

Since the *Fume-Fume* and *Columbia* five-stroke bell patterns play an important role in the complementarity analysis of the ten seven-stroke patterns it is interesting to note that both are contained in each other's wheels. The five rhythms generated by the *Fume-Fume* wheel shown in Figure 13 are all used in traditional African music. The *Fume-Fume* and *Columbia* have already been discussed in the preceeding. The *Bemba* is used in Northern Zimbabwe [35]. The *Venda* is a clapping pattern used in a children's song [35], and is also used in Central African repertoires by the *Aka*, *Gbaya* and *Nzakara* ethnic groups [9]. Finally, the *Salve* is a bell pattern found in the Dominican Republic and used in a rhythm called *Canto de Vela* in honor of the Virgin Mary [17].

5 Plylogenetic Analysis of Rhythms

5.1 Measuring the Similarity of Rhythms

At the heart of any algorithm for comparing, recognizing or classifying a rhythm lies a measure of the similarity between two rhythms. There exists a wide variety of methods for measuring the similarity of two rhythms represented by a string of symbols. When the two strings are binary sequences a natural measure of distance or non-similarity between them is the Hamming distance [19] widely used in coding theory. The Hamming distance is simply the number of places in the strings where elements do not match. For example the Soli and Tambú rhythms differ in the positions of their fifth notes. Therefore there are two locations in the 12-bit binary string where a missmatch occurs and the Hamming distance between Soli and Tambú is equal to 2. The Hamming distance is not very appropriate for our problem of rhythm similarity because although it measures a missmatch, it does not measure how far the missmatch occurs. Furthermore, if a note is moved a large distance the resulting rhythm will sound more different than if it is moved a small distance.

Some rhythm detection algorithms [29] and systems for machine recognition of music patterns [13] use inter-onset intervals as a basis for measuring similarity. These are the intervals of time between consecutive note onsets in a rhythm. Coyle and Shmulevich [13] represent a music pattern by what they call a *difference-of-rhythm vector*. If $T = (t_1, t_2, ..., t_n)$ is a vector of inter-onset time intervals for the notes of a rhythm then they define the difference-of-rhythm vector as $X = (x_1, x_2, ..., x_{n-1})$, where $x_i = t_{i+1}/t_i$. This approach is more appropriate than the Hamming distance for measuring the similarity of rhythms. However, for the phylogenetic analysis of rhythms, the *swap distance* proposed in the following is more natural.

5.2 The Swap Distance

A completely different approach to measuring the dissimilarity between two strings computes the amount of "work" required to transform one string into the other. Such an approach is common in bioinformatics where the two strings to be compared are chain polymers and the work is measured by the minimum number of basic operations required to transform one molecule into the other. The type of basic operation used varies and usually models some kind of mutation relevant to evolution (see [3], [2] and the references therein). The problem of comparing two binary strings of the same length with the same number of one's suggests an extremely simple operation that will be called a *swap*. A swap is an interchange of a one and a zero (onset interval and rest interval) that are adjacent in the string. The swap distance between two rhythms is the minimum number of swaps required to convert one rhythm to the other. For example the Bembé rhythm [x . x . x x . x . x . x] can be converted to the *Tonada* rhythm [x . x x . x x . x . x .] by a minumum of four swaps, namely interchanging the third, fifth, sixth, and seventh strokes with the corresponding rests preceeding them. Such a measure of dissimilarity appears to be more appropriate than the Hamming distance between the binary vectors or the Euclidean distance between the interval vectors, in the context of rhythm similarity. The distance matrix for the ten seven-stroke bell patterns is shown in Figure 14 where the bottom row indicates, for each rhythm, the sum of the swap distances to all the other rhythms. We see that both the *Yoruba* and the *Bemba* are matched for low scores (23), indicating that these two rhythms are more similar to all the rhythms than any other. At the other extreme lies the *Sorsonet* with a score of 51, making it the maverick in the group. The phylogenetic tree computed from this distance matrix is a more revealing structure, as we shall see in the following.

Swap Distance Matrix

	Soli	Tambú	Bembé	Bembé-2	Yoruba	Tonada	Asaadua	Sorsonet	Bemba	Ashanti
Soli		1	2	7	3	6	2	9	4	5
Tambú			1	6	2	5	1	8	3	4
Bembé				5	1	4	2	7	2	3
Bembé-2					4	1	5	2	3	2
Yoruba						3	1	6	1	2
Tonada							4	3	2	1
Asaadua								7	2	3
Sorsonet									5	4
Bemba										1
Ashanti										
Σ	39	31	32	35	23	29	27	51	23	25

Figure 14: The swap distance matrix of the ten rhythms. The bottom row indicates for each rhythm the sum of the swap distances it is from the other nine.

5.3 Phylogenetic Analysis

As was previously demonstrated with the 4/4 time *clave* and bell patterns studied in [39], phylogenetic trees provide a useful tool for visualizing the interrelationships between the rhythms as well as determining economical mechanisms for their generation. These mechanisms may in turn shed light on the evolution of such rhythms. In [39] the distance used was the Euclidean distance between the interval vectors, and the phylogenetic analysis used classical phylogenetic trees. One weakness of classical phylogenetic trees is that they impose a *tree* structure on the data even if the underlying structure is not a tree. However, one may be interested in knowing how appropriate such a tree structure is. In the bioinformatics literature there exist new techniques which provide this information in a graph that is a generalization of a tree. One notable example is the *SplitsTree* [22]. Like the more traditional phylogenetic trees, the Splits Graph is a drawing in the plane with the property that the distance in the drawing between any two nodes reflects as closely as possible the true distance between the corresponding two rhythms in the distance matrix. However, if the tree structure does not match the data perfectly then edges are split to form parallelograms whose size is proportional to the missmatch. Thus the *SplitsTree* may in fact be a graph that is not a tree and has cycles. The *SplitsTree* constructed from the distance matrix of Figure 14 shown in Figure 15 bears this out. The structure is almost a chain except for the four-cycle determined by the *Bembé*, *Yoruba*, *Asaadua*, and *Tambú* rhythms. From the *SplitsTree* several additional properties are immediately evident. Only the *Sorsonet* does not have swap distance one to any other rhythm. The diameter of the graph (two most distinct rhythms) is determined by the *Sorsonet* and the *Soli*. The *center* of the graph (i.e., the vertex that minimizes the maximum distance to any other vertex in the graph) is determined by the *Ashanti* and *Bemba* jointly. Every rhythm can be generated from one of these two by at most four swaps.

More often than not, a *SplitsTree* will have additional nodes that do not correspond to any one of the input rhythms. Such nodes determine implied "ancestral" rhythms from which their "offspring"

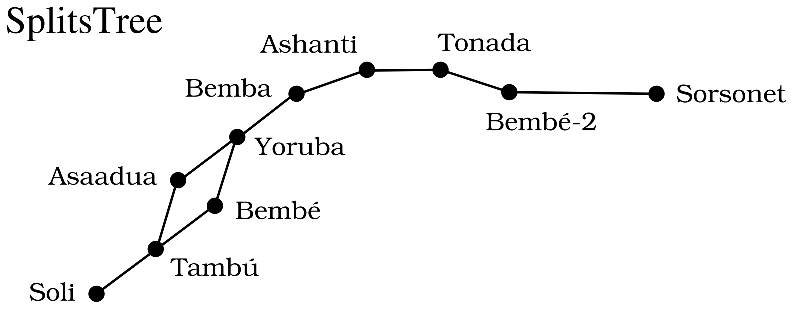

Figure 15: The SplitsTree constructed for the distance matrix in Figure 14.

may be easily derived with the fewest number of swaps (mutations). Surprisingly, in this study the *SplitsTree* computed for the ten bell patterns using the swap distance measure yields no ancestral rhythms not contained in this set. However, the *Bemba* and *Yoruba* are tied for being the *minimal* generators of all the rhythms: both can generate all other rhythms with no more than 23 swaps.

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References

[1] Kofi Agawu. *African Rhythm: A Northern Ewe Perspective.* Cambridge University Press, Cambridge, England, 1995.

[2] Greg Aloupis, Prosenjit Bose, Erik D. Demaine, Stefan Langerman, Henk Meijer, Mark Overmars, and Godfried T. Toussaint. Computing signed permutations of polygons. In *Proc. 14th Canadian Conf. Computational Geometry*, pages 68–71, Univ. of Lethbridge, Alberta, 2002.

[3] Greg Aloupis, Erik D. Demaine, Henk Meijer, Joseph O'Rourke, Ileana Streinu, and Godfried T. Toussaint. On flat-state connectivity of chains with fixed acute angles. In *Proc. 14th Canadian Conf. Computational Geometry*, pages 27–30, Univ. of Lethbridge, Alberta, 2002.

[4] Simha Arom. *African Polyphony and Polyrhythm.* Cambridge University Press, Cambridge, England, 1991.

[5] G. Balzano. The group theoretic description of 12-fold and microtonal systems. *Computer Music Journal*, 4:66–84, 1980.

[6] J. Becker. Hindu-Buddhist time in Javanese gamelan music. In *Study of Time IV*. 1981. Ed.,J. T. Fraser.

[7] H. Bettermann, D. Amponsah, D. Cysarz, and P. Van Leeuwen. Musical rhythms in heart period dynamics: a cross-cultural and interdisciplinary approach to cardiac rhythms. *Proceedings of the American Physiological Society*, pages H1762–H1770, 1999.

[8] Steven Block and Jack Douthett. Vector products and intervallic weighting. *Journal of Music Theory*, 38:21–41, 1994.

[9] Marc Chemillier and Charlotte Truchet. Computation of words satisfying the "rhythmic oddity property" (after Simha Arom's works). Technical report, GREYC, University of Caen, France, 2002.

[10] John Miller Chernoff. *African Rhythm and African Sensibility.* The University of Chicago Press, Chicago, 1979.

[11] J. Clough and J. Douthett. Maximally even sets. *Journal of Music Theory*, 35:93–173, 1991.

[12] John Collins. *Highlife Time.* Ananansesem Press, Ghana, 1996.

[13] E. J. Coyle and I. Shmulevich. A system for machine recognition of music patterns. In *Proceedings of the IEEE International Conference on Acoustics, Speech, and Signal Processing*, Seattle, Washington, 1998.

[14] Andrew Duncan. Combinatorial music theory. *Journal of the Audio Engineering Society*, 39:427–448, June 1991.

[15] William Dunham. *Journey Through Genious: The Great Theorems of Mathematics.* John Wiley and Sons, Inc., New York, 1990.

[16] A. Dworsky and B. Sansby. *A Rhythmic Vocabulary.* Dancing Hands Music, Minnetonka, 1999.

[17] Mary Farquharson. *Africa in America.* Discos Corazon, Mexico, 1992. [CD].

[18] Lennart Hallstrom. *African Drum Rhythms for Djembes, Bass Drums and Bells.* Lennart Hallstrom, Stockholm, 2000.

[19] R. W. Hamming. *Coding and Information Theory.* Prentice-Hall, Englewood Cliffs, 1986.

[20] Royal Hartigan, Abraham Adzenyah, and Freeman Donkor. *West African Rhythms for Drum Set.* Manhattan Music, Inc., 1995.

[21] T. C. Hu and B. N. Tien. Generating permutations with non-distinct items. *American Mathematical Monthly*, 83(8):193–196, October 1976.

[22] Daniel H. Huson. SplitsTree: Analyzing and visualizing evolutionary data. *Bioinformatics*, 14:68–73, 1998.

[23] A. M. Jones. *Studies in African Music.* Oxford University Press, Amen House, London, 1959.

[24] Michael Keith. *From Polychords to Pólya: Adventures in Musical Combinatorics.* Vinculum Press, Princeton, 1991.

[25] Tőm Klőwer. *The Joy of Drumming: Drums and Percussion Instruments from Around the World.* Binkey Kok Publications, Diever, Holland, 1997.

[26] Justin London. Hierarchical representations of complex meters. In *6th International Conference on Music, Perception and Cognition*, Keele University, United Kingdom, August 5-10 2000.

[27] F. Malabe and B. Weiner. *Afro-Cuban Rhythms for Drumset.* Manhattan Music, Inc., 1990.

[28] Neil McLachlan. A spatial theory of rhythmic resolution. *Leonardo*, 10:61–67, 2000.

[29] Bernard Mont-Reynaud and Malcolm Goldstein. On finding rhythmic patterns in musical lines. In *Proceedings of the International Computer Music Conference*, pages 391–397, San Francisco, California, 1985.

[30] Larry Morris. *Rhythm Catalog.* The Internet, http://www.drums.org/djembefaq, 2001.

[31] J. H. Nketia. *Drumming in Akan Communities of Ghana.* Thomas Nelson and Sons Ltd., Edinburgh, Scotland, 1963.

[32] J. H. Kwabena Nketia. *African Music in Ghana.* Northwestern Univ. Press, Britain, 1963.

[33] H. Orovio. *Diccionario de la Musica Cubana.* Letras Cubanas, Havana, Cuba, 1992.

[34] Fernando Ortiz. *La Clave.* Editorial Letras Cubanas, La Habana, Cuba, 1995.

[35] Jeff Pressing. Cognitive isomorphisms between pitch and rhythm in world musics: West Africa, the Balkans and Western tonality. *Studies in Music*, 17:38–61, 1983.

[36] Olavo Alén Rodríguez. *From Afro-Cuban Music to Salsa.* Piranha, Berlin, 1998.

[37] Rene V. Rosalia. *Migrated Rhythm: The Tambú of Curaçao.* CaribSeek, 2002.

[38] Frank Ruskey and Joe Sawada. An efficient algorithm for generating necklaces with fixed density. *SIAM Journal of Computing*, 29(2):671–684, 1999.

[39] Godfried T. Toussaint. A mathematical analysis of African, Brazilian, and Cuban *clave* rhythms. In *Proceedings of BRIDGES: Mathematical Connections in Art, Music and Science*, pages 157–168, Towson University, Towson, MD, July 27-29 2002.

[40] John Varney. *Colombian* Bambuco*: The Evolution of a National Music Style.* Grifith University, South Brisbane, Australia, 1999. Ph.D. Thesis.

[41] O. Wright. *The Modal System of Arab and Persian Music AD 1250-1300.* Oxford University Press, Oxford, England, 1978.