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Skew partitions in perfect graphs

Bruce Reed[a, b]

[a] *McGill University, Montreal, Canada*
[b] *CNRS, France*

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Abstract

We discuss some new and old results about skew partitions in perfect graphs.

Keywords: Perfect graphs; Skew partitions; Strong perfect graph conjecture

1. Introduction

A skew partition of a graph G is a partition of its vertex set into two non-empty parts A and B such that A induces a disconnected subgraph of G and B induces a disconnected subgraph of $\overline{G}$. Thus, a skew partition (A, B) of G yields a skew partition (B, A) of $\overline{G}$. It is this self-complementarity which first suggested that these partitions might be important to an understanding of the structure of perfect graphs.

Chvátal [3] introduced this notion, conjectured that no minimal imperfect graph permits a skew partition, and speculated that skew partitions might play a key role in a decomposition theorem for Berge graphs which would imply the Strong Perfect Graph Conjecture. Both his conjecture and his speculation were spot on.

Indeed, Chudnovsky et al. [1] recently proved every Berge graph either:

(a) is in one of five basic classes of perfect graphs (line graphs of bipartite graphs, their complements, bipartite graphs, their complements, or *double split graphs*), or
(b) permits one of three partitions (a proper 2-join, a homogeneous pair, or a special type of skew partition which they call *balanced*).

It was known that the first two of these three partitions could not occur in a minimal imperfect graph (see [5,4]). Chudnovsky et al. also proved that balanced skew partitions cannot occur in a smallest minimal imperfect Berge graph. These results taken together imply the Strong Perfect Graph Conjecture.

In this note, we survey the history of skew partitions. In Section 2 we discuss two special types of skew partitions: star cutsets and homogeneous sets, which were of particular importance in the study of perfect graphs before Chvátal made his definition. Indeed, the former motivated it. In Section 3, we discuss a number of special kinds of skew partitions which various researchers showed could not appear in minimal imperfect graphs after Chvátal made his definition

E-mail address: breed@cs.mcgill.ca.

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and conjecture. The section culminates with a proof that balanced skew partitions cannot occur in minimal imperfect graphs. All the results in Sections 2 and 3 appear elsewhere. Nevertheless, we felt it would be useful to bring them together and trace the links between them.

In Section 4, we discuss the computational complexity of determining if a graph has a skew partition. It turns out that there is a polynomial time algorithm for this question, but its running time is prohibitively expensive. We present some new results which suggest that for perfect graphs, this problem may be easier to resolve.

By a split of a skew partition (A, B) we mean four non-empty sets (A_1, A_2, B_1, B_2) where (A_1, A_2) is a partition of A such that no edges link A_1 to A_2 and (B_1, B_2) is a partition of B such that every vertex of B_1 is linked to every vertex of B_2. Given a split we define corresponding graphs $G_1 = A_1 \cup B$ and $G_2 = A_2 \cup B$.

We assume the reader is familiar with the standard definitions and notations of perfect graph theory which can be found in [17]. We warn her that, following the conventions of that field, by a subgraph we mean an induced subgraph.

2. Prehistory

A *homogeneous set* H in a graph G is a subset of its vertices with $2 \leqslant |H| < |V(G)|$ such that $V(G) - H$ can be partitioned into a set A of vertices which see (i.e. are adjacent to) all the vertices of H and a set N of vertices which see none of H. If N and A are both non-empty then for any vertex v of H, $(H - v + N, v + A)$ is a skew partition of G. Otherwise, G is disconnected or disconnected in the complement in which case it has a skew partition (such that one of A or B has two vertices) unless it is a stable set, a clique, a graph with three vertices, a cycle of length four or the complement of such a cycle. So except for these rather simple perfect graphs, if G has a homogeneous set it has a skew partition.

Lovász [15] proved that no minimal imperfect graph has a homogeneous set. This result was crucial to his proof of the Weak Perfect Graph Conjecture which states that a graph is perfect precisely if its complement is. Thus, though Chvátal's conjecture that minimal imperfect graphs do not contain skew partitions was motivated by the fact that perfect graphs are self-complementary, the proof of this fact requires a special case of Chvátal's conjecture.

When *substituting* a graph H for a vertex x in a graph G, we obtain a new graph on $V(H) \cup V(G - x)$ such that the vertices of H induce H, the vertices of $G - x$ induce $G - x$ and every vertex of H is adjacent to precisely those vertices of $G - x$ adjacent to x in G. We note that every subgraph of this new graph is either a subgraph of H, contains at most one vertex of H and hence is isomorphic to a subgraph of G, or has a homogeneous set (the vertices of H within it). Thus, if we substitute a perfect graph in a new perfect graph, the resultant graph cannot contain a minimal imperfect graph, and so is perfect.

Homogeneous sets and substitution were also important to Gallai's characterization of the comparability graphs [10], Seinsche's characterization of graphs with no induced P_4s [20], and Chvátal's proof [2] that a graph G is perfect if and only if, letting A_G be its clique node incidence matrix, the polytope $A_G x \leqslant 1$ has only integer vertices (see [18]).

A *clique cutset* is a cutset that induces a clique, as one might expect. If C is a clique cutset with at least two vertices then $(V - C, C)$ is a skew partition. If G has a clique cutset with one vertex then it has one with two, unless it is a stable set or a graph with at most four vertices. Thus except for some simple pathological cases, every graph with a clique cutset has a skew partition.

It is easy to show that a minimal imperfect graph cannot contain a clique cutset, we prove a stronger result below. Dirac [7] proved that every *triangulated graph*, i.e. every graph with no induced cycle of length at least four, contains a clique cutset. It follows that all such graphs are perfect.

Thus, we see that many special skew partitions played an important role in the early history of perfect graphs. In 1983, Chvátal [3] discussed a generalization of the two special types of skew partitions mentioned above as well as a number of others that had raised their ugly heads in the theory of perfect graphs. He defined a *star cutset* to be a cutset C containing a vertex v adjacent to all of $C - v$. Clearly, a clique cutset is a star cutset. Also, for any homogeneous set H and vertex v of H, $v + A$ is a star cutset unless N is empty in which case v is a star cutset in the complement of G. Chvátal proved, as we do in the next section, the Star Cutset Lemma which states that no minimal imperfect graph has a star cutset. This implies that no minimal imperfect graph has a homogeneous set or a clique cutset. Hence star cutsets can be used to prove triangulated and P_4-free graphs are perfect. Chvátal showed that the Star Cutset Lemma also implied many other known results on the perfection of special classes of graphs.

Further evidence of the importance of this notion was provided by Hayward, at about the same time. He used it to show that a new class of graphs, *the weakly triangulated graphs*, was perfect. A graph G is weakly triangulated if neither

G nor its complement has an induced subgraph isomorphic to a cycle of length at least five. Hayward [13] proved that every weakly triangulated graph with at least three vertices has a star cutset or a star cutset in the complement. By the Star Cutset Lemma and the self-complementarity of perfect graphs, it follows that all such graphs are perfect.

Since star cutsets were such a useful tool, Chvátal tried to generalize them. He came up with the notion of a skew partition and noted that if C is a star cutset with at least two vertices then it is a skew partition (so except for stable sets, and graphs of size at most four, every graph with a star cutset permits a skew partition). This led him to a generalization of the Star Cutset Lemma:

The Skew Partition Conjecture. No minimal imperfect graph has a skew partition.

3. Modern times

We begin with

The Proof of the Star Cutset Lemma. Suppose C is a star cutset of G, v is a vertex seeing all of $C - v$, and U is a component of $G - C$. Then, $G_1 = G - U$ has an $\omega(G)$ colouring as does the subgraph G_2 of G induced by $U \cup C$. Let S_i be the stable set in some $\omega(G)$ colouring of G_i which contains v. Now S_i meets all the $\omega(G)$ cliques contained in G_i because all such cliques contain a vertex of every colour. So, $S_1 \cup S_2$ meets every maximum clique of G since no clique has vertices in both U and $G - U - C$. This implies that $\omega(G - S_1 - S_2) = \omega(G) - 1$. But $S_1 \cup S_2$ is stable, as $S_1 - v$ is contained in $G - U - C$ and $S_2 - v$ is contained in U. If G were minimally imperfect than $G - S_1 - S_2$ would have an $\omega(G) - 1$ colouring and adding the stable set $S_1 \cup S_2$ would yield an $\omega(G)$ colouring of G, a contradiction. □

The key idea in this proof, which was also used by Lovász in his proof that no minimal imperfect graph has a homogeneous set, can be generalized to skew partitions in the following way (this result can be found in Hoàng [14] where it is stated in a more general framework, see also Olariu [16]).

The Colouring Lemma. *Suppose* (A_1, A_2, B_1, B_2) *is a split of a skew partition in a minimally imperfect graph* G*. Then there do not exist optimal colourings* C_1 *of* G_1 *and* C_2 *of* G_2 *such that the number of colours that appear on* B_1 *is the same in these two colourings.*

Proof. Assume to the contrary that exactly k colours appear on B_1 in both colourings. Let Y_i be the union of the k colour classes in C_i which intersect B_1 and note that $Y_i - B_1 \subseteq A_i$. Clearly Y_i does not contain the vertices of a $k+1$-clique, and so neither does $Y_1 \cup Y_2$. Thus, by the minimality of G this set is the union of k stable sets. On the other hand $\omega(G_i - Y_i)$ is at most $\omega(G) - k$, as our colouring of it shows. Thus, $\omega(G - Y_1 - Y_2)$ is at most $\omega(G) - k$ and by the minimality of G, $G - Y_1 - Y_2$ has a colouring with $\omega(G) - k$ colours. We have constructed an $\omega(G)$ colouring of G, a contradiction. □

To apply this lemma we need to find conditions on G which allow us to find the desired colourings. Hoàng used the following:

Observation. If there is a perfect graph G_i^* which is obtained from G_i by adding a vertex v^* adjacent to all of B_1, some of B_2 and none of A_i then G_i has an $\omega(G)$ colouring such that $\omega(B_1)$ colours appear on B_1.

Proof. Clearly at least $\omega(B_1)$ colours appear on B_1 in any colouring, so we need only find a colouring where at most these many colours are used on B_1. Let ω^* be the size of the largest clique in G_i^* containing v^*. Substitute a clique C^* of size $\omega(G) - \omega^* + 1$ for v^* in G_i^* to obtain a new perfect graph G_i'. Then G_i' has clique number $\omega(G)$ and C^* is contained in a clique K of G_i' of size $\omega(G)$. Now, K consists of C^*, a clique K_1 of B_1 and a clique K_2 of B_2. We note that every vertex of $K_2 \cup C^*$ is adjacent to all of B_1. So no colour used on $K_2 \cup C^*$ is used in B_1. But there are $\omega(G) - |K_1| \geqslant \omega(G) - \omega(B_1)$ colours appearing in this clique, so at most $\omega(B_1)$ colours appear on B_1. □

By applying his observation, Hoàng proved that certain special skew partitions could not appear in minimal imperfect graphs.

Definition. A skew partition is a *T*-cutset if it has a split (A_1, A_2, B_1, B_2) such that for $i \in \{1, 2\}$ there is a vertex v_i of A_i which sees all of B_1.

The T-cutset Lemma. *No minimal imperfect graph G has a T-cutset.*

Proof. Since $\overline{G}$ contains no star cutset, $|A_1| \geqslant 2$ and $|A_2| \geqslant 2$. So, for $i \in \{1, 2\}$, the subgraph obtained from G_i by adding v_{3-i} is a proper subgraph of G and hence perfect. So, by Hoàng's observation, we can use $\omega(G)$ to colour G_i so that $\omega(B_1)$ colours appear on B_1. This contradicts the Colouring Lemma. □

Roussel and Rubio found a way of demonstrating that a skew partition is a *T*-cutset by considering the paths between elements of *B* whose interior is in *A*. Note that if we choose a skew partition with *B* minimal then either it is a star cutset, or every vertex of *B* has a neighbour in every component of *A*. It follows that in every *B*-minimal skew partition of a minimal imperfect graph, each vertex of *B* sees a vertex of every component of *A*. Thus, for every pair b_1, b_2 of non-adjacent vertices of *B* in such a *B-minimal* partition, and every component *U* of *A*, there is a chordless path from b_1 to b_2 whose interior is in *U*. If two such paths between b_1 and b_2 have different parities (the parity of a path is the parity of the number of edges on it) then clearly we can find two paths of different parities whose interiors are in different components of *A* and hence an odd chordless cycle of *G* of length at least 5. Thus, for any *B*-minimal skew partition of a minimal imperfect graph *G*, we define every non-edge b_1b_2 of *B* to be either odd or even according to the parity of the paths of $A + b_1 + b_2$ between b_1 and b_2.

Roussel and Rubio [19] proved the following:

The Wonderful Lemma. *Suppose that a Berge graph G contains an odd path P whose endpoints are both adjacent to all of a set Y such that $\overline{Y}$ is connected. Then one of the following occurs*:

(a) *some internal vertex of P sees all of Y, or*
(b) *there is an odd path with its endpoints in Y and its interior in the interior of P, or*
(c) *there is an odd path of $\overline{G}$ whose endpoints are two consecutive vertices of the interior of P and whose interior is in Y.*

Roussel and Rubio applied this result to odd paths *P* with endpoints in *B* and interiors in *A*, where *Y* is a component of $\overline{B}$ not containing the endpoints of *P*. For example, they proved:

The Odd Path Lemma. *Suppose that (A, B) is a B-minimal skew partition in a Berge graph G. Suppose further that there exist three mutually non-adjacent vertices b_1, b_2 and b_3 of B and an odd b_1 to b_2 path P with its interior in A and disjoint from the neighbourhood of b_3. Then (A, B) is a T-cutset and hence G is not minimal imperfect.*

Proof. Let B_1 be a component of $\overline{B}$ which does not contain b_1, b_2, b_3. By the Wonderful Lemma, if there is no internal vertex of *P* which sees all of B_1 then either (a) there is an odd path with its endpoints in B_1 and its interior in the interior of *P*, or (b) there is an odd path of $\overline{G}$ with its endpoints adjacent vertices of *P* and its interior in B_1. But if (a) holds then adding b_3 to this path yields an odd hole in *G* and if (b) holds then adding b_3 yields an odd hole in $\overline{G}$. Since *G* is Berge, it follows that some vertex v_1 on the interior of *P* sees all of B_1. Now, let A_1 be the component of *A* containing the interior of *P*, and A_2 be the rest of *A*. We know there is an odd path P' from b_1 to b_2 whose interior is in A_2 and hence disjoint from the neighbourhood of v_1. Repeating the above argument with v_1 playing the role of b_3, we see that there is an internal vertex v_2 of P' (which is in A_2) that sees all of B_1. Hence the split $(A_1, A_2, B_1, B - B_1)$ shows that (A, B) is a *T*-cutset. □

Using similar but more complicated applications of the Wonderful Lemma, Roussel and Rubio managed to show that a minimal imperfect cannot contain a skew partition with a split (A_1, A_2, B_1, B_2) such that B_1 is stable (if $|B_1|$ is at least three, it is easy to show that a triple b_1, b_2, b_3 as in the statement of the Odd Pair Lemma must exist in B_1 using the theory of even pairs, if $|B_1| = 1$ then B is a star cutset, the case $|B_1| = 2$ is more complicated). This generalized a result of Cornuéjols and Reed [6] who had proven that a minimal imperfect graph cannot contain a complete multipartite cutset.

These results motivated researchers to investigate two questions (with limited success):

(1) Can we apply Hoàng's observation to other classes of skew partitions to prove that they do not appear in minimal imperfect graphs?, and
(2) Are there conditions on the parities of the non-edges within B for a skew partition (A, B) which ensure that it cannot occur in a minimal imperfect graph? In particular, what if all the non-edges are even?

Chudnovsky et al. had two key insights. The first was to restrict their attention to imperfect Berge Graphs of minimum order. Thus, for Hoàng's observation to be useful, we only need to prove that the relevant auxiliary graph is Berge, because it will be smaller (as, the self-complementarity of perfect graphs implies that a minimal imperfect graph has no star cutset and hence for any split, both A_1 and A_2 have at least two vertices). The second was to note that the self-complementary extension of the evenness of all non-edges of B is exactly the property that allows us to apply Hoàng's observation. Thus, they call a skew partition *balanced* if every path of G with its endpoints in B and its interior in A is even and every path of $\overline{G}$ with its endpoints in A and its interior in B is even. This is equivalent to saying that if we add a vertex adjacent to all of B and none of A then the graph remains Berge. Thus, Hoàng's observation and the Colouring Lemma implies no *minimum order* Berge graph has a balanced skew partition.

We have shown that one of the two results Chudnovsky et al. needed to prove the Strong Perfect Graph Theorem is an immediate consequence of results of Chvátal, Hoàng, and Lovász. The second result is the decomposition theorem mentioned above. Its proof requires over 100 pages. It is worth mentioning, however, that the Wonderful Lemma is used heavily throughout the proof. Thus one of the crucial tools for the proof of this theorem arose out of the study of skew partitions. I would love to give the readers a feeling for how the Wonderful Lemma is used in the proof, but that is a whole other story. . . .

4. Algorithmic considerations

There is a polynomial time algorithm to determine if a graph has a skew partition, due to de Figueiredo et al. [9], but the best bound on its running time is $\Theta(n^{101})$. Its seems plausible that there might be a faster algorithm if we restrict our attention to perfect graphs, and such an algorithm could be useful in recognizing special classes of perfect graphs.

To support this speculation we sketch fast algorithms to test if the basic classes of perfect graphs that Chudnovsky et al. use in their decomposition theorem admit a skew partition. Since a skew partition in G is also a skew partition of $\overline{G}$, we need only consider line graphs, bipartite graphs and double split graphs. As pointed out by Chvátal [3], there is an $O(nm)$ algorithm to test if a graph has a star cutset. So, we need only test for skew partitions (A, B) such that B is not a star cutset of G.

A graph is a double split graph precisely if its vertices can be partitioned into two sets, each of size at least four, one of which consists of an induced matching M of G, the other of which induces a matching N of $\overline{G}$ and such that for each pair e, f with $e \in M$ and $f \in N$, the edges between e and f form a matching of size two. Clearly, $(V(M), V(N))$ is a skew partition of G. Furthermore, every vertex of M has degree $|N| + 1$ in G whilst every vertex of N has degree $2|N| - 2 + |M|$. Thus, we can find a skew partition of a doubly split graph in linear time simply by computing the degree of every vertex.

Suppose (A, B) is a skew partition of the line graph of a graph H and that B is not a star cutset. Let (A_1, A_2, B_1, B_2) be a splitting of it. Let E_1 be the set of edges of H corresponding to the vertices of B_1 and E_2 be the set of edges of H corresponding to the vertices of B_2. So, every edge of E_1 intersects every edge of E_2. There are non-edges within both B_1 and B_2 as otherwise B is a star cutset. Thus there is a matching e_1, e_2 in E_1 and a matching f_1, f_2 in E_2. Since each e_i intersects each f_j, these two matchings are on the same four vertex set X. Furthermore, every edge of E_1 intersects both f_1 and f_2 and every edge of E_2 intersects both e_1 and e_2. Thus $E_1 \cup E_2$ consists of at most six edges, incident to four vertices of H. Furthermore, having picked a choice for e_1 and e_2 our choices for the remainder of B are a subset of a set of four vertices of G (those which see both of the corresponding two vertices of G). So there are $O(n^2)$ choices for the vertices of B. Testing if deleting one such set yields a disconnected set A takes $O(m)$ time. It follows that we can test for a skew partition of a line graph in $O(n^2 m)$ time.

We note that if (A, B) is a skew partition of a bipartite graph G then B is a complete bipartite cutset of G, a so-called *biclique cutset*. We now describe a fast algorithm for determining if a bipartite graph has such a biclique cutset. In contrast, as shown by de Figueiredo and Klein [8], it is NP-complete to determine if an arbitrary graph has a biclique cutset.

We actually present an algorithm which given a bipartite graph G with bipartition (C, D), and an integer k, either finds a skew partition of G, or determines that G has no skew partition (A, B) with $|B \cap D| = k$ and runs in $O(n^4)$ time. Applying this subroutine $|D|$ times allows us to determine if there are any biclique cutsets in G in $O(n^5)$ time. Our algorithm relies on the fact that if (b, b') is a pair of vertices of $B \cap C$ then $|N(b) \cap N(b')| \geqslant k$ while if (a, a') is a pair of vertices of $A \cap C$ with $|N(a) \cap N(a')| > k$ then a and a' must lie in the same component of $G - B$.

Our algorithm has three steps. In the first, we test for the existence of a skew partition where for some a and a' lying in different components of $A \cap C$, $|N(a) \cap N(a')| = k$. In the second we find a skew partition of any graph which contains a skew partition such that one of $C \cap A_1$, $C \cap A_2$ is empty. In the third step we search for skew partition for which neither of these conditions holds.

In the first step, we consider every pair (a, a') of vertices in C whose set S of common neighbours has exactly k elements. We let T be the set of common neighbours of the vertices of S. We assume $T - a - a'$ is non-empty as otherwise the desired skew partition cannot exist. Clearly if a and a' are in different components of $G - S - (T - a - a')$ then we have found the desired skew partition for this pair. On the other hand, if a and a' are in the same component of this graph, then there is no such partition (since in any such partition, B must contain S). This step takes at most $O(m)$ time per pair and hence $O(n^4)$ time in total.

If A_1 is disjoint from C then $B \cap C$ together with any vertex of $B \cap D$ is a star cutset. So, in the second step, we need only check if G has a star cutset.

In the third step, we create an auxiliary graph H whose vertex set is C in which two vertices are joined by an edge if they have at least k common neighbours. Clearly the desired skew partition corresponds to a clique cutset of H although the converse does not hold.

In looking for such a clique cutset, we use a *clique cutset tree*. A clique cutset tree for a graph F consists of a rooted tree T every node t of which is labelled with a subgraph F_t of F. If t is not a leaf then it is also labelled with a clique cutset K_t of F_t. If F has no clique cutset then the only clique cutset tree has one node labelled F. Otherwise, we take any clique cutset K of F, and proceed as follows to construct a clique cutset tree for F whose root is labelled by F and K. The children of the root are in one to one correspondence with the components of $F - K$. The child corresponding to U will be labelled with the subgraph of F induced by $U \cup K$, and the tree underneath it will be a clique cutset tree for this graph. As shown by Gavril [11], these trees have at most n nodes. Thus, using an algorithm of Whitesides [21], they can be constructed in $O(n^4)$ time. We construct such a tree for H.

Now, for any clique cutset K of H, we see that K is a clique in the subgraph corresponding to one of the children of the root. Thus, by induction, there is a leaf of the clique cutset tree such that K is a clique in the graph labelling this leaf. Of course, it cannot be a clique cutset of this graph, by definition. Thus, considering the path from this leaf to the root, we see that for some node t of the tree, K is not a clique cutset in the graph labelled by t but is a clique cutset in the graph labelled by the parent of t. We now present an $O(n^3)$ algorithm which for a fixed node t, either finds a biclique cutset of G or determines that there is no such clique K which corresponds to a biclique cutset K^* of G with k vertices in D. We will apply it to each non-root t of our clique cutset tree.

If F_t is a clique, we first test if F_t could be such a clique cutset. To this end, we let L_t be the set of vertices of D which see all of F_t in G. If $|L_t| = 1$ then $L_t \cup F_t$ is the only possible such cutset and so this step is trivial. So we assume $|L_t| \geqslant 2$. Clearly, $G - V(F_t) - L_t$ is non-empty as it contains some vertices of C. If this graph is disconnected then $V(F_t) \cup L_t$ induces a biclique cutset of G and we are done. In the same vein, if v is a vertex of L_t which sees none of $G - V(F_t) - L_t$ then $V(F_t) - (L_t - v)$ induces a biclique cutset. If neither of these conditions hold then for any subset L of L_t, $G - V(F_t) - L$ is connected, so F_t cannot be a clique cutset of the type we are looking for. The bulk of the work here is a connectivity check, so this step runs in $O(n^2)$ time.

Now, we can assume that if our desired clique K exists then $F_t - K$ is non-empty. Furthermore, by our choice of t, it must be connected. So, because we are in the third step, it will all live in one component of $G - K^*$. We let s be the parent of t. We see that for one component U_t of $F_s - K_s$, F_t is induced by $F_s \cup U_t$ while for every other component U of $F_s - K_s$, U is disjoint from F_t and hence K. Thus, all of U lies in one component of $G - K^*$. For at least one U, this component must be different from that containing $F_t - K$. We present an algorithm which runs in $O(n^2)$ time and determines, for a specific U whether or not there is a clique cutset K and corresponding biclique cutset K^* such that U and $F_t - K$ are in different components of $G - K^*$.

We proceed iteratively. At each step, we have a set U^* of vertices of G which must be in the same component of $G - K^*$ as U, a set K' of vertices which must be in K and a set L' of vertices which contains all the common neighbours of K' not in U^*. Initially, $U^* = U, K'$ consists of the vertices of F_t adjacent to a vertex of U, and L' consists of all their common neighbours not in U. In each iteration, if there is an edge from U^* to a vertex of F_t not in K' we can add this vertex to K' and throw all of its non-neighbours out of L'. If there is an edge from U^* to a vertex v of $G - F_t - L'$ then we can add this vertex to U^*. If $K' = V(F_t)$ then we can stop and say that the desired subgraph does not exist. If none of these three rules apply then (K', L') is a biclique cutset separating $F_t - K'$ from U^*. A straightforward implementation of this algorithm runs in $O(n^3)$ time. It is easy to implement it in $O(n^2)$ time. We omit the details.

Since we run the algorithm once for each component of $F_s - F_t$, treating each node requires $O(n^3)$ time, and hence we require $O(n^4)$ time to treat the whole tree. This complete our description of an $O(n^5)$ algorithm to test for biclique cutsets in bipartite graphs.

We have given $O(n^5)$ algorithms for testing the existence of skew partitions in the five basic classes of perfect graphs. Perhaps these observations will inspire a reader to develop a fast algorithm to find a skew partition in a Berge graph.

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