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Fractionally total colouring $G_{n,p}$ ☆

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Abstract

We study the fractional total chromatic number of $G_{n,p}$ as p varies from 0 to 1. We also present an algorithm that computes the fractional total chromatic number of a random graph in polynomial expected time.

Keywords: Graph colouring; Fractional total colouring; Random graphs

1. Introduction

A total colouring is the assignment of a colour to each vertex and edge of a graph such that no adjacent vertices or incident edges receive the same colour and no edge receives the same colour as one of its endpoints. A total stable set is the union of a matching and a stable set such that no vertex of the stable set is an endpoint of an edge in the matching. We can formulate total colouring as an integer program, in which we are trying to find the minimum number of total stable sets such that each element of $V(G) \cup E(G)$ is in at least one set. This IP's fractional relaxation is known as fractional total colouring. Formally if $\mathscr{T}(G)$ is the family of all total stable sets of a graph G then the fractional total chromatic number, $\chi_f^T(G)$, is the solution to the following LP:

$$\min \left\{ \sum_{T \in \mathscr{T}(G)} w_T : \forall T \in \mathscr{T}(G), w_T \geqslant 0; \forall u \in E(G) \cup V(G), \sum_{\{T \in \mathscr{T}(G): T \ni u\}} w_T \geqslant 1 \right\}$$

We denote the maximum degree of a vertex in G by $\Delta(G)$ and the minimum as $\delta(G)$. For each vertex v, a total stable set contains at most one element of the set composed of v and all the edges incident to it. This implies that $\chi_f^T(G) \geqslant \Delta + 1$. Kilakos and Reed [12] showed that $\chi_f^T(G) \leqslant \Delta + 2$ and this bound is tight. Thus, $\Delta(G) + \frac{3}{2}$ is an easily computable approximation to $\chi_f^T(G)$ which is never off by more than $\frac{1}{2}$. We are interested in trying to tie $\chi_f^T(G)$ down even more. We present three deterministic results which we then use to characterize the behaviour of the fractional chromatic number of almost every graph.

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Before discussing our results in this direction, we present two related questions and mention our algorithmic result. If we formulate vertex colouring as an IP and takes its fractional relaxation we get the following LP that defines the fractional chromatic number (where $\mathscr{S}(G)$ is the family of all stable sets of G)

$$\min\left\{\sum_{S\in\mathscr{S}(G)} w_S : \forall S\in\mathscr{S}(G), w_S\geqslant 0; \forall v\in V(G), \sum_{\{S\in\mathscr{S}(G):S\ni v\}} w_S\geqslant 1\right\}.$$

Similarly, we can formulate edge colouring as an IP and take its fractional relaxation to get an LP that defines the fractional chromatic index of a graph. If $\mathscr{M}(G)$ is the family of all matchings of a graph G the solution to the following LP defines the fractional chromatic index:

$$\min\left\{\sum_{M\in\mathscr{M}(G)} w_M : \forall M\in\mathscr{M}(G), w_M\geqslant 0; \forall e\in E(G), \sum_{\{M\in\mathscr{M}(G):M\ni e\}} w_M\geqslant 1\right\}.$$

It is hard to approximate the chromatic number within a multiplicative factor of $|V(G)|^{1-\varepsilon}$ for any constant $\varepsilon>0$ [7] unless $NP\subseteq ZPP$. It has also been shown that the fractional chromatic number approximates the chromatic number to within a logarithmic factor [13] which implies that calculating the fractional chromatic number is hard. Furthermore, Lund and Yannakakis [14] proved that there exists an $\varepsilon>0$ such that approximating the fractional chromatic number within a factor of $|V(G)|^{\varepsilon}$ is NP-hard. On the other hand, Edmond's characterization of the matching polytope [6] provided the means to compute the fractional chromatic index of a graph in polynomial time. More strongly he showed that we can solve weighted fractional edge colouring in polynomial time.

It is currently unknown whether or not it is NP-hard to determine the fractional total chromatic number. However, the problem is more tractable than computing the fractional chromatic number since we can approximate $\chi_f^T(G)$ to within an additive error of $\frac{1}{2}$ in $O(|E(G)|+|V(G)|)$ time. On the other hand the fractional total colouring problem is not as well behaved as the fractional edge colouring problem as the weighted version of the latter can be solved in polynomial time. However, for weighted fractional total colouring if we assigned weights of 0 to the edges and 1 to the vertices we would be attempting to compute the fractional chromatic number which is NP-hard.

We now return to our characterization of the fractional total chromatic number of random graphs.

Edmond's characterization of the matching polytope tells us that if G has no vertex of degree at least k then it has a fractional edge colouring using k colours. A theorem of Vizing tells us that in fact such graphs have an edge colouring using k colours. This suggests that in attempting to $\Delta(G)$ fractionally edge colour it is the vertices of maximum degree which are the hot-spots. Intuitively, if they are far apart this may well make finding the desired colouring easy. This intuition is borne out by the following result of Fournier.

Lemma 1.1 (*Fournier [8]*). *A simple graph G whose vertices of maximum degree induce a forest is $\Delta(G)$ edge colourable* (*i.e.* $\chi^e(G)=\Delta(G)$).

Our first result on fractional total colouring has a similar flavour.

Lemma 1.2. *If no two vertices of degree $\Delta(G)$ are joined by a path of length three or less in G and no two vertices of degree at least $\Delta(G)-1$ are joined by an edge then $\chi_f^T(G)=\Delta+1$.*

It turns out that this result shows that $\chi_f^T(G_{n,p})$ is asymptotically almost surely (a.a.s.) $\Delta(G_{n,p})+1$ unless p is very near 0 or very near 1.

To illustrate our approach when p is near 1, we consider the case $p=1$ (ie. when $G_{n,p}=K_n$). We use the fact that, since K_n has no stable set of size 2, the fractional edge colourings of K_{n+1} are in 1–1 correspondence with the fractional total colourings of K_n. (If v^* is the vertex in $V(K_{n+1})-V(K_n)$ then uv^* is an edge of a matching if and only if u is the unique vertex in the corresponding total stable set). Results from the characterization of the matching polytope then imply that for n odd $\chi_f^T(K_n)=\Delta+1$ and for n even $\chi_f^T(K_n)=\Delta+2$.

In the same way, for any graph G, the fractional total colourings of G in which no total stable set contains more than one vertex are in 1–1 correspondence with the fractional edge colourings of the graph G^* obtained from G by adding a vertex v^* adjacent to all of $V(G)$. To characterize $\chi_f^T(G_{n,p})$ when p is near 1 we first rip out some total stable sets and then finish off our colouring by adding an auxiliary vertex v^* and constructing a fractional edge colouring of the resultant auxiliary graph (this approach extends that use by Hilton [11]).

For p near 0, $G_{n,p}$ a.a.s. contains only unicyclic components (ie. each component can be made into a tree by deleting an edge, for proofs on the structure of $G_{n,p}$ that are not provided here see either [3, or 18]). We prove by induction in Section 4 the following lemma:

Lemma 1.3. *If each component of G has at most one cycle then* $\chi^T(G) = \Delta + 1$ *unless* (*a*) $\Delta(G) = 1$ *in which case* $\chi_f^T(G) = \chi^T(G) = \Delta + 2$ *or* (*b*) *it satisfies* $\Delta(G) = 2$ *and contains a cycle of length* $3k + 1$ *or* $3k + 2$ *for some integer k.*

In the next section we state our results on $G_{n,p}$ more precisely and prove them modulo Lemmas 1.2 and 1.3.

2. Characterizing $\chi_f^T(G_{n,p})$

Our analysis considers three regimes: $p \leqslant (1 - \varepsilon)/n$ for some $\varepsilon > 0$, $1/n \leqslant p < 1 - d(n)/n$ for some $d(n)$ such that $d(n)/\log(n) \to \infty$ with n, and $p \geqslant 1 - (c + o(1))\log(n)/n$ for some constant c.

If we break up the first regime, we find that $G_{n,p}$ a.a.s. has no cycles for $p = o(1/n)$ and a.a.s. $\Delta(G_{n,p}) > 3$ for $p = \Omega(n^{-4/3})$. So $\forall p$ a.a.s. $G_{n,p}$ does not satisfy both $\Delta(G_{n,p}) = 2$ and G contains a cycle. We can deduce by Lemma 1.3:

Theorem 2.1. *For* $0 \leqslant p \leqslant (1 - \varepsilon)/n$, *a.a.s.* $\chi_f^T(G_{n,p}) \in \{\Delta + 1, \Delta + 2\}$ *and* $Prob(\chi_f^T(G_{n,p}) = \Delta + 2) = Prob(\Delta(G_{n,p}) = 1) + o(1)$. *This latter probability*:

- *tends to* 0 *if* $p \leqslant o(n^{-2})$ *or* ($p = \omega(n^{-3/2})$ *and* $p \leqslant (1 - \varepsilon)/n$ *for some* $\varepsilon > 0$);
- *tends to* 1 *if* $p = \omega(n^{-2})$ *and* $p = o(n^{-3/2})$;
- *is bounded away from zero and one if* $p = (c + o(1))n^{-2}$ *or* $p = (c + o(1))n^{-3/2}$ *for some constant c*.

As we prove in Section 5, the number of degree Δ or $\Delta - 1$ vertices in $G_{n,p}$ for $1/n \leqslant p \leqslant 1 - d(n)/n$ where $\lim d(n)/\log(n) \to \infty$ (the second regime) is quite small and hence they are sufficiently spread out to allow us to apply Lemma 1.2 to prove:

Theorem 2.2. *For* $1/n \leqslant p \leqslant 1 - d(n)/n$ *where* $\lim d(n)/\log(n) \to \infty$ *a.a.s.* $\chi_f^T(G_{n,p}) = \Delta + 1$.

In the third regime, we generalized the approach shown above for $p = 1$. Here, a.a.s. $\Delta(G_{n,p}) \geqslant n - O(\log^2 n)$, usually we will take $n - \Delta$ disjoint total stable sets $T_1, \ldots, T_{n-\Delta}$ with 2 or 3 vertices (depending on the parity of n) and $\lfloor (n-2)/2 \rfloor$ edges. We then construct an auxiliary graph G^* from the original graph without the edges that appear in a T_i. We add a new vertex v^* adjacent to each vertex of the original graph that isn't in any T_i. Now we can use results from matching theory to look for a $\Delta(G^*)$ fractional edge colouring of this auxiliary graph. If we are successful, for each matching M_j in this fractional edge colouring with non-zero weight w_j and an edge (v^*, y) we create the total stable set $T'_j = M_j - (v^*, y) + y$ and give it weight w_j. If M_j does not contain an edge with v^* as an endpoint we let $T'_j = M_j$ and give it weight w_j. Combining this colouring with the total stable sets $T_1, \ldots, T_{n-\Delta}$ yields a fractional $\Delta + 1$ total colouring.

For n odd, this technique works for all p in this regime. For n even we run into some difficulty when p is near 1 (as above for n even and $p = 1$). In this case we need to consider a "fractional" variant of the above procedure. Using it, we can prove the following lemma which is a fractional analogue of a result of Hilton [11]:

Lemma 2.3. *Let J be a subgraph of* K_{2n} *with e edges whose maximum fractional matching has value j. If* $G = K_{2n} \backslash E(J)$ *has* $\Delta(G) = 2n - 1$ *then* $\chi_f^T(G = K_{2n} \backslash E(J)) = 2n + 1 - (e + j)/n$ *when* $(e + j)/n \leqslant 1$ *and we can compute this value in* $O(n^3)$ *time*.

Using Lemma 2.3 and our technique for large p we can prove the following two theorems that characterize $\chi_f^T(G_{n,p})$ for p close to 1.

Theorem 2.4. *If* $p = 1 - (c + o(1))\log(n)/n$ *and constant* c *where* $\frac{1}{3} \leqslant c < \infty$ *then a.a.s.* $\chi_f^T(G_{n,p}) = \Delta + 1$.

Theorem 2.5. *If* $1 - \log(n)/3n \leqslant p \leqslant 1$ *and* n *is odd then a.a.s.* $\chi_f^T(G_{n,p}) = \Delta + 1$. *If* n *is even then a.a.s.* $\chi_f^T(G_{n,p}) = \Delta + 2 - 2(e + j)/n$ *where* $e = \binom{n}{2} - |E(G_{n,p})|$ *and* j *is the size of the largest fractional matching of* $\overline{G_{n,p}}$.

3. An expected polynomial time algorithm

We present an expected polynomial time algorithm for computing the fractional total chromatic number of a graph (when every simple graph on n vertices is equally likely, ie. $G_{n,\frac{1}{2}}$). Our approach is to attempt to extend a $\lceil n/100 \rceil$ vertex colouring with colour classes $S_1, \ldots, S_{\lceil n/100 \rceil}$ to $\lceil n/100 \rceil$ total stable sets $S_1 \cup M_1, \ldots, S_{\lceil n/100 \rceil} \cup M_{\lceil n/100 \rceil}$ such that $\Delta(G - \bigcup_{i=1}^{\lceil n/100 \rceil} M_i) = \Delta(G) - \lceil n/100 \rceil + 1$ and such that $G - \bigcup_{i=1}^{\lceil n/100 \rceil} M_i$ has a $\Delta(G - \bigcup_{i=1}^{\lceil n/100 \rceil} M_i)$ fractional edge colouring. If this approach works we get a fractional $\Delta + 1$ total colouring. When this approach fails we use an exponential algorithm based on the ellipsoid method. We show the probability that we need to use the slow algorithm is so small that the expected runtime of the combined algorithm is polynomial.

3.1. Polynomial time algorithm for most graphs

We present a polynomial algorithm that shows a graph has a fractional $\Delta + 1$ total colouring. It fails with probability $n^{-(1/8+o(1))n}$ proportion of graphs. This algorithm is based on a procedure used by McDiarmid and Reed [15] to lower bound the proportion of graphs that have a $\Delta + 1$ total colouring.

- Assume $\Delta(G) \geqslant 49n/100$.
- $\lceil n/100 \rceil$ vertex colour G using colour classes $S_1, \ldots, S_{\lceil n/100 \rceil}$ of size at most 200.
- Set $G_1 = G$.
- For $i = \{1, \ldots, \lceil n/100 \rceil\}$ do:
  - Choose a special vertex v_i of $G_i - S_i$ of degree between $47n/100$ and $\Delta(G) - 2$ in G which was not special in any previous iteration.
  - Find a matching $M_i \in G_i - S_i$ hitting every vertex of degree at least $47n/100$ in G except possibly v_i.
  - Set $T_i = M_i \cup S_i$ and $G_{i+1} = G_i - M_i$.
- Fractionally $\Delta(G) - \lceil n/100 \rceil + 1$ edge colour $G_{\lceil n/100 \rceil + 1}$.
- Combining this fractional edge colouring with $T_1, \ldots, T_{\lceil n/100 \rceil}$ yields the desired colouring.

If each step of this algorithm is successful, it is not difficult to see we end up with a valid fractional total colouring. Note, $G_{\lceil n/100 \rceil + 1}$ has maximum degree $\Delta(G) - \lceil n/100 \rceil + 1$ because each vertex v of degree Δ or $\Delta - 1$ in G is hit by all of the M_i except for the i with $v \in S_i$. Similarly if v has degree between $\Delta - 2$ and $\Delta(G) - \lceil n/100 \rceil - 1$ it is hit by every M_i except the i such that $v \in S_i$ and possibly the i where $v_i = v$ (ie. v was chosen as the special vertex). Thus $\Delta(G_{\lceil n/100 \rceil} + 1)$ is at most $\Delta(G) - \lceil n/100 \rceil + 1$ and its maximal degree vertices had degree $\Delta(G)$ or $\Delta(G) - 1$ in G. So in the last step we are looking for a $\Delta(G_{\lceil n/100 \rceil + 1})$ fractional edge colouring of $G_{\lceil n/100 \rceil + 1}$.

This approach will work on almost all graphs. In fact, in [15] it is shown that the proportion of graphs for which any of the following properties fail is at most $n^{-(1/8+o(1))n}$.

(A) The graph has $\Delta \geqslant 49n/100$.
(B) The desired vertex colouring of G exists.
(C) The number of vertices of degree between $\Delta - 2$ and $49n/100$ is at least $2n/100$. So, regardless of our choices so far at each iteration there is a valid choice of special vertex.
(D) For every valid colouring and choice of special vertices and choice of matchings $M_1, \ldots, M_i$, the desired M_{i+1} exists.
(E) For every valid colouring, choice of special vertices, and matchings $M_1, \ldots, M_i$, $G_{\lceil n/100 \rceil + 1}$ has the desired fractional edge colouring.

It also not too hard to show by analyzing a greedy colouring algorithm that the following property:

(B′) A $O(|E(G)|)$ greedy colouring algorithm gives us our desired vertex colouring of G

fails on at most $n^{-(1/8+o(1))n}$ proportion of graphs. The reason this approach differs from that of McDiarmid and Reed is they were bounding the proportion of graphs with a $\Delta + 1$ total colouring not a fractional total colouring. This relaxation allows us to use a fractional edge colouring instead of a edge colouring and hence property (E) is much easier to verify.

We can check for property (A) in $O(|E(G)|)$ time. We can select our special vertex v_i assuming (C) holds in $O(n)$ time by choosing any element of the set. We can also get our desired matchings (D) covering vertices of large degree in polynomial time by finding a maximum matching in the graph obtained from G_i by adding a clique of $|V(G_i)|$ vertices each of which is adjacent to all the vertices of G_i which do not need to be covered (this takes $O(n^3)$ time). We can check that $G_{\lceil n/100\rceil+1}$ has a $\Delta(G_{\lceil n/100\rceil+1})$ fractional edge colouring in $O(n^4)$ time (by an algorithm of Padberg and Rao [17]). We can therefore check if a graph can be $\Delta + 1$ fractional total coloured by this algorithm in $O(n^4)$ time. In fact, with a little extra work this algorithm could be changed so that checking if the algorithm can $\Delta + 1$ colour a graph can be done in linear time (ie. $O(|V(G)| + |E(G)|)$ time) and still work of a large enough proportion of graphs.

3.2. An exponential algorithm

Results of Grötschel et al. [10] relate the problem of finding a maximum weight total stable set to computing the fractional total chromatic number. Using the ellipsoid method their results imply that optimizing over the dual of the fractional total colouring polytope can be done in polynomial time if and only if finding a maximum weight total stable set can done in polynomial time. Finding a maximum size total stable set was shown to be NP-hard by Gavril and Yannakakis [9] and the weighted version is trivially NP-hard since if we assign a weight of 0 to each edge and 1 to each vertex we are looking for a maximum cardinality stable set. Since optimizing over the dual of the fractional total colouring polytope is NP-hard it follows by linear programming duality that optimizing over the fractional total colouring polytope is NP-hard. However, this relation can still be used to find an exponential time algorithm if we have an exponential time algorithm for finding a maximum weight total stable set.

We now present such an algorithm. Given a graph G with weights $w_l, \forall l \in E(G) \cup V(G)$, consider all possible 2^n subsets of $V(G)$, and label them $V_i \subseteq V(G)$ for $1 \leqslant i \leqslant 2^n$. If V_i is a stable set then let G_i be the induced graph on $V(G)\backslash V_i$. We can then use a maximum weight matching algorithm to compute a maximum matching M_i of G_i in $O(n^3)$ time. Let T_i be the total stable set $V_i \cup M_i$. The sum of the weights associated with the elements of T_i is $w_{T_i} = \sum_{l \in T_i} w_l$. Now the T_i with that largest w_{T_i} is clearly a maximum weight total stable set of G. We can therefore solve maximum weight total stable set in $O(2^n n^3)$ time.

Let $\mathscr{T}(G)$ be the set of all total stable sets of G and let Q be the polytope defined by:

$$x_u \geqslant 0, \forall\ u \in E(G) \cup V(G),$$
$$\sum_{u \in T_i} x_u \leqslant 1, \forall\ T_i \in \mathscr{T}(G).$$

Then Q is the polytope of the dual of the LP formulation for the fractional total chromatic number. A strong optimization algorithm over the total stable set polytope (ie. maximum weight total stable set) is in fact a strong separation algorithm over Q. This implies that we have a $O(2^n n^3)$ time algorithm for strong separation over Q. We can therefore optimize the dual of fractional total colouring via ellipsoid method, hence compute the fractional total chromatic number.

The following runtime analysis from [2] will be used to bound the runtime of the ellipsoid method given this $O(2^n n^3)$ strong separation algorithm.

Lemma 3.1. *Suppose we are given a strong separation oracle that takes time T over polytope $P \in \mathbb{R}^d$. Suppose further that polytope P is contained in a sphere of radius R, and if P is non-empty it contains a sphere of radius r. Then the ellipsoid method can optimize a linear function over P in time* $O(d^2 \log(R/r)T + d^4 \log(R/r))$.

Polytope Q can be bounded by a sphere of radius $\sqrt{|E(G)| + |V(G)|}$ and must contain a sphere of radius $1/\sqrt{2^{|E(G)|+|V(G)|}}$. By Lemma 3.1, the ellipsoid method takes $O(|E(G) \cup V(G)|^2 2^n n^3 \log(\sqrt{2^{|E(G)|+|V(G)|}}(|E(G)| + |V(G)|)) = O(2^n n^9)$ time to optimize over Q. We can therefore compute the fractional total chromatic number of any graph in $O(2^n n^9)$ time.

3.3. Polynomial average time fractional total colouring

A direct result of combining the variation of McDiarmid and Reed's algorithm with our $O(2^n n^9)$ algorithm for computing the fractional total chromatic number on the at most $n^{-(1/8+o(1))n}$ proportion of graphs for which McDiarmid and Reed's algorithm fails is a polynomial average time algorithm for computing the fractional total chromatic number.

4. Deterministic proofs

Proof of Lemma 1.3. If $\Delta = 1$ or 2 the lemma is trivially true. To deal with $\Delta \geqslant 3$ we prove that $\chi^T(G) \leqslant \max\{4, \Delta(G)+1\}$. Clearly it is enough to consider connected G. The result is trivial for a graph with one vertex and easy to prove for cycles (indeed it follows from Brooks' Theorem [5]). To prove the result in general we proceed by induction. So, we can assume G is not a cycle or a singleton, and the result holds for all subgraphs of G. By counting degrees we see that G contains a pendant vertex v. Let $e = uv$ be the edge incident to v. We can total colour $G - v$ with $\max\{4, \Delta(G)+1\}$ colours, colour edge e (since at most $\Delta(G)$ colours excluded) then colour v (at most 2 colours excluded, those used on u and e). □

Proof of Lemma 1.2. We construct a special vertex colouring of G using colours $\{1, \ldots, \Delta+1\}$. We then use this vertex colouring to get a fractional total colouring of G. For each vertex v of degree $\Delta(G)$ colour v and $N(v)$ using all $\Delta(G)+1$ colours (ie. each neighbour of v gets a different colour). Since every two vertices of degree Δ are at distance at least 4, no edge has two endpoints with the same colour. We can greedily extend this partial vertex colouring of G to a complete $\Delta(G)+1$ vertex colouring C of G. We use S_i to denote the set of vertices of $V(G)$ assigned colour i by C. It takes $O(|E(G)|)$ time to get our $\Delta+1$ vertex colouring.

For all $i = 1, \ldots, \Delta(G)+1$, define G_i to be the vertex induced subgraph of $V(G)\backslash S_i$. We will bound the chromatic index of G_i, $\forall i$ using Lemma 1.1. For all v with $deg_G(v) = \Delta(G)$ and all i if $v \notin S_i$ then it has a neighbour in S_i so $\forall i$, $\Delta(G_i) \leqslant \Delta(G) - 1$. If $\Delta(G_i) < \Delta(G) - 1$ then G_i has a $\Delta(G) - 1$ edge colouring by the fact that $\chi^e(G) \leqslant \Delta(G)+1$. Otherwise $\Delta(G_i) = \Delta(G) - 1$, and so by the hypothesis, the vertices of degree $\Delta(G) - 1$ in G_i are a stable set. By Lemma 1.1, G_i has a $\Delta(G) - 1$ edge colouring in this case as well. So $\forall i$, $\chi^e(G_i) \leqslant \Delta(G) - 1$. We can use Fournier's algorithm to get the edge colouring of each subgraph.

We are going to combine the $\Delta(G)-1$ edge colourings of the G_i's with the vertex colouring C to get a total fractional colouring of G, using the approach of [12].

Let $\mathscr{M}_i = \{M_{i,1}, \ldots, M_{i,\Delta(G)-1}\}$ be the set of matchings in a $\Delta(G) - 1$ edge colouring of G_i. For $1 \leqslant i \leqslant \Delta(G)+1$ and j between 1 and $\Delta(G) - 1$, we let $T_{i,j}$ be the total stable set $S_i \cup M_{i,j}$ We assign weights of $w_{T_{i,j}} = 1/(\Delta(G) - 1)$ to each $T_{i,j}$ and a weight of zero to all the other total stable sets of G (this takes $O((\Delta+1)(\Delta-1))$ time). We now claim that w is a feasible solution to the fractional total colouring of G and $\sum_{T\in\mathscr{T}(G)} w_T = \Delta(G)+1$.

To prove that the inequalities of the fractional total LP are satisfied, we need to show that every element of $V(G)\cup E(G)$ is in $\Delta(G) - 1$ of the $T_{i,j}$'s. This is clear for $v \in V(G)$ as v is in some S_i and hence in $T_{i,j}$ for $1 \leqslant j \leqslant \Delta(G) - 1$. For each $e \in E(G)$ with one end in S_k and the other in S_l, for all $1 \leqslant i \leqslant \Delta+1$ with $i \notin S_k \cup S_l$, there is some j such that $e \in M_{i,j}$ and hence in $T_{i,j}$ so e is in at least $\Delta - 1$ of the $T_{i,j}$'s.

The second part of the claim obviously holds, as our objective function has the following value:

$$\sum_{T\in\mathscr{T}(G)} w_T = \sum_{i=1}^{\Delta(G)+1} \sum_{M_{i,j}\in\mathscr{M}_i} (1/(\Delta(G)-1))$$
$$= \sum_{i=1}^{\Delta(G)+1} 1 = \Delta(G)+1.$$

We have shown that we have a $\Delta(G)+1$ fractional total colouring of G as required. The algorithm runs in $O(n^5)$ time. □

Proof of Lemma 2.3. Let $\varepsilon = 1 - (e+j)/n$.

Necessity: (ie. $\chi_f^T(G) \geqslant 2n + \varepsilon$ then $\varepsilon \geqslant 1 - (e+j)/n$).

This part of the proof follows a proof of Hilton's [11] that deals with the total colour of such large degree graphs. We make necessary modifications to his proof to deal with fractional total colouring.

Suppose $G = K_{2n} \backslash E(J)$ has a fractional $2n + \varepsilon$ total colouring. We can assume that $\forall v \in V(G) \sum_{T \ni v} w_T = 1$. This follows since if $\sum_{T \ni v} w_T = b > 1$ for some $v \in V(G)$, we can assume removing v from some non-zero weight total stable set will drop $\sum_{T \ni v} w_T$ below 1 otherwise we would do so, we could now split a total stable set T_i of weight $w_i > 0$ into two total stable sets $T_i' = T_i$ and $T_i'' = T_i - \{v\}$ of weights $w_i' = w_i - (b-1)$ and $w_i'' = b - 1$, respectively, by repeating this process for each vertex we get a $2n + \varepsilon$ fractional total colouring where $\forall v \in V(G)$, $\sum_{T \ni v} w_T = 1$. Let $T_1, \ldots, T_l$ be those total stable sets with non-zero weights $w_1, \ldots, w_l$ in the $2n + \varepsilon$ fractional total colouring which contain at least one vertex. Let S_i be $T_i \cap V(G)$ and M_i be $T_i \cap E(G)$. Let $x_i = |S_i|$ then,

$$w_1 x_1 + w_2 x_2 + \cdots + w_l x_l = 2n.$$

There is a matching of J, whose vertices are in S_i, of size $\lfloor x_i/2 \rfloor$. Since j is the size of a largest fractional matching of J we see

$$w_1 \left\lfloor \frac{x_1}{2} \right\rfloor + w_2 \left\lfloor \frac{x_2}{2} \right\rfloor + \cdots + w_l \left\lfloor \frac{x_l}{2} \right\rfloor \leqslant j. \tag{1}$$

Let $z = \sum_{\{i : x_i \text{ is odd}\}} w_i$. Then it follows that

$$w_1 x_1 + w_2 x_2 + \cdots + w_l x_l \leqslant 2j + z.$$

Therefore,

$$z \geqslant 2n - 2j. \tag{2}$$

Call a pair (T_i, v) a *vertex total stable set pair* if either $v \in S_i$ or v is an endpoint of an edge of M_i. We associate the weight w_i of total stable set S_i with each (S_i, v) pair. We will consider the sum of the weights of all vertex total stable set pairs.

A total stable set with an odd number of vertices is in at most $2n - 1$ vertex total stable set pairs. A total stable set with an even number of vertices is in at most $2n$ vertex total stable set pairs. It follows from (2) that the sum of the weights of all vertex total stable set pairs is at most,

$$\begin{aligned}(2n-1)z + (2n)(2n + \varepsilon - z) &= 4n^2 + 2n\varepsilon - z \\ &\leqslant 4n^2 - 2n + 2j + 2n\varepsilon. \end{aligned} \tag{3}$$

Since the sum of the weights of all vertex total stable sets containing a vertex v is $1 + deg(v)$, it follows that the sum of the weights of all vertex total stable sets is

$$\begin{aligned} 2n + \sum_{v \in V(G)} deg(v) &= 2n + 2|E(G)| \\ &= (2n)^2 - 2e'. \end{aligned} \tag{4}$$

Combining Eq. (3) with Eq. (4) we get that,

$$4n^2 - 2e' \leqslant 4n^2 - 2n + 2j + 2n\varepsilon. \tag{5}$$

Rearranging gives us our desired result that $\varepsilon \geqslant 1 - (e' + j)/n$.

Sufficiency: ie. if $\varepsilon = 1 - e + j/n$ then $\chi_f^T(G) \leqslant 2n + \varepsilon$.

Let R be the set of edges in a maximal fractional matching of J which is half integral made up of a matching of weight 1 edges and disjoint odd cycles with weight $\frac{1}{2}$ on the edges (one always exists by a result of Balinski [1]) and let $r = |R|$. We enumerate R as $\{e_1 = (x_1, y_1), \ldots, e_r = (x_r, y_r)\}$ and let $w_i \in \{0, \frac{1}{2}, 1\}$ be the weight of e_i in our optimal fractional matching. We then find disjoint matchings $M_1, \ldots, M_r$ in G such that M_i is a perfect matching in $G_i = G - x_i - y_i - \bigcup_{j<i} M_j$. We have omitted the proof here but we show that it is always possible to find these perfect matchings.

Having constructed $M_1, \ldots, M_r$ we define total stable sets $T_1, \ldots, T_r$ where $T_i = M_i \cup \{x_i, y_i\}$. We give T_i the same weight, w_i, that e_i has in our optimal fractional matching of J. Since our fractional matching was half-integral and every edge with weight $\frac{1}{2}$ was in an odd cycle, it follows that for any vertex $v \in V(G)$, $\sum_{T_i \ni v} w_i \in \{0, 1\}$. So every vertex is either completely covered or not covered yet at all.

We construct an auxiliary graph G' from G by adding a vertex v^* adjacent to all of $V(G) - \cup_{e_i \in R}\{x_i, y_i\}$. We weight the edges of G' as follows:

$$\begin{array}{ll} \forall\ e' = v^*v \in G' & f(e') = 1. \\ \forall\ e' \in E(G) - \cup_{i=1}^{r} M_i & f(e') = 1. \\ \forall\ e' \in M_i & f(e') = 1 - w_i \end{array}$$

We claim that we can find matchings $\{N_1, \ldots, N_l\}$ of G' and weights $\{z_1, \ldots, z_l\}$ s.t. $\sum z_i = 2n - j + \varepsilon$ and $\forall e \in G'$, $\sum_{\{i: e' \in N_i\}} z_i = f(e')$.

Having done so, for $1 \leqslant i \leqslant l$ we define a total stable set N_i' of G as follows:

- If $v^* \notin V(N_i)$ then $N_i' = N_i$.
- If $\exists\ u$ s.t. $v^*u \in N_i$ then $N_i' = N_i - v^*u + u$.

It is an easy matter to verify that giving T_i weight w_i and N_i' weight z_i and all other total stable sets weight 0 yields a fractional $2n + \varepsilon$ total colouring of G. So it remains only to prove our claim.

By the characterization of the matching polytope [6], we know that if the claim does not hold then either:

(i) $\exists\ v \in V(G')$ s.t. $\sum_{v \in e'} f(e') > 2n - j + \varepsilon$, or
(ii) $\exists\ H \subseteq G'$, $|V(H)|$ odd s.t. $\sum_{e' \in E(H)} f(e') > (2n - j + \varepsilon)(|V(H)| - 1)/2$.

Because, each M_i is a perfect matching of $G - x_i - y_i$, $\forall\ v \in V(G)$, $\sum_{\{x \in E(G'), x \ni v\}} f(x) = deg_G(v) + 1 - j \leqslant 2n - j$. Clearly $\sum_{x \ni v^*} f(x) = 2n - 2j$. So (i) does not hold.

It remains to show that $\forall\ H \subseteq G$, where $|V(H)| \geqslant 3$ and $|V(H)|$ is odd the following inequality holds $\sum_{x \in E(H)} f(x) \leqslant (2n - j + \varepsilon)(|V(H)| - 1)/2$.

Assume that $H = G'$, then we get,

$$\begin{aligned} \sum_{x \in E(H)} f(x) &= |E(G)| - (n-1)j + (2n - 2j) \\ &= (2n(2n-1)/2 - e') - (n-1)j + (2n - 2j) \\ &= 2n^2 - (e' + j) - jn + n. \end{aligned}$$

But, the right-hand side of (ii) is, $(2n - j + \varepsilon)(|V(H)| - 1)/2 = 2n^2 - jn + n\varepsilon$ and since $(e' + j) \geqslant n - n\varepsilon$ it follows that (ii) does not hold for $H = G'$.

Now consider $H \subsetneq G'$ and assume property (ii) holds. A simple property of a subgraph satisfying (ii) is that the sum of the weights on edges leaving H is at most $(2n - j + \varepsilon) - 2$. We are going to abuse notation a little and let $deg'(v) = \sum_{x \ni v} f(x)$ and $\Delta'_{\min}(G') = \min\{deg'(v) : \forall\ v \in V(G')\}$.

For v^*, each x around it has $f(x) = 1$, so $deg'(v^*) = deg(v^*) = 2n - 2j$. For $v \in V(G)$, $deg'(v) = \sum_{x \ni v} f(x) = deg_G(v) - j + 1$.

So $\Delta'_{\min}(G') = \min\{2n - 2j, \delta(G) - j + 1\}$. Since $j \leqslant e'$ and $\delta(G) \geqslant 2n - 1 - e'$ it follows that $\Delta'_{\min}(G') \geqslant 2n - e' - j$.

Let B be the of vertices in G' not in H and let $b = |B|$, (i.e. $b = |V(G') \backslash V(H)|$). Then for H to satisfy (ii) the sum of the weights on edges leaving H going to B must be at most $(2n - j + \varepsilon) - 2$. This implies that the sum on edges leaving B is at most $(2n - j + \varepsilon) - 2$. Assume that the vertices of B form a clique, and each edge x in this clique has $f(x) = 1$ (this minimizes the possible sum of weights on edges H). The sum on edges leaving B must be at least $b\delta'(G') - b(b-1)$. But since $2 \leqslant b \leqslant j \leqslant \frac{n}{2}$ it easy to verify that,

$$b\delta'(G') - b(b-1) > (2n - j + \varepsilon) - 2. \tag{6}$$

This implies that no such H exists such that (ii) holds. This implies we have our matchings $\{N_1, \dots, N_l\}$ with weights $\{z_1, \dots, z_l\}$ as required.

It is shown in [4] how to find a half-integral maximal fractional matching in $O(n^3)$ time, it follows that we can compute $\chi_f^T(K_{2n} \backslash E(J))$ in $O(n^3)$ time. □

The following lemma will be useful in proving Theorem 2.4.

Lemma 4.1. *Every graph G with $n = |V(G)|$ even and $\delta(G) \geqslant n/2$ has a perfect matching.*

Proof. Let M be a maximum matching of G and suppose there exists u and v unmatched by M. Then $N(v)$ and $N(u)$ are contained in $V(M)$ by the maximality of M. Furthermore, for each edge e of M, $|N(v) \cap e| + |N(u) \cap e| \leqslant 2$ by the maximality of M. So $|N(v)| + |N(u)| \leqslant 2|M| < n$, a contradiction. Hence, since $|V(G)|$ is even M is perfect. □

Proof of Theorem 2.4. To prove this theorem we will assume that some structural properties hold for $G_{n,p}$. Namely, a.a.s. $G_{n,p}$ has at least $n - \Delta(G_{n,p})$ vertex independent stable sets of size 3, the minimum degree $\delta(G_{n,p}) \geqslant 3n/2 - \Delta(G_{n,p}) + 2$, the maximum degree $\Delta(G_{n,p}) \geqslant n - (c-1)\log(n)$, and lastly $|E(G_{n,p})| \leqslant \binom{n}{2} - (n-1)(c + o(1))\log(n)/2 + n$. It is not difficult (but often tedious) to prove using standard concentration bounds that these structural properties a.a.s. hold for p is this range (these properties are proved in full in [16]). Assume G is some graph satisfying these structural properties.

If n is odd, consider the $a = n - \Delta(G)$ vertex disjoint stable sets of size 3, call them $S_1, \dots, S_a$ and $S_i \cap S_j = \emptyset$ for $i \neq j$.

Consider a perfect matching, M_1 of $G \backslash S_1$. Since $\delta(G) \geqslant n/2 + 3$ we know that $G \backslash S_1$ has a perfect matching by Lemma 4.1. Define M_i to be a perfect matching of G_i where $V(G_i) = (V(G) \backslash S_i)$ and $E(G_i) = E(G \backslash S_i) \backslash (M_1 \cup M_2 \cup \dots \cup M_{i-1})$. G_i has a perfect matching M_i by Lemma 4.1, the fact that $|V(G_i)|$ is even and $\delta(G_i) \geqslant \delta(G) - 3 - (i-1) > |V(G_i)|/2$. Let T_i be the total stable set $S_i \cup M_i$.

Let $V' = S_1 \cup S_2 \cup \dots \cup S_a$ and $E' = M_1 \cup M_2 \cup \dots \cup M_a$. Consider the auxiliary graph H where $V(H) = V(G)$ plus a new vertex v^* and $E(H) = E(G) \backslash E'$ plus edges $\{(u, v^*) : \forall u \in V(G) \backslash V'\}$.

Now, $\Delta(H) = \Delta(G) - a + 1$ since $\forall v \in V'$, $deg_H(v) = deg_G(v) - a + 1$ this follows since we have removed one edge adjacent to v in each M_j except one. For all $v \in V(G) \backslash V'$, $deg_H(v) = deg_G(v) - a + 1$ since we have removed a edges incident to v and added one adjacent to v^*. Finally $deg(v^*) < \Delta(G) - 2a + 1$ since $deg(v^*) = n - 3a$. So $\Delta(H) = \Delta(G) - a + 1 = n - 2a + 1$. Also, $\delta(H) = \min\{deg_H(v^*), \delta(G) - a + 1\} \geqslant n/2 + 1$. We want to show that H has a $\Delta(H)$ fractional edge colouring. Results of Edmond's [6] imply that H has a $\Delta(H)$ fractional edge colouring if and only if H doesn't contain an odd overfull subgraph (an induced subgraph $H' \subseteq H$ such that $|E(H')| > \Delta(H)\frac{|V(H')|-1}{2}$ where $|V(H')|$ is odd).

To get a contradiction assume $H' \subset H$ is an odd overfull subgraph. By definition this means,

$$
\begin{aligned}
|E(H')| &> \Delta(H)\frac{(|V(H')| - 1)}{2} \\
&= (n - 2a + 1)\frac{(|V(H')| - 1)}{2}.
\end{aligned}
$$

This implies that $|V(H')| > \Delta(H) = n - 2a + 1$. Let $B = V(H) \backslash V(H')$, then $|B| = n + 1 - |V(H')| < 2a$. Therefore,

$$|E(H')| > (n - 2a + 1)(n - |B|)/2. \tag{7}$$

Since, $|E(H)| = |E(G)| - (n-3)a/2 + n - 3a$, we get

$$|E(H')| \leqslant |E(G)| - (n-3)a/2 + n - 3a - |B|\delta(H) + |B|(|B| - 1)/2. \tag{8}$$

Combining inequalities (7) and (8) we find that if

$$|E(G)| \leqslant \binom{n}{2} - nB/2 - an/2 + aB + 3a/2 + B\delta(H) - B^2/2 \tag{9}$$

no overfull subgraph can exists. Substituting in the bound on $\delta(H)$ we get that no overfull subgraph exists if

$$|E(G)| \leqslant \binom{n}{2} - an/2 + aB + 3a/2 + B - B^2/2. \tag{10}$$

Since $|B| < 2a$ it follows that if

$$|E(G)| \leqslant \binom{n}{2} - an/2 + 3a/2 \tag{11}$$

no overfull subgraph exists. Since $|E(G)| \leqslant \binom{n}{2} - (n-1)(c + \mathrm{o}(1)) \log(n)/2 + n$ and $a \leqslant (c-1) \log(n)$ it follows that inequality (11) holds for all n at least as large as some n_c and no overfull subgraph H' exists. This means that H has a $\Delta(H)$ fractional edge colouring.

We can now show we have a fractional $\Delta(G) + 1$ total colouring of G in a similar fashion as we did in the proof of Lemma 2.3. Give each total stable set T_i a weight of 1. Now every edge and vertex in T_i is covered. For all matchings M_j of H where M_j has weight w_j in our $\Delta(H)$ fractional edge colouring, if $(v^*, u) \in M_j$ for some vertex u then set total stable set M'_j of G to be $M_j - (v^*, u) \cup \{u\}$. Otherwise set $M'_j = M_j$. Assign M'_j the weight w_j. Now, the total stable sets $\forall i\ T_i$ and $\forall j\ M'_j$ together with their weights form a $\Delta + 1$ fractional total colouring of G.

For n even, the proof is basically the same we just date $n - \Delta$ vertex disjoint stable sets of size 2 with perfect matchings of the rest of the graph in the beginning. □

Proof of Theorem 2.5. For p in this range a.a.s. $\Delta(G_{n,p}) = n - 1$. As discussed in Section 1 the total colourings of K_n can be mapped $1-1$ to edge colourings of K_{n+1}. For n odd K_{n+1} has a $\Delta(K_{n+1}) = n$ edge colouring which implies K_n has a $\Delta(K_n) + 1 = n$ total colouring. Since $G_{n,p}$ is a subgraph of K_n and a.a.s. $\Delta(G_{n,p}) = n - 1$ it follows that a.a.s. $\chi_f^T(G_{n,p}) = \Delta(G_{n,p}) + 1 = n$.

For n even, we use Lemma 2.3 to characterize the value of the fractional total chromatic number. □

Remark. We do not show it here but the proofs in this section can easily be converted to efficient algorithms that a.a.s. compute an optimal fractional total colouring of $G_{n,p}$ in polynomial time (see [16]).

5. Probabilistic proofs

Let A be the random variable counting the number of pairs of vertices such that either both have maximum degree and have a path of length $\leqslant 3$ between them, or an edge exists between the pair and both have degree $\geqslant \Delta(G_{n,p}) - 1$. Our interest in A is explained in the second regime. We also let the random variable X_k count the number of vertices in $G_{n,p}$ of degree k.

Lemma 5.1. *If* $1/n \leqslant p = \mathrm{o}(\log(n))/n$ *then* $\Pr(A \neq 0) = \mathrm{o}(1)$.

Proof. In [3] Bollobás proved that if $1/n \leqslant p = \mathrm{o}(\log(n)/n)$ then there is some $k = k(n)$ which is between $\log(n)/\log\log(n)$ and $\log(n)$ such that a.a.s. $\Delta(G_{n,p}) = k$ and furthermore $\mathrm{Exp}[X_{k+1}] = \mathrm{o}(1)$. Now,

$$\mathrm{Exp}[X_{k+1}] = n \binom{n-1}{k+1} p^{k+1}(1-p)^{n-2-k}$$

and

$$\mathrm{Exp}[X_k] = n \binom{n-1}{k} p^k (1-p)^{n-1-k}.$$

So, $\mathrm{Exp}[X_k] = (k+1)p^{-1}(1-p)/(n-k-2)\mathrm{Exp}[X_{k+1}]$. Thus since $\mathrm{Exp}[X_{k+1}] = \mathrm{o}(1)$ and $k = \mathrm{O}(\log(n))$ we know $\mathrm{Exp}[X_k] = \mathrm{O}(\log(n))$.

Since $\mathrm{Exp}[X_k] = n \binom{n-1}{k} p^k (1-p)^{n-1-k} = \mathrm{O}(\log(n))$ it follows that

$$\binom{n-1}{k} p^k (1-p)^{n-1-k} = \mathrm{O}\left(\frac{\log(n)}{n}\right). \tag{12}$$

The probability $G_{n,p}$ has two vertices of degree Δ with an edge between them is given by,

$$\binom{n}{2}\Pr(deg(u)=deg(v)=\Delta \text{ and } (u,v)\in E(G_{n,p}))$$

$$=\binom{n}{2}\left(\binom{n-2}{k-1}p^{k-1}(1-p)^{n-1-k}\right)^2 p$$

$$=\binom{n}{2}\mathrm{O}(\log^2(n)/n)^2 p \tag{13}$$

$$=\mathrm{o}(1). \tag{14}$$

Eq. (13) follows from (12), since $\binom{n-2}{k-1}=k/(n-1)\binom{n-1}{k}$.

Similarly we can bound the probability that a vertex of degree Δ is adjacent to a vertex of degree $\Delta-1$ or a vertex of degree $\Delta-1$ is adjacent to a vertex of degree $\Delta-1$ by

$$\binom{n}{2}p\left(\left(\binom{n-2}{k-1}p^{k-1}(1-p)^{n-1-k}\right)\left(\binom{n-2}{k-2}p^{k-2}(1-p)^{n-k}\right)\right.$$
$$\left.+\left(\binom{n-2}{k-2}p^{k-2}(1-p)^{n-k}\right)^2\right)$$

$$=\binom{n}{2}p(\mathrm{O}(\log^5(n)/n^2)+\mathrm{O}(\log^6(n)/n^2)) \tag{15}$$

$$=\mathrm{o}(1). \tag{16}$$

The probability that we have a path of length two between vertices of degree Δ can be bounded by

$$\binom{n}{3}p^2\left(\binom{n-3}{k-1}p^{k-1}(1-p)^{n-2-k}\right)^2=\binom{n}{3}p^2(\mathrm{O}(\log^2/n)^2) \tag{17}$$

$$=\mathrm{o}(1). \tag{18}$$

The probability that we have a path of length three between vertices of degree Δ can be bounded by

$$\binom{n}{4}p^3\left(\binom{n-4}{k-1}p^{k-1}(1-p)^{n-3-k}\right)^2=\binom{n}{4}p^3(\mathrm{O}(\log^2/n)^2) \tag{19}$$

$$=\mathrm{o}(1). \tag{20}$$

Combining Eqs. (14), (16), (18) and (20) we get $\Pr(A\neq 0)=\mathrm{o}(1)$. □

Lemma 5.2. *If* $p=(c+\mathrm{o}(1))\log(n)/n$ *for fixed* c *and* $0<c<\infty$ *then* $\Pr(A\neq 0)=\mathrm{o}(1)$.

We will need the following lemma to prove Lemma 5.2.

Lemma 5.3. *For* $p=(c+\mathrm{o}(1))\log(n)/n$ *for some constant* c *with* $0<c<\infty$, $G_{n,p}$ *a.a.s. has* $(c/5)\log(n)\leqslant\Delta(G_{n,p})\leqslant 3ec\log(n)$.

Proof. Standard concentration bounds on the binomial distribution $(Bin(\binom{n}{2},p))$ tell us that a.a.s. $G_{n,p}$ has $(c+\mathrm{o}(1))n\log(n)(1-\mathrm{o}(1))$ edges which yields our lower bound.

To prove the upper bound let Y be the number of $\lceil 3ec\log(n)\rceil$ sets of edges around a fixed vertex v. Then $\mathrm{Exp}[Y]=\binom{n-1}{\lceil 3ec\log(n)\rceil}p^{\lceil 3ec\log(n)\rceil}$. If Z is the number of vertices of degree at least $\lceil 3ec\log(n)\rceil$, we get,

$$\mathrm{Exp}[Z]\leqslant n\,\mathrm{Exp}[Y]$$
$$=\mathrm{o}(1).$$

Since Z is integer valued, it follows that

$$\Pr(Z>0)\leqslant \mathrm{Exp}[Z] \\ = \mathrm{o}(1).$$

So a.a.s. $\Delta(G_{n,p})\leqslant 3ec\ \log(n)$.

Proof of Lemma 5.2. Bollobás proved in [3] that for $p=(c+\mathrm{o}(1))\ \log(n)/n$, where c is a constant, a.a.s. $\Delta(G_{n,p})$ cannot be confined to a finite set of values. By Lemma 5.3 we know a.a.s. $(c/5)\log(n)\leqslant\Delta(G_{n,p})\leqslant 3ec\ \log(n)$. Let j be the largest value such that $\lim\ \mathrm{Exp}[X_j]=\infty$, we know from Lemma 5.3 that $j\geqslant(c/5)\log(n)$. So for $k>j$, we get $\lim\ \mathrm{Exp}[X_k]=C$ where $0\leqslant C<\infty$. Now, $\mathrm{Exp}[X_j]=\mathrm{Exp}[X_{j+1}](j+1)p^{-1}(1-p)/(n-1-j)$ since $\lim\ \mathrm{Exp}[X_{j+1}]=C$ and $j=\mathrm{O}(\log(n))$ it follows that $\mathrm{Exp}[X_j]=\mathrm{O}(\log(n))$. So for $k\geqslant j$ we have

$$\binom{n-1}{k}p^k(1-p)^{n-1-k}=\mathrm{O}(\log(n)/n).$$

Now we can bound the probability that two vertices have degree Δ and are adjacent by Eq. (13) summed over $\mathrm{O}(\log(n))$ possible values of k.

$$\mathrm{O}(\log(n))\binom{n}{2}\mathrm{O}(\log^2(n)/n)^2p=\mathrm{o}(1).$$

Similarly we can sum Eqs. (15), (17) and (19) over the $\mathrm{O}(\log(n))$ possible values of Δ to get, $\Pr(A\neq 0)=\mathrm{o}(1)$. □

Lemma 5.4. *For* $pn/\log(n)\to\infty$ *and* $(1-p)n/\log(n)\to\infty$ *a.a.s.* $\Pr(A\neq 0)=\mathrm{o}(1)$.

To prove this lemma we use the following theorem from [3]:

Theorem 5.5 (*Bollobás [3]*). *Given a labeling of vertices* $x_1,x_2,\ldots,x_n\in V(G_{n,p})$ *and, respectively, corresponding degrees* $d_1,d_2,\ldots,d_n$, *such that* $d_1\geqslant d_2\geqslant\cdots\geqslant d_n$. *If* $m=\mathrm{o}(p(1-p)n/\log n)^{1/4}$, $m\to\infty$, *and* $\alpha(n)\to 0$ *then a.a.s.*

$$d_i-d_{i+1}\geqslant\frac{\alpha(n)}{m^2}\left(\frac{p(1-p)n}{\log n}\right)^{1/2}\quad\forall i<m$$

Proof of Lemma 5.4. Let $m=(p(1-p)n/\log(n))^{1/8}$, since p satisfies both $np/\log(n)\to\infty$ and $(1-p)n/\log(n)\to\infty$ it follows that $m\to\infty$. Let $\alpha(n)=2m^2(\log(n)/p(1-p)n)^{1/2}$, then it is easy to see that $\alpha(n)\to 0$. We can now use the Theorem 5.5 to show that asymptotically almost surely,

$$d_i-d_{i+1}\geqslant\frac{2m^2(\log(n)/p(1-p)n)^{1/2}}{m^2}\left(\frac{p(1-p)n}{\log(n)}\right)^{1/2}\quad\forall i<m.$$

Simplifying we get that,

$$d_i\geqslant d_{i+1}+2\quad\forall i<m.$$

In particular this gives us our desired result that $d_1\geqslant d_2+2$. Therefore if $v^*\in V(G_{n,p})$ has degree $\Delta(G_{n,p})$ then $\forall u\in V(G_{n,p})-v^*$, a.a.s. $deg(u)\leqslant\Delta(G_{n,p})-2$. It follows that $\Pr(A\neq 0)=\mathrm{o}(1)$. □

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