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Fractional coloring and the odd Hadwiger's conjecture

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Abstract

Gerards and Seymour (see [T.R. Jensen, B. Toft, Graph Coloring Problems, Wiley-Interscience, 1995], page 115) conjectured that if a graph has no odd complete minor of order p, then it is $(p-1)$-colorable. This is an analogue of the well known conjecture of Hadwiger, and in fact, this would immediately imply Hadwiger's conjecture. The current best known bound for the chromatic number of graphs without an odd complete minor of order p is $O(p\sqrt{\log p})$ by the recent result by Geelen et al. [J. Geelen, B. Gerards, B. Reed, P. Seymour, A. Vetta, On the odd variant of Hadwiger's conjecture (submitted for publication)], and by Kawarabayashi [K. Kawarabayashi, Note on coloring graphs without odd K_k-minors (submitted for publication)] (but later). But, it seems very hard to improve this bound since this would also improve the current best known bound for the chromatic number of graphs without a complete minor of order p.

Motivated by this problem, we prove that the "fractional chromatic number" of a graph G without odd K_p-minor is at most $2p$; that is, it is possible to assign a rational $q(S) \geq 0$ to every stable set $S \subseteq V(G)$ so that $\sum_{S \ni v} q(S) = 1$ for every vertex v, and $\sum_S q(S) \leq 2p$.

This generalizes the result of Reed and Seymour [B. Reed, P.D. Seymour, Fractional chromatic number and Hadwiger's conjecture, J. Combin. Theory Ser. B 74 (1998) 147–152] who proved that the fractional chromatic number of a graph with no K_{p+1}-minor is at most $2p$.

1. Hadwiger's conjecture and the odd Hadwiger's conjecture

Hadwiger's conjecture from 1943 suggests a far-reaching generalization of the Four Color Theorem [1,2,17] and is considered by many as the deepest open problems in graph theory. Hadwiger's conjecture states the following.

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Conjecture 1.1. *For all $p \geq 1$, every p-chromatic graph has a K_p minor.*

Conjecture 1.1 is trivially true for $p \leq 3$, and reasonably easy for $p = 4$, as shown by Dirac [4] and Hadwiger himself [5]. However, for $p \geq 5$, Conjecture 1.1 implies the Four Color Theorem. In 1937, Wagner [20] proved that the case $k = 5$ of Conjecture 1.1 is, in fact, equivalent to the Four Color Theorem. In 1993, Robertson, Seymour and Thomas [16] proved that a minimal counterexample to the case $p = 6$ is a graph G which has a vertex v such that $G - v$ is planar. By the Four Color Theorem, this implies Conjecture 1.1 for $p = 6$. Hence the cases $p = 5, 6$ are each equivalent to the Four Color Theorem [1,2,17]. Conjecture 1.1 is open for $p \geq 7$. For the case $p = 7$, Toft and the first author [12] proved that any 7-chromatic graph has K_7 or $K_{4,4}$ as a minor. Recently, the first author [9] proved that any 7-chromatic graph has K_7 or $K_{3,5}$ as a minor.

It is not known if there exists an absolute constant c such that any cp-chromatic graph has a K_p-minor. So far, it is known that there exists a constant c such that any $cp\sqrt{\log p}$-chromatic graph has a K_p-minor. This follows from the results of Kostochka [13,14] or Thomason [18,19].

For that reason it would be of great interest to decide whether there is a constant C so that every graph with no K_p minor is Cp-colorable. This is still open, but it was proved in [15] that fractional coloring exists for $C = 2$. Let us give the formal definition of the fractional chromatic number.

Let $k \geq 0$ be a rational. A *fractional k-coloring* of a graph G means a map $q : \mathcal{S} \to \mathbf{Q}_+$ (where $\mathbf{Q}_+$ is the set of non-negative rationals, and $\mathcal{S}$ is the set of all stable subsets of $V(G)$) such that

1. for every vertex v, $\sum(q(S) : S \in \mathcal{S} \text{ and } v \in S) = 1$
2. $\sum(q(S) : S \in \mathcal{S}) \leq k$.

Thus G is k-colorable, where k is an integer, if and only if it has a fractional k-coloring q which is (0, 1)-valued. Consequently, Hadwiger's conjecture implies that every graph with no K_{p+1} minor has a fractional p-coloring; but this too remains open. The following is the main result in [15].

Theorem 1.2 (*Reed and Seymour [15]*). *For every integer $p \geq 1$, every graph with no K_p minor has a fractional $2p$-coloring.*

Recently, the concept "odd minor" has been considered by many researchers.

Let us give the definition of odd minor. We say that H has an *odd complete minor* of size at least p if there are p vertex disjoint trees in H such that every two of them are joined by an edge, and in addition, all the vertices of trees are two-colored in such a way that the edges within the trees are bichromatic, but the edges between trees are monochromatic. We say that H has an *odd K_l-minor*, if H has an odd complete minor of size at least l. It is easy to see that any graph which has an odd K_p-minor certainly contains K_p as a minor.

Gerards and Seymour (see [8], page 115) conjectured the following.

Conjecture 1.3. *For all $p \geq 1$, every graph with no odd K_{p+1}-minor is p-colorable.*

This is an analogue of Conjecture 1.1. In fact, it is easy to see that Conjecture 1.3 is strictly stronger than Conjecture 1.1. Again, Conjecture 1.3 is trivially true when $l = 1, 2$. In fact, when $p = 2$, this means that if a graph has no odd cycles, then it is 2-colorable. This is easy since such a graph must be bipartite. The $p = 3$ case was proved by Catlin [3]. Recently, Guenin [7] announced a solution of the $l = 4$ case. This result would imply the Four Color Theorem because a graph having an odd K_5-minor certainly contains a K_5-minor. Conjecture 1.3 is open for $p \geq 5$.

Recently, Geelen et al. [6] gave a structural theorem with graphs without an odd K_p-minor, but having a K_{16p}-minor. This result is enough to prove that there exists a constant c such that any graph with no odd K_p-minors is $cp\sqrt{\log p}$-colorable. Shortly after that, a shorter and simpler proof without using the structural theorem is obtained by the first author [10]. This is an analogue of the results of Kostochka [13,14] or Thomason [18,19]. In fact, this immediately implies the result of the results of Kostochka [13,14] or Thomason [18,19]. But it seems that improving the chromatic number of graphs with no odd K_p-minors is very hard, since we do not even know if there exists a constant c such that any cp-chromatic graph contains K_p as a minor.

Motivated by this problem and Theorem 1.2, we prove the following result.

Theorem 1.4. *For every integer $p \geq 1$, every graph with no odd K_p minor has a fractional $2p$-coloring.*

From linear programming duality, to prove Theorem 1.4, it suffices to prove the following:

Theorem 1.5. *Let G have no odd K_p minor, and let $w : V(G) \to \mathbf{Q}_+$ be some map. There is a stable set $X \subseteq V(G)$ such that $\sum_{v \in X} w(v) \geq \frac{1}{2p} \sum_{v \in V(G)} w(v)$.*

For $w \equiv 1$ this was already proved ; indeed, this was generalized by Kawarabayashi and Song [11] who proved that every graph with no odd K_{p+1}-minor has a stable set of size at least $|V(G)|/2p$. This motivated the present paper.

Our approach is very similar to that in [15]. A graph is *chordal* if it has no induced circuit of length ≥ 4. It is elementary that the chromatic number of a chordal graph H equals the maximum size of a complete subgraph of H. If G is a graph and $X \subseteq V(G)$, we denote by $G|X$ the subgraph induced on X, that is, the subgraph obtained by deleting $V(G)-X$. If $X, Y \subseteq V(G)$ are disjoint, we say that X, Y *touch* (in G) if some vertex in X is adjacent in G to some vertex in Y. If $\mathcal{P} = \{X_1, \ldots, X_n\}$ is a partition of $V(G)$, its *touching pattern* $H(\mathcal{P})$ is the graph with vertex set $\mathcal{P}$ in which X_i, X_j are adjacent if $i \neq j$ and they touch.

A *chordal decomposition* of G means a partition $\mathcal{P}$ of $V(G)$ such that

1. $G|X$ is non-null and connected for each $X \in \mathcal{P}$, and
2. $H(\mathcal{P})$ is chordal.

We call the subgraphs $G|X (X \in \mathcal{P})$ the *pieces* of the decomposition. Evidently every graph has a chordal decomposition (take the pieces to be the components of G).

We do not know whether every graph has a chordal decomposition with bipartite pieces, but the main step in our proof of Theorem 1.5 is proving a related statement, the following. Let G be a graph and let $w : V(G) \to \mathbf{Q}_+$. If $X \subseteq V(G)$, $w(X)$ denotes $\sum(w(v) : v \in X)$. If $X \subseteq V(G)$, a *yolk* of X means a subset $Y \subseteq X$ such that Y is stable and $w(Y) \geq \frac{1}{2}w(X)$. We say $X \subseteq V(G)$ is an *egg* (of (G, w)) if $G|X$ is non-null and connected, and X has a yolk. We say that an egg X is a *strong egg* if X has a yolk Y such that the edges between $X - Y$ and Y induce a connected spanning subgraph of $G|X$. In other words, if we delete all the edges with both endpoints in X, then the resulting graph is a connected bipartite graph with the partite set (X, Y). We shall prove the following in the next section.

Theorem 1.6. *For every graph G and every map $w : V(G) \to \mathbf{Q}_+$, there is a chordal decomposition $\mathcal{P}$ of G so that each $X \in \mathcal{P}$ is a strong egg.*

Proof of Theorem 1.5 (*Assuming Theorem* 1.6). Let G be a graph and $w : V(G) \to \mathbf{Q}_+$ some map. By Theorem 1.6, there is a chordal decomposition $\mathcal{P}$ of G so that every member of $\mathcal{P}$ is a strong egg. Let $\mathcal{P} = \{X_1, \ldots, X_q\}$ be the strong eggs of this chordal decomposition. Now we do the following operation.

Suppose Y_i, Y_j are yolks of X_i, X_j. If the only edges between X_i and X_j join either $X_i - Y_i$ and Y_j or $X_j - Y_j$ and X_i, then we merge these two strong eggs X_i, X_j into one strong egg.

It is easy to see that the merged graph $X_i \cup X_j$ is a strong egg, since clearly $Y_i \cup Y_j$ is a yolk for $X_i \cup X_j$. Note that there are no edges between Y_i and Y_j. In addition, the edges between $X_i \cup X_j$ and $(X_i \cup X_j) - (Y_i \cup Y_j)$ induce a connected spanning subgraph, since both X_i and X_j are strong eggs.

We claim that after doing the above operation as long as possible, the resulting strong egg decomposition is still chordal. This is clear, since the above operation just corresponds to an edge contraction in the chordal graph. Since the edge contraction of a chordal graph still yields a chordal graph, so the resulting graph is still a chordal strong egg decomposition.

Let $\mathcal{P} = \{X'_1, \ldots, X'_w\}$ be the resulting strong eggs of this chordal decomposition. Let $\{Y'_1, \ldots, Y'_w\}$ be the yolks of $\{X'_1, \ldots, X'_w\}$, respectively. Now the edge between X'_i and X'_j are either between Y'_i and Y'_j or between $X'_i - Y'_i$ and $X'_j - Y'_j$. So, we can color all the vertices of $\{Y'_1, \ldots, Y'_w\}$ by 1, and all the vertices of $\{X'_1 - Y'_1, \ldots, X'_w - Y'_w\}$ by 2, after deleting all the edges in $X'_i - Y'_i$ for all i.

If there is a clique of size p in this strong egg decomposition, then that would clearly give an odd clique minor of order p, since all the edges between X'_i and X'_j are monochromatic. Hence we may assume that there is no clique of order at least p.

As before, $H(\mathcal{P})$ is p-colorable; let $\{S_1, \ldots, S_p\}$ be a partition of $\mathcal{P}$ into stable sets. For $1 \le i \le p$, let

$$n_i = \sum_{X \in S_i} w(X).$$

Then $n_1 + \cdots + n_p = w(V(G))$, so we may assume that $n_1 \ge p^{-1} w(V(G))$. Let $S_1 = \{X_1, \ldots, X_k\}$ say, and for $1 \le j \le k$ let Y_j be a yolk of X_j. Since S_1 is stable in $H(\mathcal{P})$ it follows that $Y_1 \cup \cdots \cup Y_k$ is stable in G; but

$$w(Y_1 \cup \cdots \cup Y_k) = \sum_{1 \le j \le k} w(Y_j) \ge \sum_{1 \le j \le k} \frac{1}{2} w(X_j) = \frac{1}{2} n_1 \ge (2p)^{-1} w(V(G)),$$

as required. ■

2. Strong egg decomposition

In this section we prove Theorem 1.6. Our proof follows that in [15], but for our purpose, we need to modify the proof. In particular, we need to study the yolks, and change the decomposition, accordingly.

For inductive purposes we need a slightly more general kind of decomposition. A vertex v of H is *simplicial* if every two neighbours of v are adjacent. Let G be a graph and $w : V(G) \to \mathbf{Q}_+$ a map, as before. An *egg decomposition* of G is a chordal decomposition $\mathcal{P}$ such that for every $X \in \mathcal{P}$, either X is a strong egg, or X is simplicial in $H(\mathcal{P})$ and all its neighbours in $H(\mathcal{P})$ are strong eggs. The *support* $\mathcal{S}(\mathcal{P})$ of an egg decomposition $\mathcal{P}$ is the union of all $X \in \mathcal{P}$ that are strong eggs.

Proof of Theorem 1.6. We may assume that G is connected and non-null. Now G has an egg decomposition with non-empty support (to see this, choose any vertex v, let $X_1, \dots, X_k$ be the vertex sets of the components of $G \setminus v$, and let $\mathcal{P} = \{\{v\}, X_1, \dots, X_k\}$; then $\mathcal{P}$ is an egg decomposition with non-empty support). Consequently there is an egg decomposition $\mathcal{P}$ with maximal support and in particular it satisfies $\mathcal{S}(\mathcal{P}) \neq \emptyset$.

Suppose, for a contradiction, that $X_0 \in \mathcal{P}$ is not a strong egg. Let $X_1, \dots, X_n$ be the neighbours of X_0 in $H(\mathcal{P})$; thus, $X_1, \dots, X_n$ are all strong eggs, and pairwise touch. Since $\mathcal{S}(\mathcal{P}) \neq \emptyset$ and G is connected, it follows that $n \geq 1$. For $1 \leq i \leq n$, choose a yolk Y_i for X_i, and define N_i to be the set of all $v \in X_0$ with a neighbour in X_i.

We claim the following.

Claim 2.1. *If there is a vertex v in X_0 such that v is adjacent to a vertex in X_i, then v has a neighbor in Y_i.*

Proof. Suppose that v is adjacent to a vertex in $X_i - Y_i$ but not adjacent to any vertex in Y_i. Then we simply add v to Y_i, and consider the partition $\mathcal{P}'$ of $V(G)$ defined by

$$\mathcal{P}' = (\mathcal{P} - \{X_0, X_i\}) \cup \{X_i \cup \{v\}\} \cup \{Z_1, \dots, Z_r\}$$

where $Z_1, \dots, Z_r$ are the vertex sets of the components of $G|(X_0 - v)$. Clearly, $Y_i \cup \{v\}$ is a yolk for $X_i \cup \{v\}$, and hence $X_i \cup \{v\}$ is a strong egg. Since $Z_1, \dots, Z_r$ do not touch each other, and do not touch any member of $\mathcal{P}$ different from $X_0, X_1, \dots, X_n$, it follows that $H(\mathcal{P}')$ is chordal. Hence $\mathcal{P}'$ is an egg decomposition with

$$\mathcal{S}(\mathcal{P}) \cup V(Q) \subseteq \mathcal{S}(\mathcal{P}')$$

contradicting the maximality of $\mathcal{S}(\mathcal{P})$. This proves Claim 2.1. ∎

So for any vertex $v \in X_0$, if v has a neighbor in X_i, then either v has only neighbor in Y_i, or v has neighbors both in $X_i - Y_i$ and in Y_i. In both cases, v has a neighbor in Y_i.

Similarly, we can prove the following.

Claim 2.2. *For all i, j with $i \neq j$ and $1 \leq i, j \leq n$, every induced path of $G|X_0$ with first vertex in N_i, no other vertex in N_i, last vertex in N_j, and no other vertex in N_j, has an even number of edges.*

Proof. Suppose Q is an induced path of $G|X_0$ with vertices $v_1, \dots, v_{2k}$ in order, where $V(Q) \cap N_i = \{v_1\}$ and $V(Q) \cap N_j = \{v_{2k}\}$. By exchanging i and j and reversing the numbering of $V(Q)$, we may assume that

$$w(v_1) + w(v_3) + \cdots + w(v_{2k-1}) \leq w(v_2) + w(v_4) + \cdots + w(v_{2k}).$$

Now $Y_i \cup \{v_2, v_4, \dots, v_{2k}\}$ is a stable set, because $v_2, v_4, \dots, v_{2k}$ are pairwise non-adjacent (since Q is induced) and have no neighbours in Y_i (since they are not in N_i). But

$$w(Y_i) \geq \frac{1}{2} w(X_i)$$

$$w(\{v_2, v_4, \dots, v_{2k}\}) \geq \frac{1}{2} w(V(Q))$$

and so $Y_i \cup \{v_2, v_4, \dots, v_{2k}\}$ is a yolk of $X_i \cup V(Q)$. Since $X_i \cup V(Q)$ is connected and v_1 has a neighbor in Y_i (by Claim 2.1), it is therefore a strong egg. Let $\mathcal{P}'$ be the partition of $V(G)$ defined by

$$\mathcal{P}' = (\mathcal{P} - \{X_0, X_i\}) \cup \{X_i \cup V(Q)\} \cup \{Z_1, \dots, Z_r\}$$

where $Z_1, \dots, Z_r$ are the vertex sets of the components of $G|(X_0 - V(Q))$. Since $X_1, \dots, X_n$ pairwise touch, and $V(Q)$ touches no member of $\mathcal{P}$ different from $X_0, X_1, \dots, X_n$, it follows that

$$H(\mathcal{P}') \setminus \{Z_1, \dots, Z_r\} = H(\mathcal{P}) \setminus \{X_0\}.$$

Moreover, since $Z_1, \dots, Z_r$ do not touch each other, and do not touch any member of $\mathcal{P}$ different from $X_0, X_1, \dots, X_n$, it follows that $H(\mathcal{P}')$ is chordal. Hence $\mathcal{P}'$ is an egg decomposition with

$$\mathcal{S}(\mathcal{P}) \cup V(Q) \subseteq \mathcal{S}(\mathcal{P}')$$

contradicting the maximality of $\mathcal{S}(\mathcal{P})$. This proves Claim 2.2. ■

Choose a minimal $U \subseteq X_0$ such that $G|U$ is connected and $U \cap N_i \neq \emptyset$ for $1 \leq i \leq n$. (This is possible, since $G|X_0$ is connected and $X_0 \cap N_i \neq \emptyset$ for $1 \leq i \leq n$.) Suppose that $G|U$ has a circuit C of odd length. From the minimality of U, for each $v \in V(C)$ there exists i $(1 \leq i \leq n)$ so that the component of $G|(U - \{v\})$ which contains $C \setminus v$ does not intersect N_i; and consequently v belongs to every path of $G|U$ between N_i and $V(C)$. Let $f(v) = i$ say.

We claim that $f(u) \neq f(v)$ if $u, v \in V(C)$ are distinct. For suppose $f(u) = f(v) = i$ say. Choose a minimal path Q of $G|U$ between N_i and $V(C)$. Then Q has only one vertex in $V(C)$, and so does not contain both u and v, a contradiction. Hence $f(u) \neq f(v)$ for all distinct $u, v \in V(C)$. We may therefore arrange the numbering so that $V(C) = \{v_1, \dots, v_k\}$ say, in order, and $f(v_i) = i$ $(1 \leq i \leq k)$. For $1 \leq i \leq k$, let Q_i be a minimal path of $G|U$ between N_i and $V(C)$; then Q_i has one end v_i, and the other end, u_i say, in N_i, and $V(Q_i) \cap N_i = \{u_i\}$, $V(Q_i) \cap V(C) = \{v_i\}$.

Since C has odd length, there are two vertices v_i, v_j, adjacent in C, so that $|E(Q_i)| \equiv |E(Q_j)|$ (modulo 2); so we may assume that $|E(Q_1)| \equiv |E(Q_2)|$ (modulo 2). If Q_1 meets Q_2, or some vertex of Q_1 different from v_1 has a neighbour in $V(Q_2)$, then there is a path of $G|U$ from u_1 to v_2 not containing v_1, contradicting that $f(v_1) = 1$. Hence $Q_1 \cap Q_2$ is null and no vertex of Q_1 except v_1 has a neighbour in $V(Q_2)$; and similarly no vertex of Q_2 except v_2 has a neighbour in $V(Q_1)$. Consequently the subgraph consisting of Q_1, Q_2 and the edge v_1v_2 is an induced path of $G|X_0$ from N_1 to N_2 with an odd number of edges. Moreover, it has no vertex in N_1 except its first, and no vertex in N_2 except its last, contrary to Claim 2.2.

This proves that $G|U$ has no circuit of odd length, and so is bipartite. Since $n \neq 0$ and hence $U \neq \emptyset$, it follows that U is a strong egg (and connected). Let

$$\mathcal{P}' = (\mathcal{P} - \{X_0\}) \cup \{U, Z_1, \dots, Z_r\}$$

where $Z_1, \dots, Z_r$ are the vertex sets of the connected components of $G|(X_0 - U)$; then $\mathcal{P}'$ is an egg decomposition with

$$\mathcal{S}(\mathcal{P}) \cup U \subseteq \mathcal{S}(\mathcal{P}'),$$

a contradiction, since $U \neq \emptyset$.

This proves that every $X_0 \in \mathcal{P}$ is a strong egg, and so $\mathcal{P}$ satisfies Theorem 1.6, as required. ■

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