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ON PLANAR QUASI-PARITY GRAPHS*

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Abstract. A graph G is strict quasi parity (SQP) if every induced subgraph of G that is not a clique contains a pair of vertices with no odd chordless path between them (an even pair). Hougardy conjectured that the minimal forbidden subgraphs for the class of SQP graphs are the odd chordless cycles, the complements of odd or even chordless cycles, and some line-graphs of bipartite graphs. Here we prove this conjecture for planar graphs. We also give a constructive characterization of all the planar minimal forbidden subgraphs for the class of SQP graphs.

Key words. perfect graphs, even pairs, planar graphs

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1. Introduction.

A graph G is *perfect* if, for every induced subgraph H of G, the chromatic number of H is equal to the size of its largest clique. A *hole* is any chordless cycle of length at least five. A hole is odd or even according to its length. The most famous problem in the context of perfect graphs was Berge's perfect graph conjecture: *A graph G is perfect if and only if neither G nor $\bar{G}$ contains an odd hole.* This was proved by Chudnovsky et al. [1]. However, other problems concerning perfect graphs remain open, some of which are related to the concept of an *even pair*. Two nonadjacent vertices of a graph G form an *even pair* (resp., *odd pair*) if every chordless path between them has an even (resp., odd) number of edges. A graph is *strict quasi parity* (SQP) if each of its induced subgraphs either contains an even pair or is a clique. Fonlupt and Uhry [3] introduced even pairs and showed that the graph obtained by contracting an even pair in a perfect graph is perfect. Meyniel also showed that no minimal imperfect graph contains an even pair [11]. He proposed the definition of SQP graphs, which, by his result, are perfect. The main interest of this class lies in the fact that many classical families of perfect graphs are SQP, as proved by Meyniel (see also the more recent survey [2]). Meyniel also suspected that the minimal non-SQP graphs must have a simple structural characterization. More formally, Hougardy [5] proposed the following conjecture.

Conjecture 1. *Every minimal non-SQP graph is either an odd hole, or the complement of a hole, or the line graph of a bipartite graph.*

Let us call *obstruction* any graph G which is not a clique, has no even pair, and is such that every proper induced subgraph of G has an even pair or is a clique. We will prove Hougardy's conjecture for planar graphs; namely, we will prove the following theorem.

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THEOREM 1 (main theorem). *Every planar obstruction is an odd hole or the line-graph of a bipartite graph.*

We say that the neighborhood of a vertex is *correct* if it consists of either (a) three vertices with exactly one edge between them, or (b) four vertices with exactly two, nonincident, edges between them. To prove the main theorem it suffices to establish the following.

THEOREM 2. *In a planar obstruction that is not an odd hole, every neighborhood is correct.*

Let us recall a classical result which we will use several times. Call *diamond* the graph that consists of a clique of size four minus one edge. Call *claw* the graph on four vertices with one vertex of degree three and three vertices of degree one.

THEOREM 3 (see [4]). *A graph G is the line-graph of a simple bipartite graph if and only if G contains no odd hole, no diamond, and no claw.*

Proof of Theorem 1 *from Theorem* 2. Let G be a planar obstruction different from an odd hole. By Theorem 2, every vertex belongs to exactly two maximal cliques, one of size three and the other of size three or two. Hence G contains no claw and no diamond. Thus G is the line-graph of a bipartite graph by Theorem 3.

The proof of Theorem 2 is the object of the next sections. It will be convenient to first prove it for 3-connected obstructions (Theorem 4), and then to derive it for all obstructions via some decomposition technique. In section 4 we will give a more precise description of planar obstructions.

For an introduction to planar graphs, see [12]. We finish this section with a few definitions and some simple lemmas.

We will frequently use "sees" and "misses" instead of "is adjacent to" and "is not adjacent to." The subgraph of a graph G induced by a set $X \subseteq V(G)$ is denoted by $G[X]$. For vertices $x, y \in V(G)$, an x, y-path is a path whose endvertices are x and y. If P is a path and u, v are vertices in P, we let $P[u, v]$ denote the subpath of P between u and v. If two paths P, Q have a common endvertex, we use $P + Q$ to denote the path obtained by their concatenation. For any vertex x we let $N(x)$ denote the neighborhood of x. For a set $X \subseteq V(G)$, we let $N(X) = \{u \in V(G) - X \mid u \text{ sees some } x \in X\}$.

LEMMA 1. *Let G be a graph that contains no odd hole. Then*

1. *if P is a chordless path in G and x is a vertex of $G - P$ that sees the two endvertices of P, then P has even length if and only if it contains an even number of edges of $N(x)$.*

2. *if H is a hole in G and x is a vertex of $G - H$ that sees at least two nonadjacent vertices of H, then H contains an even number of edges of $N(x)$.*

Proof. If 1 fails, it is a routine matter to check that $P + x$ contains an odd hole. To prove 2, consider two nonadjacent vertices u, v of H that are neighbors of x and apply 1 to the two u, v-paths of H. ∎

LEMMA 2. *An obstruction contains no clique cutset.*

Proof. Suppose that G is an obstruction which has a clique cutset C. First, assume that for some component B of $G - C$ the induced subgraph $G[B \cup C]$ is not a clique. So $G[B \cup C]$ has an even pair a, b. No chordless path between a and b can go through another component of $G - C$ because C is a clique; it follows that a, b is an even pair of G, a contradiction. Now assume that $G[B \cup C]$ is a clique for every component B of $G - C$. Pick any a, b in two different components of $G - C$: clearly a, b is an even pair of G, a contradiction. ∎

LEMMA 3. *Let x, y be nonadjacent vertices in a minimal cutset C in a graph that contains no odd hole. Then all chordless x, y-paths in $G - (C - \{x, y\})$ have the same parity.*

Proof. Since C is a minimal cutset, every component D of $G - C$ contains a neighbor of x and a neighbor of y, and so there is a chordless x, y-path with interior in D. If the lemma fails, there must exist an even chordless x, y-path P whose interior lies in one component of $G - C$ and an odd chordless x, y-path Q whose interior lies in another component of $G-C$, and then $P+Q$ is an odd hole, a contradiction. □

LEMMA 4. *In an obstruction different from an odd hole, every vertex has degree at least three and lies in a triangle.*

Proof. Let x be any vertex in such an obstruction G. If x either has degree one, or has degree two and its neighbors are adjacent, then $N(x)$ is a clique cutset, contradicting Lemma 2. Suppose x has degree at least two and its neighbors are pairwise nonadjacent. Let u, v be any two neighbors of x. If P is an odd chordless u, v-path in G, then $P + x$ contains an odd hole by Lemma 1. So there is no such path; but then u, v form an even pair, a contradiction. □

2. 3-connected planar obstructions.

Here we prove Theorem 2 for 3-connected planar obstructions.

THEOREM 4. *In a* 3*-connected planar obstruction every neighborhood is correct.*

Proof. Let G be a 3-connected planar obstruction, and let us fix one planar representation of G. We shall arrive at the desired conclusion of Theorem 4 through a succession of lemmas that gradually restrict the possible types of neighborhood in G. □

LEMMA 5. *In G the neighborhood of every vertex induces either a chordless cycle or a collection of vertex-disjoint paths.*

Proof. Let x be any vertex of G. If $N(x)$ contains a vertex y of degree three in $N(x)$, then planarity implies that x and y are in a triangle cutset of G, contradicting Lemma 2. Thus the maximum degree in $N(x)$ is two. Moreover, if $N(x)$ properly contains a set S of vertices which induce a cycle, then planarity implies that $S \cup \{x\}$ contains a triangle cutset separating $N(x) - S$ from $V(G) - (N(x) \cup \{x\})$, again a contradiction. Lemma 5 follows. □

LEMMA 6. *G has no vertex x such that $N(x) \cup \{x\}$ induces a diamond.*

Proof. Suppose on the contrary that there exists such a vertex x, and let y, z be the nonadjacent vertices in $N(x)$. Since G is an obstruction, there is an odd chordless path P between y and z. But then $P + x$ is an odd hole, a contradiction. □

A *star-cutset* is a cutset C such that some vertex $x \in C$ sees all vertices of $C - x$.

LEMMA 7. *G has no star-cutset.*

Proof. Suppose that G contains a star*-cutset C, and let x be a vertex of C that sees all of $C - x$. We claim that each component K of $G - C$ has a nonneighbor of x. Suppose the contrary, that is, $K \subseteq N(x)$. By Lemma 5, K induces a cycle or a path. Let y be any vertex of K. Then $N(y) \cup \{y\}$ induces a diamond, a contradiction to Lemma 6. So our claim holds, and consequently we can assume that $C = \{x\} \cup N(x)$. As G is 3-connected, each component of $G - C$ is adjacent to at least three vertices of C and hence three vertices of $N(x)$. We can enumerate the neighbors of x as $x_1, \ldots, x_d$ clockwise around x and choose a component K of $G - C$ with $x_1 \in N(K), x_j \in N(K), N(K) \subseteq \{x_1, \ldots, x_j\}$, and such that j is as small as possible over all clockwise enumerations and all choices of K. This choice, the fact that $|N(K)| \geq 3$, and planarity imply that any other component L of $G - C$ satisfies $N_L \subseteq \{x_j, x_{j+1}, \ldots, x_d, x_1\}$. It follows that $\{x, x_1, x_j\}$ is a cutset of G. Then x_1x_j is not an edge of G, for otherwise G has a clique cutset $\{x, x_1, x_j\}$. Furthermore, $\{x, x_1, x_j\}$ is a minimal cutset since G is 3-connected. Thus for every component L of $G - \{x, x_1, x_j\}$ there is a chordless x_1, x_j-path in $L \cup \{x_1, x_j\}$. Such paths must all have the same parity, for otherwise we could find two paths of different parity in different components of $G - \{x, x_1, x_j\}$, and their union would be an odd hole in G. Actually all these paths are odd, for otherwise $\{x_1, x_j\}$ would be an even pair. There is a chordless x_1, x_j-path P in K since $\{x_1, x_j\} \subseteq N(K)$, and by the above remark, P is odd. Now, $P + x$ is an odd hole, a contradiction. This completes the proof of Lemma 7. □

LEMMA 8. *If G has a cutset C of size three which induces a subgraph with just one edge, then there is a vertex w with $N(w) = C$.*

Proof. Assume that $C = \{x, y, z\}$ with $xy \in E(G)$ and $yz, xz \notin E(G)$. Since G is 3-connected, C is a minimal cutset. By Lemma 3, all chordless x, z-paths in $G - y$ have the same parity p_{xz} and all chordless y, z-paths in $G - x$ have the same parity p_{yz}. If p_{xz} is even and p_{yz} is odd (or vice versa), then x, z (or y, z) form an even pair, a contradiction. So p_{xz} and p_{yz} are the same. Since G has no star-cutset, there is a chordless x, z-path P_1 in $G - (N(y) \cup \{y\})$. The interior of P_1 is in some component K_1 of $G - C$. Similarly, there is a y, z-path P_2 in $G - (N(x) \cup \{x\})$. The interior of P_2 is also in K_1, for otherwise $P_1 + xy + P_2$ forms an odd hole. Let K be any component of $G - C - K_1$, and let P be any chordless y, z-path with interior in K. Call w the neighbor of y along P. Let w' be the last neighbor of x along P, starting from y. If $w' \neq w$, then one of the paths $xw' + P[w', z] + P_1$ and $xw' + P[w', z] + P_2 + yx$ is chordless and odd, a contradiction. It follows that x sees w and no other vertex of $P - y$. Thus the set $W = N(x) \cap N(y) \cap K$ separates x and y from z in $K \cup \{x, y, z\}$. Suppose that W has two vertices w, w'. Let P_3 (resp., P_4) be a w, z-path (w', z-path) in $G \backslash (K \cup \{x, y\})$. If w, w' are adjacent, then it is easy to see that the subgraph induced by $P_1 \cup P_2 \cup P_3 \cup P_4$ contains a subdivision of K_5, which contradicts the planarity. If w, w' are not adjacent, then, since they do not form an even pair, there is an odd w, w'-path Q in G, and any such path must be in $G - (K \cup \{x, y\})$. But then it is easy to see that the subgraph induced by $P_1 \cup P_2 \cup P_3 \cup P_4 \cup Q$ contains a subdivision of K_5 or of $K_{3,3}$, which contradicts planarity. So $|W| \leq 1$, say $W = \{w\}$. Now $K = \{w\}$; otherwise $\{w, z\}$ would be a cutset of size two in G. So $N(w) = \{x, y, z\}$, which completes the proof of Lemma 8. □

LEMMA 9. *For every vertex x of degree three, the neighborhood of x contains exactly one edge.*

Proof. Clearly, $N(x)$ does not induce a triangle K_3, for otherwise either $N(x)$ is a clique cutset or G is K_4, both of which are impossible. Lemma 6 implies that $N(x)$ does not induce a path of length two. On the other hand, $N(x)$ must contain an edge, for otherwise any two vertices in $N(x)$ would form an even pair. This proves the lemma. □

LEMMA 10. *Every face of G is a hole.*

Proof. Every face of G is a cycle because G is 2-connected. If a face F has a chord xy, planarity implies that $\{x, y\}$ is a clique cutset of G that separates one component of $F - \{x, y\}$ from the other, a contradiction to the fact that G is 3-connected. □

LEMMA 11. *Any two intersecting faces in G intersect in an edge or a vertex.*

Proof. Suppose that two faces F, F' intersect in at least two nonadjacent vertices. Then it is easy to see that there are two vertices of $F \cap F'$ that form a cutset, a contradiction to the fact that G is 3-connected. □

For any vertex x of G, we denote by F_x the face of $G - x$ such that x lies in the interior of F_x. Note that F_x is a cycle since $G - x$ is 2-connected.

LEMMA 12. *For every vertex x, if yz is an edge between two vertices of F_x that are not consecutive along F_x, then x misses y and z and there is a vertex w of F_x such that $N(w) = \{x, y, z\}$.*

Proof. In the opposite case, $\{x, y, z\}$ would be either a clique cutset or a cutset, contradicting Lemma 8. □

LEMMA 13. *For every vertex x of degree at least four, F_x is a hole.*

Proof. Consider any such vertex x and assume that F_x is not chordless. By Lemma 12, there is a vertex w on F_x with $N(w) = \{x, y, z\}$, $y, z \in F_x$, edges $yz, yw, wz \in E(G)$, and $xy, xz \notin E(G)$. Let F_1 and F_2 be the two faces of G containing the edge wx. By Lemma 10, F_1 and F_2 both induce holes of G. Since each of F_1 and F_2 contains one of y or z, neither is a triangle and hence both are even. By Lemma 11, $F_1 \cap F_2 = \{w, x\}$. By Lemma 12 and the fact that $|N(x)| \geq 4$, there is no edge from $F_1 - \{w, x\}$ to $F_2 - \{w, x\}$. But now $F_1 + F_2 - w$ is an odd hole in G, a contradiction. □

LEMMA 14. *For every vertex x of degree at least four in G, $N(x)$ induces a graph with an even number of edges.*

Proof. By Lemma 13, F_x is a hole. If $N(x)$ induces a disjoint set of paths with an odd number of edges, then $F_x \cup \{x\}$ contains an odd hole by Lemma 1, a contradiction. □

LEMMA 15. *If G has a diamond on vertices $\{a, b, c, d\}$ such that $ad \notin E(G)$, then there is no path from a to d in $G - [N(b) \cup N(c) - \{a, d\}]$.*

Proof. Since $\{a, d\}$ is not an even pair there exists an odd chordless a, d-path in $G - b - c$. Choose such a path P_1 with $P_1 \cap (N(b) \cup N(c))$ minimal over all such paths. Suppose indirectly that there exists a chordless a, d-path P_2 in $G - [N(b) \cup N(c) - \{a, d\}]$. Let $x \in P_1 - a - d$ be such that $xb \in E(G)$; such an x exists or else $P_1 + b$ induces an odd hole. Let P_3, P_4 be the two subpaths of $P_1 - x$ such that $a \in P_3$ and $d \in P_4$. Clearly there is a chordless path from P_3 to P_4 whose interior vertices are in $P_2 - P_1$. Choose a shortest such path P_5 so that its endpoint f in P_3 is as close as possible to a along P_3 and its endpoint g in P_4 is as close as possible to d along P_4. Then the path $P_6 = P_3[a, f] + P_5 + P_4[g, d]$ is a chordless path. Now, as $P_5 - f - g \subseteq P_2$ we know that $(P_5 - f - g) \cap (N(b) \cup N(c)) = \emptyset$. Observe that P_6 contains strictly fewer vertices of $N(b) \cup N(c)$ than P_1. Thus, by the minimality in our choice of P_1, P_6 is even. It follows that the subpath $P_1[f, g]$ is of different parity from P_5. Note that by planarity $(P_1[f, g] - f - g) \cap N(c) = \emptyset$. Let r (resp., s) be the neighbor of c closest to f (resp., to g) along P_3 (resp., P_4). Then either $c + P_1[r, s]$ or $c + P_1[r, s] - P_1[f, g] + P_5$ is an odd hole. This contradiction completes the proof of Lemma 15. □

LEMMA 16. *G contains no diamond.*

Proof. Suppose on the contrary that $\{a, b, c, d\}$ is a diamond in G, with $ad \notin E(G)$.

First, suppose that $N(a) \subseteq N(b) \cup N(c)$. Let $y_1, \ldots, y_k$ $(k \geq 3)$ be the neighbors of a clockwise in the planar representation of G, with $b = y_1$. If c is none of y_2, y_k, then $\{a, b, c\}$ is a triangle cutset, a contradiction to Lemma 2. So we may assume that $c = y_k$. Then b misses every y_i with $3 \leq i \leq k - 1$, for otherwise $\{a, b, y_i\}$ is a triangle cutset, again a contradiction. Likewise c misses y_j with $2 \leq j \leq k - 2$. It follows that $k \leq 4$. If $k = 3$, then Lemma 9 is contradicted by a. So $k = 4$, b sees y_2, c sees y_3, and by Lemma 14, $N(a)$ is the 4-cycle b-y_2-y_3-c. Since y_2, y_3 must not form an even pair, there is an odd path P between them; but then $P + a$ is an odd hole, a contradiction. Therefore $N(a) \nsubseteq N(b) \cup N(c)$.

Consider any $y \in N(a) - (N(b) \cup N(c))$, and let K be the component of $G - (N(b) \cup N(c))$ that contains y. By Lemma 15, no vertex of K sees d. By Lemma 7, $N(c) \cup \{c\}$ is not a cutset, and so there is a vertex in $N(b) - N(c)$ adjacent to K. Likewise there is a vertex in $N(c) - N(b)$ adjacent to K. By planarity, we can choose $e \in N(b) - N(c)$ and $f \in N(c) - N(b)$ such that both e and f see elements of K

and $\{e, b, c, f\}$ is a cutset separating K from d. (More precisely, if $z_1, \ldots, z_k$ are the neighbors of b clockwise in the planar representation of G, with $z_1 = d$, $z_2 = c$, $z_3 = a$, then e is the vertex z_j that has a neighbor in K and such that j is the largest such integer. Vertex f is defined similarly.) There is a chordless e, f-path P_1 with interior in K by definition. This path must be odd or else $P_1 + b + c$ would be an odd hole.

We claim that $S = \{e, b, c, f\}$ is a minimal cutset. Suppose that some proper subset S' of S is a cutset. We have $|S'| = 3$ since G is 3-connected. Lemma 7 implies that S' is not $S - e$ or $S - f$. Thus, and by symmetry, S' is $S - b = \{e, c, f\}$. Then Lemmas 7 and 8 imply that e does not see f and that there is a vertex w with $N(w) = \{e, c, f\}$. But then $P_1 + w$ form an odd hole. So the claim holds.

Now, by Lemma 3, all chordless e, f-paths in $G - b - c$ have the same parity, which is odd since P_1 is one such path. By the same lemma, all chordless b, f-paths in $G - c - e$ have the same parity, which is odd, for otherwise $\{b, f\}$ would be an even pair. Similarly all chordless c, e-paths in $G - b - f$ are odd.

Suppose that e and f have no common neighbor in $G - K - a$. Since G is 3-connected, there exist paths from d to $\{e, f\}$ in $G - (K \cup \{a, b, c\})$. Consider a shortest one P_2; it can be assumed by symmetry that P_2 is from d to e. Note that there are no neighbors of f in $P_2 - e$ by the minimality of P_2 and the fact that e and f have no common neighbor in $G - K - a$. The path $P_2 + c$ contains a shortest c, e-path P_2', and P_2' is odd as mentioned above. But now $P_1 + P_2'$ induces an odd hole (even if $P_1 = ef$), a contradiction. So e and f have a common neighbor in $G - K - a$. It follows that $ef \in E(G)$ since every chordless e, f-path in $G - b - c$ must be odd.

Let g be the common neighbor of e and f such that the cycle $gebcf$ bounds a region of the plane containing d and no other common neighbor of e and f. If there exists a path from d to $\{e, f\}$ in $G - \{b, c, g\}$, then, as in the above case where e and f had no common neighbor, we obtain a contradiction. So $\{b, c, g\}$ is a cutset, and Lemma 8 implies $N(d) = \{b, c, g\}$. But then b, c, d, e, f, g induce a proper subgraph of G which is a $\bar{C}_6$, a contradiction. This completes the proof of Lemma 16. □

LEMMA 17. *G contains no vertex of degree five or greater.*

Proof. Suppose that x is a vertex of degree at least five in G. By Lemmas 13, 14, and 16, F_x is an even hole, it contains an even number of edges of $N(x)$, and no three consecutive vertices of F_x are neighbors of x. We claim that

(1) *no vertex of $G - x - F_x$ sees two nonadjacent vertices of F_x.*

Assume on the contrary that there exists a triple $\{a, s, t\}$ such that a is in $G - x - F_x$ and s, t are nonadjacent neighbors of a in F_x. Let P_1 and P_2 be the two chordless s, t-paths in F_x, where the length of P_1 is less than or equal to the length of P_2. We choose a triple $\{a, s, t\}$ that minimizes the length of P_1 over all appropriate choices. Path P_1 has even length, for otherwise either $P_1 + a$ would be an odd hole or the minimality of P_1 would be contradicted. Then P_2 is also even since F_x is an even hole. Note that $\{a, s, t, x\}$ is a cutset that separates $P_1 - s - t$ from $P_2 - s - t$. For $i = 1, 2$, call K_i the component of $G - \{a, s, t, x\}$ that contains $P_i - s - t$. There is a neighbor of x on $P_1 - s - t$, or else $\{a, s, t\}$ would be a cutset contradicting Lemma 7. Call u (resp., v) the neighbor of x in $P_1 - s - t$ that is closest to s (resp., to t). Consider the chordless paths $P_1^s = P_1[s, u] + ux$ and $P_1^t = P_1[t, v] + vx$. Note that these two paths have their interiors in K_1. Likewise there is a neighbor of x on $P_2 - s - t$. Call y (resp., z) the neighbor of x that is closest to s (resp., t) on $P_2 - s - t$. Consider the chordless paths $P_2^s = P_2[s, y] + yx$ and $P_2^t = P_2[t, z] + zx$. Note that these two paths have their interior in K_2.

First suppose that either (a) $u = v$ or (b) u, v are different and not adjacent. In case (a), clearly P_1^s and P_1^t have the same parity p since P_1 has even length. In case (b), $P_1^s + sa + at + P_1^t$ is a hole, which is even, so P_1^s and P_1^t again have the same parity p. In either case, both P_2^s, P_2^t also have parity p, for otherwise either $P_2^s + P_1^s$ or $P_2^t + P_1^t$ would be an odd hole. Since $\{s,t\}$ is not an even pair, there is an odd chordless s,t-path Q in G. Clearly Q does not use a. If Q also does not use x, then its interior lies in some component of $G - \{a, s, t, x\}$ and so one of $Q + P_1$ or $Q + P_2$ is an odd hole, a contradiction. Thus Q contains x. So Q consists of a chordless s,x-path Q' in $G - a - t$ plus a chordless x,t-path Q'' in $G - a - s$, and the paths Q', Q'' have different parities. Note that each of Q', Q'' has its interior in some component of $G - \{a, s, t, x\}$. Up to symmetry, Q' is odd and Q'' is even, and so x does not see t. If the parity p is even, then x does not see s, and at least one of $Q' + P_1^s$ or $Q' + P_2^s$ is an odd hole, a contradiction. If p is odd, then at least one of $Q'' + P_1^t$ or $Q'' + P_2^t$ is an odd hole, a contradiction.

Now suppose that u, v are different and adjacent. Then P_1^s and P_1^t have different parities. By Lemma 14, P_2 has an odd number of edges of $N(x)$. Call s' and t', respectively, the neighbor of s and t on P_2. Vertex a sees no vertex of $P_2 - \{s, s', t, t'\}$, for otherwise there would exist a chordless a,x-path R in $a + x + P_2 - \{s, s', t, t'\}$ and then either $R + as + P_1^s$ or $R + at + P_1^t$ would be an odd hole. Moreover, if a sees one of s', t', then it sees both, or else $F_x + a$ would contain an odd hole by Lemma 1 applied to P_2 and a. Suppose that a sees both s', t'. Let $F'_x = F_x + a - P_1$. So F'_x is a hole and must be even. Since G contains no diamond, x misses both s', t'. So x sees at least three vertices of F'_x (since x has degree at least five), and F'_x contains an odd number of edges of $N(x)$, so $x + F'_x$ contains an odd hole by Lemma 1, a contradiction. Thus a misses both s', t'. Call P'_1 the interior of P_1 and let $F'_x = F_x + a - P'_1$. A similar argument shows that $x + F'_x$ contains an odd hole, a contradiction. Thus (1) holds.

Let $E_x = \{e_0, \dots, e_{k-1}\}$ be the set of edges in $N(x)$, with $e_i = u_i v_i$. Lemmas 1, 4, 14, and 16 imply that E_x is a matching of even size $k \geq 2$, so we can enumerate the components of $G[F_x] - E_x$ as $C_0, \dots, C_{k-1}$ so that for $0 \leq i \leq k-1$ the endpoints of C_i are v_{i-1} and u_i (modulo k). By Lemma 1, each C_i is an even path, so it has a nonempty interior. Let W be the set of vertices of $G - x - F_x$ that have a neighbor in F_x. By Lemma 16 and (1), for every vertex $w \in W$ the set $N(w) \cap F_x$ consists of one or two consecutive vertices of some C_i. Thus W can be partitioned into $W_0, \dots, W_{k-1}$, where $W_i = \{w \in W \mid N(w) \cap F_x \subset C_i\}$. Let $W_i^* = \{w \in W_i \mid w$ has a neighbor in the interior of $C_i\}$, and $X_i = W_i - W_i^*$ (so every vertex $w \in X_i$ satisfies $N(w) \cap F_x = \{v_{i-1}\}$ or $\{u_i\}$). For $0 \leq i, j \leq k-1$ with $i \neq j$, call (i,j)-*connection* any path $P = f \cdots g$ in $G - x - F_x$ such that $f \in W_i$, $g \in W_j$, there are nonadjacent vertices $y \in C_i \cap N(f)$ and $z \in C_j \cap N(g)$, and the interior vertices of P are in $G - (W_0^* \cup \cdots \cup W_{k-1}^*)$. Say that an (i,j)-connection is *pure* if its interior vertices are in $G - (W - X_i - X_j)$. We claim that

(2) *there exists a pure* $(i, i+1)$*-connection for some* i.

By Lemma 7, the graph $G - \{x, u_0, v_0, \dots, u_{k-1}, v_{k-1}\}$ is connected, which implies that for each $i = 0, \dots, k-1$ there exists an (i,j)-connection for some j.

We remark that, for every $i \neq j$, every (i,j)-connection $P = f \cdots g$ contains a pure (i', j')-connection for some i', j'. We prove this remark by induction on the length of P. Suppose that P itself is not pure. So there is a vertex $h \in (P - f - g) \cap X_l$ for some $l \neq i, j$. Pick any $y \in F_x \cap N(f)$, $z \in F_x \cap N(g)$, $t \in F_x \cap N(h)$. It cannot be that t sees both y, z, so either $P[f,h]$ or $P[h,g]$ is a connection, which, by induction, contains a pure connection. So the remark is proved, and in particular there exists a pure connection.

Let $P = f \cdots g$ be a pure (i,j)-connection. Let P' be any path of F_x between vertices $y \in F_x \cap N(f)$, $z \in F_x \cap N(g)$, and let R be the region of the plane that is bounded by $P \cup P'$ and does not contain x. We choose i, j, P, P' such that the length of P' is minimized. If $|j - i| = 1$, we are done. Otherwise, there is some l such that $e_{l-1} \cup C_l \cup e_l \subseteq P'$. We know that there exists an (l, m)-connection P'' for some m. By the planarity of G, either P'' lies entirely inside R, and then every pure connection contained in P'' contradicts the choice of P'; or P'' intersects P, and then $P \cup P''$ contains an (l,i)- or an (l,j)-connection, which contains a pure connection that contradicts the minimality of P'. So (2) holds.

By (2) and up to symmetry we can assume that there exists a pure $(1,2)$-connection. We choose a shortest one $P = f \cdots g$, with $f \in W_1$ and $g \in W_2$. Let y (resp., y') be the neighbor of f on C_1 closest to v_0 (resp., to u_1). Recall that y, y' are either equal or adjacent. Likewise let z (resp., z') be the neighbor of g along C_2 closest to u_2 (resp., to v_1). Let Q be the y,z-path of F_x that does not contain u_1v_1, and let Q' be the y',z'-path of F_x that contains u_1v_1. Suppose $|E_x| \geq 4$. Then no interior vertex h of P sees v_0 or u_2, for otherwise $P[h,g]$ or $P[f,h]$ would contradict the choice of P. Thus $P \cup Q$ induces a hole, which contains $|E_x| - 1$ edges of E_x; so, by Lemma 1, G contains an odd hole, a contradiction. So $|E_x| = 2$. Now Q and Q' play a symmetric role. If no interior vertex of P sees any of u_0, v_0, u_1, v_1, then $P \cup Q$ and $P \cup Q'$ induce two holes, each of them contains one edge of E_x, and at least one of them contains three neighbors of x; so, by Lemma 1, G contains an odd hole, a contradiction. So we may assume, up to symmetry, that some interior vertex h of P is adjacent to u_1. Then $z = v_1$, for otherwise $P[h,g]$ would contradict the choice of P; no vertex g' of $P - g$ is adjacent to v_1, for otherwise $P[f,g']$ would contradict the choice of P; no vertex f' of $P - f$ is adjacent to v_0, for otherwise $P[f',g]$ would contradict the choice of P; and no vertex d of P is adjacent to u_0, for otherwise $P[d,h]$ would contradict the choice of P. Thus $P \cup Q$ induces a hole, which contains one edge of E_x and at least three neighbors (v_1, u_0, v_0) of x, and so by Lemma 1, G contains an odd hole, a contradiction. This completes the proof of Lemma 17. □

Now, since G is 3-connected, it has minimum degree three. This and the above lemmas yield that every vertex of G has a correct neighborhood, and Theorem 4 is established.

3. Obstructions that have a 2-cutset. In this section we consider obstructions that are not necessarily 3-connected. For this purpose we need some new definitions.

DEFINITION 1 (candidate). *A candidate is any planar graph G such that*

(c1) *G is the line-graph of a bipartite graph;*

(c2) *every neighborhood is correct;*

(c3) *G is 2-connected;*

(c4) *every 2-cutset is an odd pair.*

LEMMA 18. *Any candidate satisfies the following additional properties:*

(c5) *If $\{a,b\}$ is a 2-cutset in a candidate G, then $G - \{a,b\}$ has exactly two components;*

(c6) *a candidate has no clique-cutset;*

(c7) *in a 3-connected candidate, every 3-cutset C contains at most one edge, and if it contains one edge, then C is the neighborhood of a vertex.*

Proof. If (c5) fails, then the neighborhood of a or b is incorrect.

To prove (c6), assume C is a clique cutset in a candidate G. By (c3) and (c4), C has at least three vertices. By (c2), G has no clique of size four, so $|C| = 3$. Let a be a vertex of C, and let B_1 and B_2 be components of $G - C$. Vertex a must have a neighbor in B_1 and a neighbor in B_2, or else $C - \{a\}$ would be a 2-cutset violating (c4). But then the neighborhood of a is incorrect, a contradiction.

To prove (c7), consider a 3-cutset $C = \{a, b, c\}$ in a 3-connected candidate G. Let B_1 and B_2 be two components of $G - C$. By (c6) we can assume that a, c are not adjacent. Suppose ab is an edge. Vertex a must have a neighbor x in B_1, as well as a neighbor y in B_2, or else $\{b, c\}$ would be a cutset. Since the neighborhood of a is correct, we may assume by symmetry that b sees x and misses y. Vertex b must have a neighbor z in B_2, or else $\{a, c\}$ would be a cutset separating $B_1 \cup \{b\}$ from B_2. Then b has no neighbor in $B_1 - \{x\}$, or else the neighborhood of b would be incorrect. The same holds for a. Now if $B_1 - \{x\}$ is not empty, then $\{c, x\}$ is a cutset, a contradiction. So, $B_1 = \{x\}$, and then neighborhood correctness implies that x sees c and c misses b. This proves (c7). ∎

An edge is called *flat* if it does not lie in a triangle. It is easy to see that an edge ab is flat in a graph G containing no odd hole if and only if the pair $\{a, b\}$ is an odd pair in the graph $G - ab$.

DEFINITION 2 (glueing). *Let $M_1 = (V_1, E_1), \ldots, M_k = (V_k, E_k)$ be some graphs. Assume that there exist two vertices a, b such that $V_i \cap V_j = \{a, b\}$ for all i, j with $1 \le i < j \le k$, where ab is an edge of each M_i. We will say that the graphs are glueable (along ab). Given such graphs as above, we can build a graph with vertex-set $V_1 \cup \cdots \cup V_k$ and edge-set $E_1 \cup \cdots \cup E_k - \{ab\}$. This construction will be called glueing the graphs $M_1, \ldots, M_k$ along edge ab.*

DEFINITION 3 (unglueing). *Let G be a graph with a 2-cutset $\{a, b\}$, where a and b are nonadjacent, and let $B_1, \ldots, B_k$ be the components of $G - \{a, b\}$. For each i, we can build a graph with vertex-set $B_i \cup \{a, b\}$ and with all the edges of $G[B_i \cup \{a, b\}]$ plus the edge ab. This will be called unglueing the graph G along ab.*

It is easy to see that the operations of glueing and unglueing preserve planarity, and that unglueing preserves the absence of odd holes. Conversely, if $M_1, \ldots, M_k$ contain no odd hole and ab is a flat edge in each of them, then glueing them along ab produces a graph with no odd hole. Hence they are perfection-preserving for planar graphs. (The fact that they are perfection-preserving for general graphs is also true and is an easy consequence of a lemma of Tucker [14].)

LEMMA 19 (the glueing lemma for candidates).

1. *Let M_1 and M_2 be two graphs that are glueable along an edge ab which is a flat edge in each of them. If M_1 and M_2 are candidates, then the graph obtained from glueing them along ab is a candidate.*

2. *Conversely, let M be a candidate which admits a 2-cutset $\{a, b\}$. Let M_1, M_2 be the two graphs obtained by unglueing M along this cutset. Then each of M_1, M_2 is a candidate.*

Proof. To prove the first part, let H_1, H_2 be vertex-disjoint bipartite graphs such that $M_1 = L(H_1)$ and $M_2 = L(H_2)$. Since ab is an edge of M_i $(i = 1, 2)$, the edges a and b in H_i are adjacent. We call z_i their common endpoint and a_i, b_i, respectively, the other endpoint of a and b. Vertex z_i has degree two in H_i because ab is a flat edge of M_i. Let H be the bipartite graph obtained from the union of H_1 and H_2 by removing z_1 and z_2 and adding edges a_1a_2 and b_1b_2. It is a routine matter to check that $M = L(H)$ and that every neighborhood is correct in M. Also, M has no clique cutset Q, for otherwise Q would obviously be a clique cutset of either M_1 or M_2. Thus (c1), (c2), (c3) hold for M. To check (c4), suppose on the contrary that

$\{x_1, x_2\}$ is an even 2-cutset of G. If x_1, x_2 both lie in some M_i, then $\{x_1, x_2\}$ would be an even 2-cutset of M_i, which is impossible. So we may assume $x_1 \in M_1 - \{a, b\}$ and $x_2 \in M_2 - \{a, b\}$. Let P_i (resp., Q_i) be any chordless x_i, a-path (resp., x_i, b-path) in $M_i - b$ (resp., in $M_i - a$). Note that P_1 and P_2 have the same parity since $P_1 + P_2$ is a chordless x_1, x_2-path. Likewise Q_1 and Q_2 have the same parity. If P_1 is even and Q_1 is odd, then every chordless x_1, a-path in M_1 is even, which is impossible. It follows that P_1 and Q_1 have the same parity. There must be a chord from P_1 and Q_1, for otherwise $P_1 + Q_1$ would be an even chordless a, b-path, which is impossible. This chord implies that $(P_1 + P_2 + Q_1 + Q_2) - \{x_1, x_2\}$ is a connected subgraph of $G - \{x_1, x_2\}$ that contains a and b. Thus a, b lie in the same component of $G - \{x_1, x_2\}$, which in turn implies that x_1, x_2 lie in the same component of $G - \{a, b\}$, a possibility that was excluded earlier.

To prove the second part, let M be a candidate which admits a 2-cutset $\{a, b\}$. Suppose that a has degree three and call a' the vertex such that aa' is a flat edge; then both $\{a, b\}$ and $\{a', b\}$ are 2-cutsets of M, so one of them is not an odd pair, contradicting (c3). Thus a and, similarly, b have degree four. It follows that in M_1 as well as in M_2 both a and b have a correct neighborhood (of degree three); note that any other vertex of M_1 and M_2 has the same neighborhood as in M; so M_1 and M_2 have property (c2). Since the unglueing operation is perfection-preserving, M_1 and M_2 contain no odd hole. It follows that they are line-graphs of bipartite graphs by Theorem 3; i.e., (c1) holds for both of them. Suppose now that M_1 has a cut vertex x. Thus, x is one of a, b (say $x = a$), for otherwise x would be a cut vertex of M, a contradiction. There is a component C of $M_1 - a$ such that $b \notin C$ and there is no path in $M_1 - a$ joining b to any vertex of C. Hence a is also a cut vertex in M (separating b from C), a contradiction to M being a candidate. The same argument works for M_2. So (c3) holds for M_1 and M_2. Finally, suppose that $\{x, y\}$ is any 2-cutset of M_1. Thus $\{x, y\}$ is also a 2-cutset of M, and by (c4) all chordless x, y-paths in M have odd length. Consider a chordless x, y-path P in M_1. If P does not use the edge ab, then P is a path in M and it is odd by the argument in the preceding sentence. So assume P uses the edge ab. Suppose that x, y do not lie in the same component of $M - \{a, b\}$. This implies that any two vertices from $\{a, b, x, y\}$ form a 2-cutset of M. It is easy to see that at least one of these six cutsets would violate (c4) in M. So x, y must lie in the same component of $M - \{a, b\}$. Let Q be a chordless a, b-path of M whose interior vertices are in the component of $M - \{a, b\}$ that does not contain x and y; then the path $P - ab + Q$ must be odd, and since Q is odd, this implies that P is odd. So M_1 and M_2 satisfy (c4), and the lemma is proved. ∎

THEOREM 5. *Every candidate is even pair-free.*

Proof. This theorem is proved by induction on the number of vertices of the candidate. We distinguish between the candidates that are 3-connected and those that are not. For 3-connected planar candidates, the desired result is a consequence of a result of Hsu [8, Theorem 9.2] which we formulate as follows.

LEMMA 20 (Hsu [8]). *Let G be a 3-inseparable planar perfect graph such that the neighborhood of each vertex is correct. Then between any two nonadjacent vertices there are chordless paths of both parities.*

In this lemma, "3-inseparable" is a weaker condition than 3-connected, and it is easy to check that every 3-connected candidate is 3-inseparable. So Lemma 20 gives immediately the even pair-freeness of every 3-connected candidate. Now we look at candidates that are not 3-connected. A property of candidates will be useful.

LEMMA 21. *Let x, y be two nonadjacent vertices in a graph G that satisfies* (c2). *Let $v_0 v_1 \cdots v_k v_{k+1}$ and $w_0 w_1 \cdots w_l w_{l+1}$ be internally disjoint chordless paths from*

$x = v_0 = w_0$ to $y = v_{k+1} = w_{l+1}$. Then the only possible chords between these two paths are v_1w_1 and v_kw_l.

Proof. Let v_iw_j be any chord with $1 \leq i \leq k$, $1 \leq j \leq l$. Neighborhood correctness for v_i implies that there is exactly one edge induced by $\{v_{i-1}, v_{i+1}, w_j\}$; since there is no edge $v_{i-1}v_{i+1}$, by symmetry we can assume that w_j sees v_{i-1} and misses v_{i+1}. Now if $j \geq 2$, then neighborhood correctness is violated because w_j has four neighbors and there is no edge $w_{j-1}w_{j+1}$. If $j \leq 1$, the same holds unless $w_{j-1} = v_{i-1}$, i.e., $i = 1$, and so the chord is v_1w_1. ∎

LEMMA 22. *Any candidate G that admits a 2-cutset is even pair-free.*

Proof. Let $\{a, b\}$ be a 2-cutset of G. Recall from (c4) that $\{a, b\}$ is an odd pair and from (c5) that $G - \{a, b\}$ has exactly two components B_1, B_2. Let G_1, G_2 be the two graphs obtained by unglueing G along ab. By the second part of the glueing lemma, G_1 and G_2 are candidates. By the induction hypothesis, G_1 and G_2 are even pair-free.

Consider any pair of nonadjacent vertices x, y in G. If x, y are both in one of G_1, G_2, say in G_1, then by the induction hypothesis there exists an odd chordless x, y-path P in G_1. If P does not use the edge ab, then P is the desired odd chordless x, y-path in G. If P uses the edge ab, then replace in P the edge ab by any chordless a, b-path lying in $B_2 \cup \{a, b\}$. Since a, b is an odd pair the parity is not changed by this replacement and we get an odd chordless x, y-path in G.

Now, assume that x is in B_1 and y is in B_2. Since G is 2-connected there exist two internally vertex-disjoint paths Q_1 and Q_2 from y to a and b, respectively. Clearly, Q_1 and Q_2 are entirely in $B_2 \cup \{a, b\}$. Call y_1 and y_2 the neighbors of y along Q_1 and Q_2, respectively. Let R be any (odd) chordless a, b-path entirely in $B_1 \cup \{a, b\}$. By Lemma 21, there is no chord between the chordless y, a-paths Q_1 and $Q_2 + R$ except maybe for the chord y_1y_2. We claim that we can assume that y_1y_2 is not an edge. Indeed, suppose that y_1 and y_2 are adjacent. By (c4), the set $\{y_1, y_2\}$ is not a cutset of G, so there exists a shortest path Q from y to $Q_1 \cup Q_2 - \{y, y_1, y_2\}$. By symmetry we can assume that the other endpoint of Q is on $Q_1 - \{y, y_1\}$. Then $Q \cup Q_1 - \{y_1\}$ connects y to a and is disjoint from Q_2, so there exists a shortest path Q' in $Q \cup Q_1$ which connects y to a and is disjoint from Q_2. The neighbor of y on Q' is not adjacent to y_2, or else the neighborhood of y would be incorrect. Then we replace the pair of paths Q_1, Q_2 by the pair Q', Q_2. So the claim holds.

Now, since y_1, y_2 are not adjacent, $Q_1 + Q_2 + R$ is a hole. Since R is odd, we may assume by symmetry that Q_1 is odd and Q_2 is even. By the induction hypothesis, G_1 contains an odd path P from x to b. If P does not use vertex a, then $P + Q_2$ is the desired odd chordless path from x to y. If P uses a, then $P - b$ is an even chordless x, a-path, and so $P - b + Q_1$ is the desired odd path. ∎

Now the proof of Theorem 5 is complete. ∎

THEOREM 6. *Every planar obstruction is an odd hole or a candidate.*

Proof. Let G be a planar obstruction different from an odd hole. We prove Theorem 6 by induction on the number of vertices of G. If G is 3-connected, the result holds by Theorem 4. Now assume G is not 3-connected. By Lemma 2, G is 2-connected. Let $\{a, b\}$ be a 2-cutset of G. By Lemma 3 and since G has no even pair, every chordless a, b-path in G is odd. Let $B_1, \ldots, B_k$ be the connected components of $G - \{a, b\}$, and put $G_i = G[B_i \cup \{a, b\}] + ab$. Note that G_i is not a clique (because a, b have no common neighbor in G). Let Q_i be any chordless a, b-path in $B_i \cup \{a, b\}$.

We first claim that each G_i has no even pair. Indeed, let x, y be any two nonadjacent vertices in G_i. In G there exists an odd chordless x, y-path P from x to y. If P does not use both a, b, then P lies entirely in G_i and we are done. If P uses both

a, b, then $P[a, b]$ is odd; so $P[x, a] + ab + P[b, y]$ is an odd chordless x, y-path in G_i and we are done again.

Since each G_i is even pair-free, it contains an obstruction G'_i. For each i, both a, b lie in G'_i, or else G'_i would be a proper induced subgraph of G, which is impossible. Moreover, G'_i is not an odd hole, for otherwise replacing the edge ab in this hole by Q_j with $j \neq i$ would produce an odd hole in G, which is impossible. The induction hypothesis (and the fact that each G'_i has strictly fewer vertices than G) implies that each G'_i is a candidate. Note that the edge ab is flat in each G'_i, because a, b have no common neighbor in G. Let G' be the graph obtained by glueing G'_1 and G'_2 along the edge ab. The glueing lemma for candidates implies that G' is a candidate. Theorem 5 implies that G' has no even pair, and so it contains an obstruction G''. Since G itself is an obstruction, we must actually have $G'' = G' = G$. In particular, G is a candidate. ◻

Proof of Theorem 2. Let G be a planar obstruction different from an odd hole. Theorem 6 implies that G is a candidate. Thus every neighborhood is correct, and Theorem 2 is established.

LEMMA 23 (the glueing lemma for obstructions).

1. *Let M_1 and M_2 be two obstructions that are glueable along an edge ab such that ab is a flat edge in each M_i. Then the graph M obtained by glueing them along ab is an obstruction.*

2. *Conversely, let M be an obstruction which admits a 2-cutset. Let M_1, M_2 be the two graphs obtained by unglueing M along this cutset. Then each of M_1, M_2 is an obstruction.*

Proof. To prove the first part of the lemma, suppose that M is not an obstruction. We know by the glueing lemma for candidates that M is a candidate. So M contains strictly an obstruction M'. Note that M' cannot be entirely in $M_1 - ab$ or in $M_2 - ab$, for otherwise M' would be entirely in either $M_1 - \{a, b\}$ or $M_2 - \{a, b\}$, contradicting the fact that M_1 and M_2 are obstructions, or a or b would be an incorrect vertex in M'. Now $\{a, b\} \cap M'$ is a cutset of M', and hence $\{a, b\} \subset M'$. Let M'_1 and M'_2 be the graphs obtained by unglueing M' along $\{a, b\}$. We know that M'_1 and M'_2 are candidates. Since M' is a proper subgraph of M, and by symmetry, we may assume that M'_1 is a proper subgraph of M_1. But this is a contradiction to the fact that M_1 is an obstruction.

For the second part of the lemma, suppose that M_1 is not an obstruction. We know that M_1 is a candidate. So M_1 contains strictly an obstruction M'_1. If M'_1 does not contain the edge ab, then M'_1 is a subgraph of M, contradicting the fact that M is an obstruction. If M'_1 contains the edge ab, then we can glue M'_1 with M_2. We obtain a candidate which is a proper subgraph of M, again a contradiction. ◻

Now we have proved that every planar obstruction is an odd hole or a candidate and that every candidate contains an obstruction, and thus *a planar perfect graph is an SQP graph if and only if it contains no candidate.* However, it is not true that every candidate is an obstruction: a counterexample is the line-graph of $K_{3,3}$. In the next section we study the difference between obstructions and mere candidates.

4. Distinguishing obstructions from candidates. Here we show how to find a subset of vertices of a candidate whose removal yields an obstruction.

DEFINITION 4 (danger). *In a candidate G, we denote by* danger *any proper subset of vertices D such that each component of $G - D$ is a candidate.*

For example, in the line-graph of $K_{3,3} - e$, two adjacent vertices of degree three form a danger.

The definition entails the following lemma, which is immediate.

LEMMA 24. *A candidate is an obstruction if and only if it contains no danger.*

The question now is how to find a danger in a graph. Given a vertex x, we can define iteratively a subset $D(x)$ as follows. First, remove x from G and let $H = G - x$. Then apply the following procedure:

Step 1. If H has an incorrect vertex y, then remove y and iterate with each component of $H - y$.

Step 2. If H has a cut vertex y, then remove y and iterate with each component of $H - y$.

Step 3. If H has a 2-cutset C that violates (c4) in H, then remove C and iterate with each component of $H - C$.

Step 2 (resp., Step 3) is applied only when there is no possibility of applying Step 1 (resp., Step 2). This procedure is applied as long as possible. We let $D(x)$ denote the set of all removed vertices at the end of the procedure. Clearly, this definition entails immediately that if $D(x)$ does not contain all vertices of G, then $D(x)$ is a danger. Furthermore, we have the next lemma.

LEMMA 25. *Let G be a candidate containing a danger D, and let x be any vertex of D. Then $D(x) \subseteq D$ and $D(x)$ is a danger.*

Proof. We prove that $D(x) \subseteq D$. This will imply that $D(x)$ does not consist of all vertices of G, and so $D(x)$ will be a danger. Consider any subset R of D. We claim that

If $G - R$ has an incorrect vertex y, then y is in D.

For otherwise, y would be an incorrect vertex in $G - D$, contradicting the fact that each component of $G - D$ is a candidate.

If a component H of $G - R$ has only correct vertices and has a cut vertex y, then y must be in D.

For otherwise, let B_1 and B_2 be the components of $H - y$, and let H' be the component of $G - D$ containing y. Clearly H' is a subgraph of H. By neighborhood correctness and by symmetry, we may assume that y has two neighbors in B_1 and one or two neighbors in B_2. At least one vertex from $N(y) \cap B_1$ and one vertex from $N(y) \cap B_2$ must be in H', or else y would have an incorrect neighborhood in H'. But then y is a cut vertex in H', a contradiction.

If a component H of $G - R$ has only correct vertices, has no cut vertex, and has a cutset C violating (c4) in H, then $C \subseteq D$.

For otherwise, call B_1 and B_2 the components of $H - C$, assume that some vertex y in C is in $G - D$, and call H' the component of $G - D$ containing y. As above, at least one vertex from $N(y) \cap B_1$ and one vertex from $N(y) \cap B_2$ must be in H', or else either y would be incorrect or $C - \{y\}$ would be a 1-cutset in H. But then $C \cap H'$ is either a 1-cutset or a 2-cutset violating (c4) in H', a contradiction.

Now the fact that $D(x) \subseteq D$ follows from the definition of $D(x)$ and from the three points above, applied iteratively, starting with $R = \{x\}$. □

The following lemma is now obvious.

LEMMA 26. *A candidate G is an obstruction if and only if $D(x) = V(G)$ for every vertex x of G.*

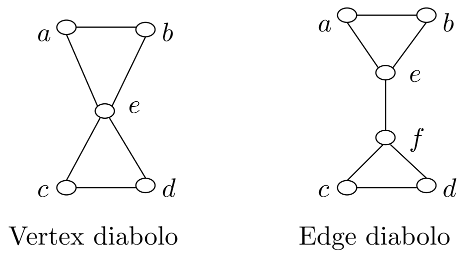

FIG. 1. *The two types of diabolo.*

As a consequence, we can decide if a candidate contains a danger, and if it does, we can find a danger efficiently; we have only to build $D(x)$ for each vertex until we find one such that $D(x)$ does not contain all vertices. Notice that the construction of $D(x)$ is polynomial, as it involves only degree and connectivity tests.

Finally, this allows us to find an obstruction in any given candidate G. If G contains no danger D, then G is an obstruction; if not, then find such a danger D and continue with the components of $G - D$, which is a candidate.

4.1. Constructing all candidates. We are going to describe a method for constructing a family $\mathcal{F}$ of graphs starting from $\bar{C}_6$. We will show that $\mathcal{F}$ is exactly the class of all candidates.

Let G be the line-graph of a connected bipartite graph H, where the vertices are colored blue or red along the natural bipartition. Using the one-to-one correspondence between the vertices of H and the maximal cliques of G, we can consider the edges of G colored blue or red accordingly.

DEFINITION 5 (diabolo). *The left and right graphs depicted in Figure* 1 *will be called, respectively,* vertex-diabolo *and* edge-diabolo. *Edges ab and cd are the bases of the diabolo, and vertex e (and f) is (are) the center(s) of the diabolo.*

DEFINITION 6 (splitting elements). *Let G be a planar graph without an odd hole and where every neighborhood is correct (so G is the line-graph of a bipartite graph). We call a* splitting element *of G any flat edge and any vertex of degree four. Then a* splitting operation *is the following:*

1. *Choose two splitting elements on the same face.*
2. *If one of the chosen elements is a vertex f, with neighbors* x_1, x_2, x_3, x_4 *such that* x_1x_2 *and* x_3x_4 *are edges, then replace it by a chordless path* $f_1f_2f_3$ *such that* f_2 *has degree two,* f_1 *is linked to* x_1 *and* x_2*, and* f_3 *is linked to* x_3 *and* x_4*. Edges* f_1f_2 *and* f_2f_3 *are called new.*
3. *If one of the chosen elements is an edge, then either call it new, or subdivide it into a path of length three (in this case, only the middle edge is called new).*
4. *A diabolo is introduced whose two bases are two new edges, one from each splitting element. The type of diabolo (edge or vertex) will be chosen so as to respect the blue/red bipartition of maximal cliques.*

Figures 2–4 illustrate the splitting operations, which allow us to create new planar graphs from old.

We can also view the splitting operations on the bipartite graph H, of which G is the line-graph, rather than on G. For example, in the first row in Figure 2, the hypothesis means that in H there exists a vertex x_{ab} of degree two incident with the edges a and b, there exists a vertex x_{cd} of degree two incident with the edges c and d, and the vertices x_{ab}, x_{cd} are on the same side of the bipartition of H. (In the

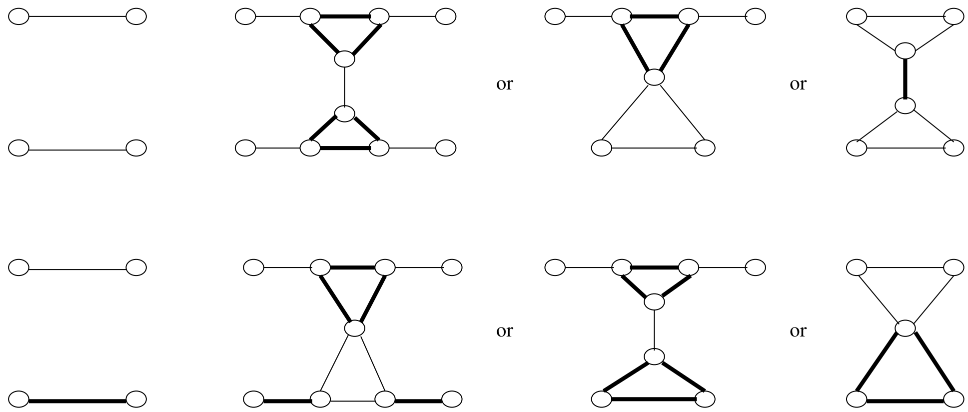

FIG. 2. *Edge-edge operations.*

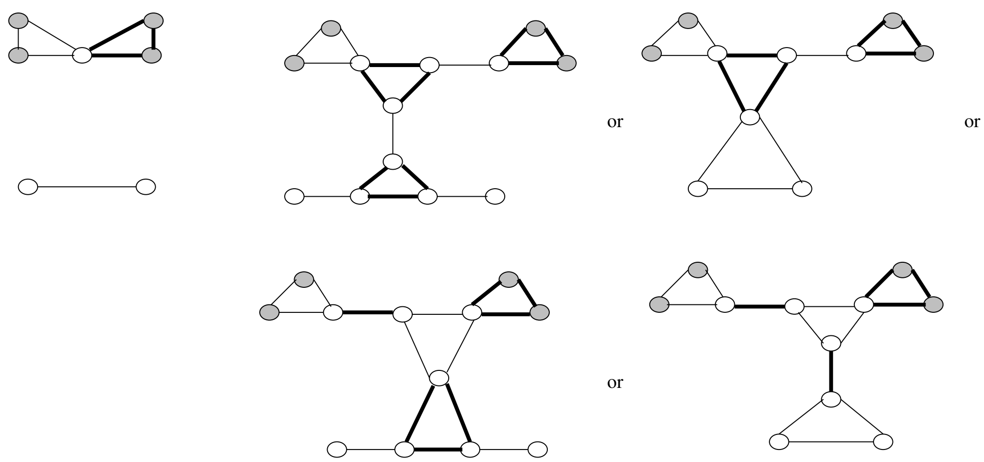

FIG. 3. *Edge-vertex operations.*

second row of Figure 2, the vertices x_{ab}, x_{cd} are on different sides of the bipartition.) For the first transformation illustrated in the first row of Figure 2, in H split the vertex x_{ab} into two vertices x_a and x_b to be the new endpoints of the edges a and b, respectively; likewise split x_{cd} into x_c and x_d. Add new vertices y_{ab}, y_{cd}, x and edges $x_a y_{ab}, x_b y_{ab}, x_c y_{cd}, x_d y_{cd}, x y_{ab}, x y_{cd}$. It is easy to check that this transformation on H corresponds exactly to the transformation on G. A similar transformation on H can be defined for each splitting operation illustrated in Figures 2–4.

DEFINITION 7 (class $\mathcal{F}$). *We define the class $\mathcal{F}$ of graphs by the following inductive scheme:*

- $\bar{C}_6 \in \mathcal{F}$;
- *if a graph G is in $\mathcal{F}$, then any graph obtained from G by splitting is in $\mathcal{F}$;*
- *if two graphs of $\mathcal{F}$ are glueable along a flat edge, then the graph resulting from the corresponding glueing is in $\mathcal{F}$.*

See Figure 6.

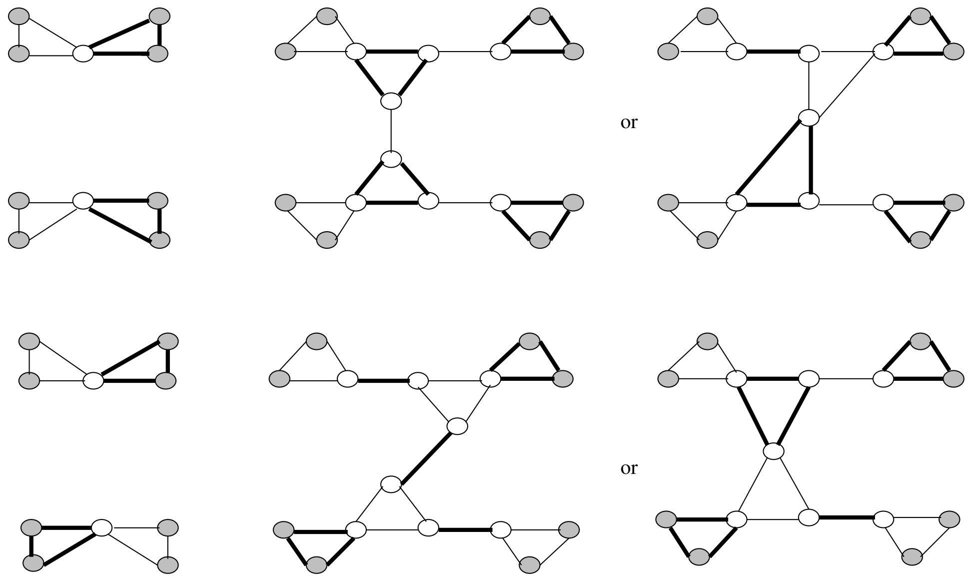

FIG. 4. *Vertex-vertex operations.*

THEOREM 7. *All graphs in $\mathcal{F}$ are candidates.*

Proof. The proof is by induction on the number of vertices. The theorem is true for $\bar{C}_6$. Now let us consider a graph $G' \neq \bar{C}_6$ in $\mathcal{F}$ and assume that the theorem is true for every graph smaller than G'. We know that G' is obtained either by splitting a graph G from $\mathcal{F}$ or by glueing two graphs G_1, G_2 from $\mathcal{F}$ along a common flat edge. If G' results from glueing, the desired conclusion is immediate from the glueing lemma and the induction hypothesis. So we may assume that G' results from splitting G. By the induction hypothesis, G is a candidate. Let H be the bipartite graph such that $G = L(H)$, with the vertices of H and the edges of G colored blue or red according to the bipartition. It is easy to see that G' is also the line-graph of a bipartite graph H', because the splitting operation preserves the bipartition of the maximum cliques, and to see that G' is 2-connected. Clearly, splitting also preserves the fact that each neighborhood is correct. Also, it is a routine matter to check that the splitting operation maintains all path parities of G in G'; i.e., if there exists a path between two vertices x, y of G, then G' contains a path of the same parity between the corresponding vertices. So we have established properties (c1)–(c3) for G'. Finally, suppose that G' has a cutset C which violates property (c4). Then C is necessarily a cutset of G, which contradicts the fact that G is a candidate. ◻

It remains to prove that the operations generate all planar candidates. Thus we must prove that for each candidate we can, by performing inverse operations, obtain a bunch of $\bar{C}_6$'s.

We will say that G' arises from G by a *reduction* if G arises from G' via a splitting.

Recall that each neighborhood in a candidate must be correct. This implies that every vertex x and every flat edge e of a candidate corresponds to a possible reduction. Unfortunately, this reduction may fail to yield a proper smaller candidate, as we may create a graph that violates (c2) or (c4). We are going to show that these problems can be avoided by an appropriate choice of x or e. That is, we will show that if G is a 3-connected candidate, we can choose a reduction that yields a 3-connected candidate. If G is not 3-connected, then G arises from smaller candidates by glueing.

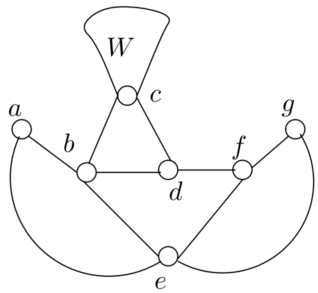

FIG. 5. *A diabolo $\{b, c, d, W\}$ whose reduction generates a diamond*

Putting these results together will give the desired result that every planar candidate is in $\mathcal{F}$.

Call *trivial cutset* a cutset C such that one component of $G - C$ is a vertex or an edge or a triangle. Let C be a trivial 3-cutset. Any diabolo associated with a vertex of C defines a reduction which does not change the connectivity of G.

Observe that only the vertex-edge and vertex-vertex reverse operations can produce an incorrect neighborhood for a candidate, by creating a diamond, and that they do so only if the vertices to be identified belong to a face of length four. We can see this situation in Figure 5.

LEMMA 27. *Let G be a 3-connected candidate. Let W be a diabolo whose reduction would generate a diamond. Then, there is another diabolo whose reduction will generate a 3-connected candidate.*

Proof. Let us label the vertices as in Figure 5. We must examine two cases. First, suppose that the face containing c, d, f, g is not of length four. Since the identification of d and g cannot produce a diamond, consider the diabolo W' having vertex e as the center. We claim that the reduction corresponding to W' generates a 3-connected candidate. The nontriangular face containing a and c cannot be of length four with a of degree three, since g and a neighbor of c in the other nontriangular face will form a 2-cutset which is an even pair. Also, if e is in a 3-cutset, this cutset is trivial (by a tedious but easy analysis) and so the graph obtained by the reduction remains 3-connected and is a candidate. Now, if the face containing c, d, f, g has length four, then cg must be an edge. Note that cg cannot be a flat edge, for otherwise $\{c, a\}$ would be a 2-cutset. So the reduction corresponding to the diabolo centered on df generates a 3-connected and diamond-free graph (it is easy to see that, in this case, all 3-cutsets containing f are trivial). □

LEMMA 28. *If a vertex x of a 3-connected candidate G belongs to a nontrivial cutset, then one of the two nontriangular faces containing x has a vertex y such that any 3-cutset containing y is trivial.*

Proof. Let F_1 and F_2 be the nontriangular faces containing x. Let y' and z be the vertices of F_1 and F_2, respectively, such that $G - \{x, y', z\}$ has the smallest component between all nontrivial 3-cutsets. Let y be a vertex of F_1 in $G - \{x, y', z\}$ such that y is a neighbor of y' and such that the edge $y'y$ belongs to a triangle. The vertex y is contained in a diabolo W which contains the intersection of F_1 and another nontriangular face F_2'. We claim that every 3-cutset containing y is trivial. Otherwise, there exists a 3-cutset $\{y, y'', x'\}$ with y'' in F_1 and x' in F_2' such that there is no component included in W. Therefore, either y' is not in the external face or $\{y'', y'\}$ or $\{x', x''\}$ is a 2-cutset, where x'' is the other vertex of W' that is on the external face, which in both cases is a contradiction. □

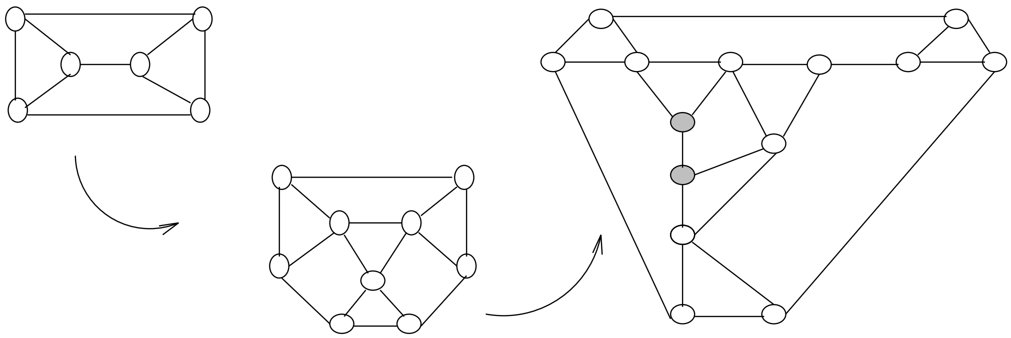

FIG. 6. *A graph G in $\mathcal{F}$. The gray vertices form a danger.*

LEMMA 29. *In every 3-connected candidate, there exists a diabolo whose reduction generates a candidate.*

Proof. Consider a candidate G. Choose some vertex x of G. If the reduction on the diabolo defined by this vertex produces a 2-connected graph G' in which every neighborhood is correct, and in which all 2-cutsets form odd pairs, then clearly G' is a candidate and the lemma holds. Otherwise, by Lemma 28, there is a diabolo in G centered in a vertex y such that the corresponding reduction produces a 3-connected graph. By Lemma 27, this diabolo may be replaced by one whose corresponding reduction produces a graph with a correct neighborhood. This lemma also guarantees that this reduction will keep the 3-connectivity. So, the lemma follows. ∎

THEOREM 8. *All candidates belong to $\mathcal{F}$.*

Proof. This proof is by induction on the number of vertices. The theorem is trivially true for the smallest candidate $\bar{C}_6$. Suppose it is true for candidates with less than n vertices and consider a candidate G with n vertices. If G has a 2-cutset, let G_1 and G_2 be the graphs obtained by unglueing G along this 2-cutset. Clearly, G_1 and G_2 are smaller than G and are candidates. So G is contained in $\mathcal{F}$. If G is 3-connected, by Lemma 29, there is an inverse operation that generates a candidate G' smaller than G. By the induction hypothesis, G' can be reduced to a set of $\bar{C}_6$'s. So, G can be obtained from G' by an operation of Definition 7 and then belongs to $\mathcal{F}$. ∎

LEMMA 30. *A graph G in $\mathcal{F}$ is an obstruction if and only if G has no dangers.*

This is a mere reformulation of Lemma 24.

Let $\mathcal{F}^*$ be the subclass of $\mathcal{F}$ made of the graphs containing no danger. The following is a corollary of all the results in this section.

THEOREM 9. *The class $\mathcal{F}^*$ consists exactly of all obstructions.*

Proof. It is clear that every graph from $\mathcal{F}^*$ is an obstruction. By Theorem 8, we can reduce any candidate to a collection of $\bar{C}_6$'s, and so from this collection we can obtain the original graph by performing in reverse order the same operations. So every obstruction is in $\mathcal{F}^*$. ∎

5. Conclusion and comments. We proved that Hougardy's conjecture is true for planar graphs. A related problem is to find a polynomial-time algorithm that decides if a graph is a planar SQP graph. We solved this problem in [9] using a decomposition technique due to Hsu [6], which is very different from the results presented here. Another related problem is to find an even pair (or decide that there is none) in a planar perfect graph. This problem was solved by Hsu and Porto [7] and by Linhares Sales and Sampaio [10]. Let us also recall that coloring can be done in polynomial time for all planar perfect graphs with an algorithm due to Hsu [6].

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