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Computing a High Depth Point in the Plane

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Abstract Given a set $S = \{P_1, \ldots, P_n\}$ of n points in R^d, the *depth* $\delta(Q)$ of a point $Q \in R^d$ is the minimum number of points of S that must be in a closed halfspace containing Q. A high depth point is a point whose depth is at least $\max_i [\delta(P_i)]$. For dimension $d = 2$ we give a simple, easily implementable $O(n(\log n)^2)$ deterministic algorithm to compute a high depth point and we give an $\Omega(n \log n)$ lower bound for this task.

1 Introduction and Summary

Given a set $S = \{P_1, \ldots, P_n\}$ of n real numbers, the depth of $x \in R$ is defined to be

$$d(x) = \min(|\{P_i \leq x\}|, |\{P_i \geq x\}|), \tag{1}$$

and may be thought of as the "size" of the smallest halfspace containing x. A median is a point of maximal depth ($\lceil (n+1)/2 \rceil$) and may be found in $O(n)$ time.

There are many situations where multivariate generalizations of ranks and order statistics are useful, and several suggestions for depth in R^d have been made (Chazelle [2], Gill et. al. [8], Rousseeuw and Hubert [14], Rousseeuw and Hubert [15]). One of the more familiar ones was proposed by John Tukey [17], a natural extension to $d > 1$ of the notion in (1). Given a set $S = \{P_1, \ldots, P_n\}$ of n points in R^d, the Tukey depth, or simply depth of $x \in R^d$ is defined by

$$\delta_S(x) = \min_{u:\|u\|=1} (d_u(x)) \tag{2}$$

where $d_u(x)$ is the directional depth of x in direction u and is defined as the depth (as in (1)) of the P_i projected orthogonally onto the line $\{x + tu, t \in R\}$. Clearly $\delta_S(x)$ measures the smallest number of points of S in any halfspace containing x, and we could take

$$\delta_S(x) = \min_{\text{halfspace } h \ni x} |h \cap S| \tag{3}$$

as the definition of Tukey depth. When the set that defines depth is clear we write δ for δ_S. A *median* is a point μ of maximal depth, and we write $\delta^* = \delta^*_S$ for the Tukey depth of a median.

For integer $k > 0$ let $C_k(S) \subset R^d$ denote the set of points of depth at least k. Then

$$\delta^* = \max\,(k : C_k(S) \neq \phi). \tag{4}$$

It follows from (3) that $C_k(S)$ is convex. A well known consequence of Helly's Theorem (e.g., Edelsbrunner [7]) is that there is a point $x \in R^d$

(not necessarily in S) of depth at least $\lceil n/(d+1)\rceil$. Such a point is called a *centerpoint*. The *center* is the set of all centerpoints.

The question addressed in this paper is related to the computational complexity of selection; that is, finding points of a given depth k, or finding points of maximal depth $\delta^* = \delta^*_S$, along with the value of δ^*. Clearly

$$\lceil n/(d+1)\rceil \leq \delta^* \leq \lceil (n+d)/2\rceil.$$

We use the unit cost RAM model of computation in which every arithmetic operation $(+, -, *, /)$ and each binary comparison is assigned a unit cost. A centerpoint in R^d can be found by solving a set of $\Theta(n^d)$ linear inequalities, using linear programming (Clarkson et. al. [3]). When $d = 2$ the situation improves. Cole, Sharir and Yap [6] described an $O(n(\log n)^5)$ algorithm to construct a centerpoint, and subsequent ideas of Cole [4] could be used to lower the complexity to $O(n(\log n)^3)$. Recently, Jadhav and Mukhopadhyay [9] described a linear time algorithm to find a Tukey centerpoint.

Matoušek [11] attacked the harder problem of computing a Tukey median. Let us first observe that a "brute-force" approach could compute a point of maximum depth in $O(n^4)$: The $N = \binom{n}{2}$ pairs of points define N lines which subdivide the plane into $\binom{N}{2}$ convex cells. The points in each cell have the same Tukey depth and the max can be found in $O(n^4)$ using Lemma 1 from the next section. Matoušek first described an $O(n(\log n)^4)$ algorithm that gives a description of $C_k(S)$, $k \geq \lceil n/3\rceil$. Then, using this in a binary search on $k \in [\lceil n/3\rceil, \lceil (n+2)/2\rceil]$, the largest k for which $C_k(S) \neq \phi$ is found in $O(n(\log n)^5)$. Langerman and Steiger [10] described an algorithm that can compute a median in $O(n(\log n)^4)$, and recently improved it to $O(n(\log n)^3)$. It would not be easy to implement any of these algorithms and all have large constants hidden in the $O()$.

The contribution of the present paper is a simple, easily implemented algorithm to compute a deep point in the plane. We call a point Q "deep" if $\delta(Q) \geq \delta(P_i)$ for all points $P_i \in S$. Note that Q need not be a point in S. Deep points should be useful, in view of the interest in maximal depth points and in the apparent difficulty in obtaining them. Specifically we prove

Theorem 1 *Given n points in general position in R^2, in time $O(n(\log n)^2)$ a deep point Q can be found along with $\delta(Q)$, its depth.*

There is not much scope for improvement in view of the $\Omega(n \log n)$ lower bound we establish for finding a deep point and reporting its depth.

Theorem 2 *An algebraic decision tree to decide whether a deep point has depth $\geq (n+1)/2$ has height $\Omega(n \log n)$.*

Theorem 1 is proved via an algorithm having the asserted complexity. It is quite simple and easy to implement, using (i) the $O(n)$ centerpoint algorithm or some randomized variant, (ii) an $O(n \log n)$ algorithm to compute the depth of a given point, and a pruning strategy. The lower bound is via a connected components argument.

If the points of S are in convex position a deep point Q could have depth $\delta(Q) = 1$, not really deep. It would be easy to modify Theorem 1 to return a deep point of depth at least $n/3$ by computing in $O(n)$ time a centerpoint

A and then running the algorithm on $S \cup A$. Finally, we describe a version of these results for points in R^d, $d > 2$.

2 Finding a Deep Point in the Plane

Given $S = \{P_1, \ldots, P_n\}$, n points in general position in the plane, the goal is to construct a point whose depth is at least that of any point in S. As is often the case with selection problems, it will be useful to look at the corresponding ranking problem, in this case computing the Tukey depth $\delta(Q)$ for some input point Q Here is a natural algorithm, described by Rousseeuw and Ruts [16], but which appeared implicitly in the parametric search in Cole, Sharir, and Yap [6]:

1. Sort the points in S in the order of the slopes of the lines joining them to Q in $O(n \log n)$ time.
2. Rotate a line about Q, keeping track of the number of points on each side of the line. Record the minimum number during the rotation, in $O(n)$ time.
3. When the line has returned to its original position, the recorded number is the depth of Q.

This algorithm runs in $O(n \log n)$ time. Note that if $\delta(Q) = k$, then there must be a closed halfspace through Q containing exactly k points of S. Such a halfspace will be called a *witness halfspace* for Q.

If the Tukey depth of a point is thought of as an indicator of quality for this point as a representative of a set, a well behaved set might be expected to contain a good representative of itself, and thus a point of high Tukey depth. This motivates the notion of a deep point, which we restate here:

Definition 1 *A point $Q \in R^2$ is a* deep point *for a set S if $\delta(Q) \geq \delta(P)$ for every $P \in S$.*

A deep point could be found by the brute force algorithm in $O(n^2 \log n)$ by computing the depth of every point in S. In order to reduce that running time, we use the following:

Fact 1 *If h is a witness halfspace for the point Q, then for any point $P \in h$, $\delta(Q) \geq \delta(P)$.*

This follows from equation (3): $\delta(q) \leq |h \cap S| = \delta(p)$. We now describe an algorithm that actually solves a slightly more general problem: given a set S of n points, and a set A^* of candidate points, the algorithm returns a point Q such that $\delta_S(Q) \geq \delta_S(P)$ for every $P \in A^*$. If A^* is set to S, the algorithm returns a deep point of S. The algorithm maintains a set A of points "to beat" and works as follows:

Algorithm Deep(S,A^*)

$A \leftarrow A^*; P^* \leftarrow$ **any point of** A^***.**

while $A \neq \phi$ **repeat**

find a point $Q \in A$ **such that** $\delta_A(Q) \geq c|A|$ **for some constant** c**,**

$0 < c \leq 1/3$**. (e.g. a centerpoint of** A**)**

compute $\delta_S(Q)$**. Let** h **be a witness halfspace for** Q **in** S**.**

if $\delta_S(Q) \geq \delta_S(P^*)$ **then** $P^* \leftarrow Q$ **endif.**

$A \leftarrow A - (A \cap h)$

endwhile

Lemma 1 *In time* $O(|A^*| + n \log n \log |A^*|)$ *the algorithm returns a point* Q *of depth* $\delta(Q) \geq \delta(P)$*, all* $P \in A^*$*.*

Proof: The invariant of the algorithm is that $\delta_S(P^*) \geq \delta_S(P)$ for all $P \in A^* - A$. The invariant is true at the beginning since $A^* - A = \phi$, and the invariant is preserved after each step of the while loop: by Fact 1, $\delta_S(P^*) \geq \delta_S(P)$ for all $P \in (A \cap h)$.

For the running time of the algorithm, note that $|A \cap h| \geq c|A|$ because $\delta_A(Q) \geq c|A|$. This means that the size of A reduces by a constant factor at every step of the loop, and so there are at most $O(\log |A^*|)$ loop traversals. Each traversal takes time $O(T(A) + n \log n)$; $T(A)$ is the time needed to find a point Q, guaranteed to have depth $\delta_A(Q) \geq c|A|$, and $O(n \log n)$, the time to compute its exact depth as defined by the points in S. We may take $T(A) = O(|A|)$ using the centerpoint algorithm of Jadhav and Mukhopadhyay and observe that the overall running time is

$$\leq \sum_i K(c^i |A^*| + n \log n),$$

$K > 0$ a constant, the sum starting at $i = 0$ with $O(log|A^*|)$ terms. $\square$

As an example if $A^* = S$, then the algorithm will construct a high depth point in $O(n(\log n)^2)$ time, as claimed in Theorem 1.

This method can be easily generalized to find high depth points in R^d. Finding a point $Q \in A$ such that $\delta_A(Q) \geq c|A|$ can be done in any fixed dimension in time $O(|A|)$ for some suitably small positive constant c [3]. The next step of the algorithm, finding a witness halfspace, can be done in $O(n^{d-1} + n \log n)$ [1], and all the other steps take $O(n)$ time, so the total running time for finding a high depth point in R^d is $O(n^{d-1} \log n + n(\log n)^2)$.

Remark: These algorithms are practical. The $O(n)$ centerpoint algorithm of Jadhav and Mukhopadhyay is easily implemented using the linear-time, separated ham-sandwich cut algorithm of Megiddo [12]. Alternatively, we could compute in constant time, a centerpoint of a random sample of size (say) 100 and this would have depth at least $n/4$ with high probability. Using this for the pruning in the algorithm for Theorem 1, we will have a simply implemented randomized algorithm which has the asserted running time with high probability.

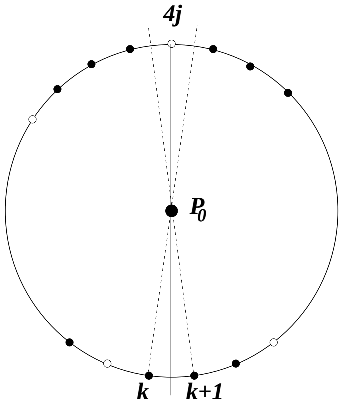
3 The Lower Bound

Given a set S of n points in the plane, let δ^* denote the depth of a median. We show that

Lemma 2 *The height of an algebraic decision tree T that can decide if $\delta^* > n/2$ is $\Omega(n \log n)$.*

Lemma 2 implies Theorem 2 because if a median were actually a point of S, a deep point would have to have the same depth.

Proof: Let C denote the circumference of the unit circle in $\mathbb{R}^2$ and let T denote an algebraic decision tree (see e.g., Preparata and Shamos [13]) that can decide whether $\delta(S) \geq \lceil (m+1)/2 \rceil$ for a set S of m points in $\mathbb{R}^2$. We restrict attention to sets S with $P_0 = (0,0)$ and $P_i = (\cos z_i, \sin z_i) \in C$, $i = 1, \ldots, 4n+3,\ z_i \in [0, 2\pi]$. These restricted inputs can be encoded by points $(z_1, \ldots, z_{m-1}) \in [0, 2\pi]^{m-1}$. We argue that the subset $Y \subseteq [0, 2\pi]^m$ of restricted inputs where T returns YES has $\Omega(n!)$ path connected components.



Let

$$\theta_i = \left(\frac{i}{4n+3} \right) 2\pi,\ i = 1, \ldots, 4n+3. \tag{5}$$

This encodes a set S with $m = 4n+4$ points, $4n+3$ of them equally spaced on C, and no two of them diametric in C. The diameter containing P_i, $i > 0$, is a halving line and both its closed halfspaces contain $2n+3$ points of S. A diameter containing no P_i, $i > 0$, is a non-halving diameter and has one closed halfspace with $2n+2$ points and the other with $2n+3$ points, so $\delta(P_0)$ is $2n+2$. This is maximal because for any $x \neq P_0$ the line through x perpendicular to $\overline{xP_0}$ has a closed halfspace with less than $2n+2$ points of S.

For inputs to T, we take P_0 and $4n+3$ other points encoded via the set I, where $z = (z_1, \ldots, z_{m-1}) \in I$ if all $z_i \in [0, 2\pi]$ and $z_j = \theta_j$, $j \equiv 1, 2, 3 \bmod 4\}$. Let π and ρ be two different permutations of the integers $4, 8, \ldots, 4n$, and define inputs $z_\pi \in I$ by

$$(\theta_1, \theta_2, \theta_3, \theta_{\pi_1}, \ldots, \theta_{4n-1}, \theta_{\pi_n}, \theta_{4n+1}, \theta_{4n+2}, \theta_{4n+3})$$

and input $z_\rho \in I$ by

$$(\theta_1, \theta_2, \theta_3, \theta_{\rho_1}, \ldots, \theta_{4n-1}, \theta_{\rho_n}, \theta_{4n+1}, \theta_{4n+2}, \theta_{4n+3}).$$

Both z_π and z_ρ are in $Y \cap I$, as they describe the m points in S, but in different order. They are in different components because on a continuous path from z_π to z_ρ in I, some z_{4j} must "cross" (i.e., reverse its radial ordering) a neighbor, θ_{4j-1} or θ_{4j+1} (see figure). Write $k = 4j + 2n + 1 \bmod (m+1)$; θ_k and θ_{k+1} are the neighbors of the diameter through θ_{4j}. If θ_{4j} crosses a neighbor, it first will cross the diameter through θ_k or the diameter through θ_{k+1}. After this occurs, that diameter will have closed halfspaces with $2n+2$ points on one side and $2n+4$ on the other, and the depth of P_0 described by this input $z \in I$ will be $\delta(S) = 2n+1$, so $z \notin Y$. Therefore $Y \cap I$ has $\geq n!$ different components. As $m = 4n+3$, the height of T is $\Omega(m \log m)$. $\square$

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