

Fractional Colouring and Hadwiger's Conjecture

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Let G be a loopless graph with no K_{p+1} minor. We prove that the “fractional chromatic number” of G is at most $2p$; that is, it is possible to assign a rational $q(S) \geq 0$ to every stable set $S \subseteq V(G)$ so that $\sum_{S \ni v} q(S) = 1$ for every vertex v , and $\sum_S q(S) \leq 2p$. © 1998 Academic Press

1. INTRODUCTION

In this paper, all graphs are finite and without loops or multiple edges; K_{p+1} denotes the complete graph with $p+1$ vertices; and H is a *minor* of G if H can be obtained from a subgraph of G by contracting edges. A famous conjecture of Hadwiger [4], dating from 1943, asserts

(1.1) (Conjecture) *For all $p \geq 1$, every graph with no K_{p+1} minor is p -colorable.*

This is easy for $p \leq 3$, true but difficult for $p = 4$ and 5 (in both cases [9, 6] it is equivalent to the four-colour theorem [1, 2, 7]), and open for $p \geq 6$. It is therefore of interest to examine weakenings of (1.1).

If G has no K_{p+1} minor, the best upper bound known on its chromatic number is $O(p(\log p)^{1/2})$. This follows easily from the following theorem of Kostochka [5] or Thomason [8].

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(1.2) *If G has no K_{p+1} minor then*

$$|E(G)|/|V(G)| \leq O(p(\log p)^{1/2}).$$

Since (1.2) is best possible, and indeed there are graphs with no K_{p+1} minor with minimum degree $O(p(\log p)^{1/2})$, it is a commonly expressed “counter-conjecture” that for large p , (1.1) is false and the best possible bound on chromatic number is $O(p(\log p)^{1/2})$.

For that reason it would be of great interest to decide whether there is a constant C such that every graph with no K_{p+1} minor is Cp -colorable. This is still open, but in this paper we prove a “fractional” version with $C=2$.

Let $k \geq 0$ be a rational. A *fractional k -colouring* of a graph G means a map $q: \mathcal{S} \rightarrow \mathbf{Q}_+$ (where $\mathbf{Q}_+$ is the set of non-negative rationals, and $\mathcal{S}$ is the set of all stable subsets of $V(G)$) such that

- (i) for every vertex v , $\sum (q(S): S \in \mathcal{S} \text{ and } v \in S) = 1$, and
- (ii) $\sum (q(S): S \in \mathcal{S}) \leq k$.

Thus G is k -colourable, where k is an integer, if and only if it has a fractional k -colouring q which is $(0,1)$ -valued. Consequently, Hadwiger’s conjecture (1.1) implies that every graph with no K_{p+1} minor has a fractional p -colouring; but this too remains open. Our main result is:

(1.3) *For every integer $p \geq 1$, every graph with no K_{p+1} minor has a fractional $2p$ -colouring.*

From linear programming duality, to prove (1.3) it suffices to prove the following:

(1.4) *Let G have no K_{p+1} minor, and let $w: V(G) \rightarrow \mathbf{Q}_+$ be some map. There is a stable set $X \subseteq V(G)$ such that*

$$\sum_{v \in X} w(v) \geq \frac{1}{2p} \sum_{v \in V(G)} w(v).$$

For $w \equiv 1$ this was already proved [3]; indeed, it was the rediscovery of that result by Eldar Fischer that motivated the present paper.

Our approach is the following. A graph is *chordal* if it has no induced circuit of length ≥ 4 . It is elementary that the chromatic number of a chordal graph H equals the maximum size of a complete subgraph of H . If G is a graph and $X \subseteq V(G)$, we denote by $G|X$ the subgraph induced on X , that is, the subgraph obtained by deleting $V(G) - X$. If $X, Y \subseteq V(G)$ are disjoint, we say that X, Y *touch* (in G) if some vertex in X is adjacent in G to some vertex in Y . If $\mathcal{P} = \{X_1, \dots, X_n\}$ is a partition of $V(G)$, its

touching pattern $H(\mathcal{P})$ is the graph with vertex set $\mathcal{P}$ in which X_i, X_j are adjacent if $i \neq j$ and they touch.

A *chordal decomposition* of G means a partition $\mathcal{P}$ of $V(G)$ such that

- (i) $G|X$ is non-null and connected for each $X \in \mathcal{P}$, and
- (ii) $H(\mathcal{P})$ is chordal.

We call the subgraphs $G|X$ ($X \in \mathcal{P}$) the *pieces* of the decomposition. Evidently every graph has a chordal decomposition (take the pieces to be the components of G). But it seems an interesting question how much one can restrict the type of pieces in such a decomposition.

For instance, perhaps every graph has a chordal decomposition in which every piece is bipartite. If this is true, then the graph G of (1.1) is $2p$ -colourable. To see this, let $\mathcal{P}$ be a chordal decomposition of G with bipartite pieces. Since G has no K_{p+1} minor and the pieces are connected, it follows that $H(\mathcal{P})$ has no K_{p+1} subgraph. Since $H(\mathcal{P})$ is chordal, it is p -colourable. Since every piece is bipartite, it follows that G is $2p$ -colourable, as required.

We do not know whether every graph has a chordal decomposition with bipartite pieces, but the main step in our proof of (1.3) is proving a related statement, the following. Let G be a graph and let $w: V(G) \rightarrow \mathbf{Q}_+$. If $X \subseteq V(G)$, $w(X)$ denotes $\sum (w(v): v \in X)$. If $X \subseteq V(G)$, a *yolk* of X means a subset $Y \subseteq X$ such that Y is stable and $w(Y) \geq \frac{1}{2} w(X)$. We say $X \subseteq V(G)$ is an *egg* (of (G, w)) if $G|X$ is non-null and connected, and X has a yolk. We shall prove the following in the next section.

(1.5) *For every graph G and every map $w: V(G) \rightarrow \mathbf{Q}_+$, there is a chordal decomposition $\mathcal{P}$ of G such that each $X \in \mathcal{P}$ is an egg.*

Proof of (1.3), assuming (1.5). It suffices to prove (1.4). Let G be a graph and $w: V(G) \rightarrow \mathbf{Q}_+$ some map. By (1.5) there is a chordal decomposition $\mathcal{P}$ of G such that every member of $\mathcal{P}$ is an egg. As before, $H(\mathcal{P})$ is p -colourable; let $\{S_1, \dots, S_p\}$ be a partition of $\mathcal{P}$ into stable sets. For $1 \leq i \leq p$, let

$$n_i = \sum_{X \in S_i} w(X).$$

Then $n_1 + \dots + n_p = w(V(G))$, so we may assume that $n_1 \geq p^{-1} w(V(G))$. Let $S_1 = \{X_1, \dots, X_k\}$ say, and for $1 \leq j \leq k$ let Y_j be a yolk of X_j . Since S_1 is stable in $H(\mathcal{P})$ it follows that $Y_1 \cup \dots \cup Y_k$ is stable in G ; but

$$w(Y_1 \cup \dots \cup Y_k) = \sum_{1 \leq j \leq k} w(Y_j) \geq \sum_{1 \leq j \leq k} \frac{1}{2} w(X_j) = \frac{1}{2} n_1 \geq (2p)^{-1} w(V(G)),$$

as required. ■

2. EGG-DECOMPOSITIONS

In this section we prove (1.5). For inductive purposes we need a slightly more general kind of decomposition. A vertex v of H is *simplicial* if every two neighbours of v are adjacent. Let G be a graph and $w: V(G) \rightarrow \mathbf{Q}_+$ a map, as before. An *egg-decomposition* of G is a chordal decomposition $\mathcal{P}$ such that for every $X \in \mathcal{P}$, either X is an egg, or X is simplicial in $H(\mathcal{P})$ and all its neighbours in $H(\mathcal{P})$ are eggs. The *support* $\mathcal{S}(\mathcal{P})$ of an egg-decomposition $\mathcal{P}$ is the union of all $X \in \mathcal{P}$ that are eggs.

Proof of (1.5). We may assume that G is connected and non-null. Now G has an egg-decomposition with non-empty support (to see this, choose any vertex v , let $X_1, \dots, X_k$ be the vertex sets of the components of $G \setminus v$, and let $P = \{\{v\}, X_1, \dots, X_k\}$; then $\mathcal{P}$ is an egg-decomposition with non-empty support). Consequently there is an egg-decomposition $\mathcal{P}$ with maximal support and in particular it satisfies $\mathcal{S}(\mathcal{P}) \neq \emptyset$.

Suppose, for a contradiction, that $X_0 \in \mathcal{P}$ is not an egg. Let $X_1, \dots, X_n$ be the neighbours of X_0 in $H(\mathcal{P})$; thus, $X_1, \dots, X_n$ are all eggs, and pairwise touch. Since $\mathcal{S}(\mathcal{P}) \neq \emptyset$ and G is connected, it follows that $n \geq 1$. For $1 \leq i \leq n$, choose a yolk Y_i for X_i , and define N_i to be the set of all $v \in X_0$ with a neighbour in X_i .

(1) *For all i, j with $i \neq j$ and $1 \leq i, j \leq n$, every induced path of $G \mid X_0$ with first vertex in N_i , no other vertex in N_i , last vertex in N_j , and no other vertex in N_j has an even number of edges.*

Subproof. Suppose Q is an induced path of $G \mid X_0$ with vertices $v_1, \dots, v_{2k}$ in order, where $V(Q) \cap N_i = \{v_1\}$ and $V(Q) \cap N_j = \{v_{2k}\}$. By exchanging i and j and reversing the numbering of $V(Q)$, we may assume that

$$w(v_1) + w(v_3) + \dots + w(v_{2k-1}) \leq w(v_2) + w(v_4) + \dots + w(v_{2k}).$$

Now $Y_i \cup \{v_2, v_4, \dots, v_{2k}\}$ is a stable set, because $v_2, v_4, \dots, v_{2k}$ are pairwise non-adjacent (since Q is induced) and have no neighbours in Y_i (since they are not in N_i). But

$$w(Y_i) \geq \frac{1}{2} w(X_i)$$

$$w(\{v_2, v_4, \dots, v_{2k}\}) \geq \frac{1}{2} w(V(Q))$$

and so $Y_i \cup \{v_2, v_4, \dots, v_{2k}\}$ is a yolk of $X_i \cup V(Q)$. Since $X_i \cup V(Q)$ is connected it is therefore an egg. Let $\mathcal{P}'$ be the partition of $V(G)$ defined by

$$\mathcal{P}' = (\mathcal{P} - \{X_0, X_i\}) \cup \{X_i \cup V(Q)\} \cup \{Z_1, \dots, Z_r\},$$

where $Z_1, \dots, Z_r$ are the vertex sets of the components of $G \mid (X_0 - V(Q))$. Since $X_1, \dots, X_n$ pairwise touch, and $V(Q)$ touches no member of $\mathcal{P}$ different from $X_0, X_1, \dots, X_n$ it follows that

$$H(\mathcal{P}') \setminus \{Z_1, \dots, Z_r\} = H(\mathcal{P}) \setminus \{X_0\}$$

except that the vertex X_i has been replaced by a new vertex $X_i \cup V(Q)$ with the same neighbours. Moreover, since $Z_1, \dots, Z_r$ do not touch each other, and do not touch any member of $\mathcal{P}$ different from $X_0, X_1, \dots, X_n$, it follows that $H(\mathcal{P}')$ is chordal. Hence $\mathcal{P}'$ is an egg-decomposition with

$$\mathcal{S}(\mathcal{P}) \cup V(Q) \subseteq \mathcal{S}(\mathcal{P}'),$$

contradicting the maximality of $\mathcal{S}(\mathcal{P})$. This proves (1).

Choose a minimal $U \subseteq X_0$ such that $G \mid U$ is connected and $U \cap N_i \neq \emptyset$ for $1 \leq i \leq n$. (This is possible, since $G \mid X_0$ is connected and $X_0 \cap N_i \neq \emptyset$ for $1 \leq i \leq n$.) Suppose that $G \mid U$ has a circuit C of odd length. From the minimality of U , for each $v \in V(C)$ there exists i ($1 \leq i \leq n$) such that the component of $G \mid (U - \{v\})$ which contains $C \setminus v$ does not intersect N_i ; and consequently v belongs to every path of $G \mid U$ between N_i and $V(C)$. Let $f(v) = i$, say.

We claim that $f(u) \neq f(v)$ if $u, v \in V(C)$ are distinct. For suppose $f(u) = f(v) = i$, say. Choose a minimal path Q of $G \mid U$ between N_i and $V(C)$. Then Q has only one vertex in $V(C)$, and so does not contain both u and v , a contradiction. Hence $f(u) \neq f(v)$ for all distinct $u, v \in V(C)$. We may therefore arrange the numbering so that $V(C) = \{v_1, \dots, v_k\}$ say, in order, and $f(v_i) = i$ ($1 \leq i \leq k$). For $1 \leq i \leq k$, let Q_i be a minimal path of $G \mid U$ between N_i and $V(C)$; then Q_i has one end v_i , and the other end, u_i say, in N_i , and $V(Q_i) \cap N_i = \{u_i\}$, $V(Q_i) \cap V(C) = \{v_i\}$.

Since C has odd length, there are two vertices v_i, v_j , adjacent in C , so that $|E(Q_i)| \equiv |E(Q_j)|$ (modulo 2); so we may assume that $|E(Q_1)| \equiv |E(Q_2)|$ (modulo 2). If Q_1 meets Q_2 , or some vertex of Q_1 different from v_1 has a neighbour in $V(Q_2)$, then there is a path of $G \mid U$ from u_1 to v_2 not containing v_1 , contradicting $f(v_1) = 1$. Hence $Q_1 \cap Q_2$ is null and no vertex of Q_1 except v_1 has a neighbour in $V(Q_2)$; and similarly no vertex of Q_2 except v_2 has a neighbour in $V(Q_1)$. Consequently the subgraph consisting of Q_1, Q_2 and the edge $v_1 v_2$ is an induced path of $G \mid X_0$ from N_1 to N_2 with an odd number of edges. Moreover, it has no vertex in N_1 except its first, and no vertex in N_2 except its last, contrary to (1).

This proves that $G \mid U$ has no circuit of odd length, and so is bipartite. Since $n \neq 0$ and hence $U \neq \emptyset$, it follows that U is an egg. Let

$$\mathcal{P}' = (\mathcal{P} - \{X_0\}) \cup \{U, Z_1, \dots, Z_r\},$$

where $Z_1, \dots, Z_r$ are the vertex sets of the connected components of $G|(X_0 - U)$; then $\mathcal{P}'$ is an egg-decomposition with

$$\mathcal{S}(\mathcal{P}) \cup U \subseteq \mathcal{S}(\mathcal{P}'),$$

a contradiction, since $U \neq \emptyset$.

This proves that every $X_0 \in \mathcal{P}$ is an egg, and so $\mathcal{P}$ satisfies (1.5), as required. ■

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