On Polyhedra Induced by Point Sets in Space*

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Abstract

Given a set S of $n \geq 3$ points in the plane (not all on a line) it is well known that it is always possible to *polygonize* S, i.e., construct a simple polygon P such that the vertices of P are precisely the given points in S. In 1994 Grünbaum showed that an analogous theorem holds in 3-dimensional space. More precisely, if S is a set of n points in space (not all of which are coplanar) then it is always possible to *polyhedronize* S, i.e., construct a simple (sphere-like) polyhedron P such that the vertices of P are precisely the given points in S. Grünbaum's constructive proof may yield *Schönhardt* polyhedra that cannot be triangulated. In this paper several alternative algorithms are proposed for constructing polyhedra induced by a set of points in space, which may always be triangulated, and which enjoy several other useful properties as well. Such properties include polyhedra that are star-shaped, have Hamiltonian skeletons, and admit efficient point location queries. Furthermore, we show that *polyhedronizations* with a variety of these properties can be computed in $O(n \log n)$ time.

1 Introduction

In 1964 Hugo Steinhaus posed the following problem [11]. Consider a set S of $n \geq 3$ points in the plane such that no three of them lie on the same straight line. Is it always possible to find a closed polygon with n non-intersecting sides whose vertices are these n points? Then he proceeded to give a clever proof by induction that this is true. Since then several mathematicians have provided alternate proofs (see for example Gemignani [3], Quintas and Supnick [8] and Grünbaum [6]). This is now a well known result in computational geometry. Gemignani's proof yields immediately an algorithm that runs in $O(n \log n)$ time, which is optimal since Shamos [10] proved an $\Omega(n \log n)$ lower bound on this problem. In fact this problem has often been tackled in computational geometry as a stepping stone to solving other problems. For example, in 1972 Ron Graham [5] proposed a simple optimal $O(n \log n)$ time algorithm for computing the convex hull of S by first computing a *star-shaped* polygonization of S. Grünbaum's constructive proof yields a *monotone* polygonization. Abellanas et al. show that an *onion* polygonization is always possible [1].

The planar polygonization problem can be generalized in at least two ways to 3-dimensional space. We can ask for a closed polygonal chain that is "simple" in the sense that it is not knotted. We call this the 3D-polygonization problem. This problem can be solved using the planar polygonization procedures by suitably projecting the points of S onto a plane and then "lifting" the planar polygonization obtained back into space. In the more interesting generalization we can ask for a simple polyhedron the vertices of which are the given point set. We call this problem the *polyhedronization* problem. Surprisingly this problem has received little attention.

In this paper we study various methods for generating polyhedronizations that have a variety of desirable properties: monotonicity, star-shapedness, admitting a tetrahedralization, possibly with nice dual structure, possessing a good 1-skeleton from the viewpoint of graph theory and affording fast point location queries. The 3D-polygonization problem is solved along the way in that a polyhedronization with the property that it admits a Hamiltonian 1-skeleton yields a 3D-polygonization when one of its Hamiltonian cycles is reported. Before presenting the 3D results we introduce a new polygonization method that combines the desirable properties of both the

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monotonic and star-shaped polygonizations, yielding in $O(n \log n)$ time, polygonizations that are: (1) monotonic, (2) serpentine and (3) triangulated in a serpentine manner at no extra cost. The method is also interesting because, surprisingly, it does not extend to 3D.

2 Serpentine Polygonizations

The main idea of the planar polygonization algorithm is quite simple and consists of sorting all the points along some direction such as the x-axis, connecting the first three points to form a triangle, and subsequently processing one point at a time in the sorted list, creating a new triangle that is "glued" on to a suitable visible edge of the existing polygonization. We describe next the algorithm more formally.

INITIALIZATION

(1) Sort lexicographically the points along their $+x$ and $+y$ coordinates to obtain the list $p_1, p_2, \ldots, p_n$.

(2) If p_1, p_2, and p_3 are not collinear, connect them to form triangle T_3, which would also be the initial polygon Q_3 and define an initial value ℓ to be $\ell = 4$.

Otherwise let p_j be the first point non-collinear with its preceding points in the list, $p_1, \ldots, p_{j-1}$. Discard temporarily the points $p_2, \ldots, p_{j-2}$, consider $T_j = Q_j = p_1 p_{j-1} p_j$ as the initial triangle and polygon, and let $\ell = j + 1$.

ITERATION

for $i = \ell$ to $i = n$ do:

Connect point p_i to a visible edge incident to p_{i-1} of triangle T_{i-1} in the polygon constructed thus far, denote this new triangle by T_i and the updated polygon by Q_i.

end for

FINALIZATION

If $\ell > 3$, split the triangle $p_1 p_j p_{j-1}$ into triangles $T_3 = p_1 p_j p_2, T_4 = p_2 p_j p_3, \ldots, T_j = p_{j-2} p_j p_{j-1}$.

The *finalization* step of the algorithm is only necessary when p_1, p_2 and p_3 are collinear, and its only meaning is to split the initial triangle into a path of sub-triangles. As the complexity of the algorithm is dominated by the sorting step, we have obtained the following result:

Theorem 2.1 *A planar set of points S admits a* serpentine *polygonization and a triangulated serpentine polygonization can be obtained in $O(n \log n)$ time.*

At first glance it may appear that this algorithm extends to three dimensions by "gluing" a new tetrahedron to one of the three faces incident on the last vertex of the polyhedron constructed thus far. Unfortunately, it may happen that none of the three faces is completely visible from the new point to be inserted, and therefore the method fails. An example of a set of points for which this procedure fails is shown in Figure 1.

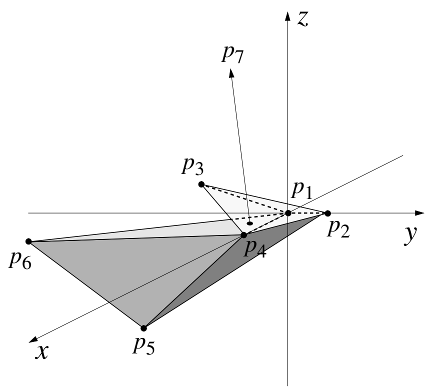

Figure 1: Illustrating the polyhedron constructed from the first three tetrahedra and the line on which the seventh point lies.

First consider the six points ordered by non-decreasing x-coordinate: $P_1 = (0,0,0)$, $P_2 = (0,1,0)$, $P_3 = (0,-3,1)$, $P_4 = (1,0,0)$, $P_5 = (2,-5/2,-3)$ and $P_6 = (3,-4,1)$. The first, second, and third tetrahedra glued in the construction are given, respectively, by $P_1P_2P_3P_4$, $P_1P_2P_4P_5$, $P_1P_4P_5P_6$.

When viewed from the top ($+z$ direction) the projection of P_5 on the xy plane lies in the interior of the projection of the triangle $P_1P_6P_4$. Therefore the outer normals of faces $P_4P_5P_6$ and $P_1P_5P_6$ are pointing in the negative z direction. Furthermore, any point P_7 above the planes $P_1P_4P_6$, $P_1P_5P_6$ and $P_4P_5P_6$ cannot see faces $P_4P_5P_6$ and $P_1P_5P_6$. Finally, if P_7 is high enough it will not see face $P_1P_4P_6$ either, and if P_7 lies on a nearly vertical line slightly slanted towards the positive x axis, its x-coordinate can be made to be larger than that of P_6, as required.

3 Polyhedronization of Point Sets in Space

In this section we outline a proof that every set of $n \geq 4$ points in three dimensional space, not all of which are coplanar, admits a polyhedronization that can be tetrahedralized. We describe several different types of polyhedronizations and analyze their properties as well as algorithms for their computation. Additional types of polyhedronizations are described in the full paper.

Monotonic Polyhedronizations

In 1994 Grünbaum [6] gave a constructive proof that a set of points S, not all of which are coplanar, can always be polyhedronized. However, he was not concerned with either its properties or its computational complexity. As it turns out his idea leads to a monotonic polyhedronization, as defined below. In the following we present a minor simplification of his approach and show that it can be efficiently computed.

Definition 3.1 *A polyhedron is xy-monotonic provided that its intersection with every line parallel to the z-axis is either empty or a connected set.*

Theorem 3.1 *A set of points S in space, not all coplanar, admits a monotonic polyhedronization that can be obtained in $O(n \log n)$ time.*

Proof: First rotate S so that it has a *regular* projection on the xy-plane (no two points lie on a vertical line). Next compute the convex hull $CH(S)$ of S. If no points of S lie inside this convex hull we are done because the polyhedron $CH(S)$ is monotonic. Otherwise, let $CH_L(S)$ and $CH_U(S)$ be the lower and upper convex hull of S, respectively. Let B be the shadow boundary of the convex hull of S, in other words the set of edges common to $CH_L(S)$ and $CH_U(S)$. Let S_U be the subset of points in S which are not vertices of $CH_L(S)$. Triangulate the projection of the set $S_U \cup B$ on the xy-plane in such a way that no edge in this triangulation connects two vertices of B. Lift each triangle of the triangulation to the points that projected onto its vertices. By gluing along B this terrain with $CH_L(S)$ we obtain the desired monotonic polyhedron.

Computing a regular projection of n points can be done in $O(n \log n)$ time with the algorithm of Gómez et al. [4]. Computing the convex hull and constructing a triangulation of the projected points can also be done in time $O(n \log n)$ [7], which is the overall running time since $O(n)$ time is sufficient for the lifting step. ∎

A drawback of Grünbaum's construction is that the resulting polyhedronization may not admit a tetrahedralization, as is demonstrated in the following. Although a polygon in the plane can always be triangulated, it is well known that the analogous theorem in 3D does not hold. In 1928 Schönhardt [9], (see also [2]) constructed a polyhedron with six vertices that does not admit even a single internal diagonal.

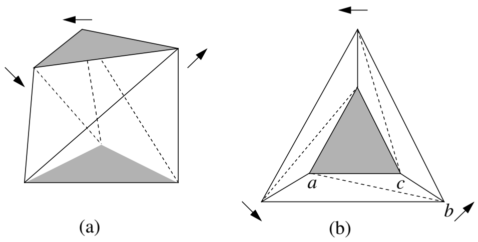

Figure 2: Constructing a polyhedral terrain that cannot be tetrahedralized.

Schönhardt's six-vertex polyhedron is constructed by starting with a triangular prism (refer to Fig. 2 (a)) and "twisting" the top face in the direction shown by some small amount. The three side faces cannot remain planar and so "buckle" inwards along the diagonals to produce non-coplanar triangular faces. No two non-adjacent vertices of this polyhedron are internally visible from each other.

To create a terrain polyhedron that cannot be tetrahedralized, simply construct Schönhardt's six-vertex polyhedron with the base triangle larger than the top triangle (see top view in Fig. 2(b)). If the vertices of the resulting Schönhardt terrain are the input to Grünbaum's construction, and the projected points are triangulated as in Fig. 2(b), the Schönhardt terrain will be reconstructed.

Although our simplification of Grünbaum's construction, described in the preceding, will result in a convex polyhedronization, which is therefore tetrahedralizable, one can obtain a set of nine points (by adding an even larger triangle below Schönhardt's example) for which our simplification also yields a polyhedronization that cannot be tetrahedralized.

In the long version of the paper we present an alternate polyhedronization algorithm, more complicated than Grünbaum's, that always admits a tetrahedralization. It is based on the following lemmas, where for simplicity of explanation we assume that no three points of S lie on a vertical plane.

Lemma 3.2 *Given a monotonic tetrahedralized polyhedron, and points that lie above or below the polyhedron such that a vertical line through each of them intersects the boundary of the polyhedron twice, it is possible to enlarge the polyhedron and its tetrahedralization to encompass these points, while remaining monotonic and tetrahedralized.*

Lemma 3.3 *Let P be a triangulated convex polygon with vertex set V lying in the xy-plane, and let S be a point set in 3-space such that every point in S has positive z coordinate and projects vertically inside P (strictly). Then it is possible to construct a tetrahedralized monotonic polyhedron with vertex set $V \cup S$ such that its lower terrain is P.*

Theorem 3.4 *A set of points S in space admits a tetrahedralizable monotonic polyhedronization.*

Star-shaped Polyhedronizations

In this section we show that if we assume that S is in general position in the sense that no four points are coplanar, then it is possible to polyhedronize S in several easy ways that yield nice properties.

Hinge Polyhedronizations

We give the name *hinge polyhedronization* to the following construction. Start with any pair of points $x, y \in S$ for which xy is an edge of the convex hull $CH(S)$, consider a plane H that supports S at xy, and let H^* be a halfplane in H bounded by the line $r = xy$. Sort all the remaining points in the order they are encountered when H^* is rotated around r. Connect all these points in sorted order obtaining an open polygonal chain, and finally connect every vertex of this chain to both x and y.

Theorem 3.5 *A hinge polyhedronization can be constructed in $O(n \log n)$ time and has the following properties:*

1. *star-shaped (fan, edge-visible);*
2. *serpentine;*
3. *Hamiltonian 1-skeleton;*
4. *affords easy $O(\log n)$ point-location queries.*

Orange Polyhedronizations

An *orange polyhedronization*, is a slight modification of a hinge polyhedronization. Start with any pair of points $x, y \in S$ for which xy is *not* an edge of the convex hull $CH(S)$ (we leave out the trivial case of four points in convex position), consider a plane H through xy, and let H^* be a halfplane in H bounded by the line $r = xy$. Sort all the remaining points in the cyclic order they are encountered when H^* is rotated around r. Connect all these points in sorted order obtaining a *closed* polygonal chain, and connect every vertex of this chain to both x and y.

Theorem 3.6 *An orange polyhedronization can be constructed in $O(n \log n)$ time and has the following properties:*

1. *star-shaped (from a diagonal);*
2. *admits a tetrahedralization whose dual is a cycle;*
3. *Hamiltonian 1-skeleton;*
4. *has an Eulerian 1-skeleton for even n;*
5. *affords easy $O(\log n)$ point-location queries.*

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