

The Evolution of the Mixing Rate of a Simple Random Walk on the Giant Component of a Random Graph*

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ABSTRACT: In this article we present a study of the mixing time of a random walk on the largest component of a supercritical random graph, also known as the giant component. We identify local obstructions that slow down the random walk, when the average degree d is at most $O(\sqrt{\ln n})$, proving that the mixing time in this case is $\Theta((\ln n/d)^2)$ asymptotically almost surely. As the average degree grows these become negligible and it is the diameter of the largest component that takes over, yielding mixing time $\Theta(\ln n / \ln d)$ a.a.s.. We proved these results during the 2003–04 academic year. Similar results but for constant d were later proved independently by Benjamini et al. in [3]. © 2008 Wiley Periodicals, Inc. *Random Struct. Alg.*, 33, 68–86, 2008

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1. INTRODUCTION

Given a graph G with vertex set $V_n = \{1, 2, \dots, n\}$, the simple random walk on G is the Markov chain where for every edge ij of G , the transition probability $p_{i,j}$ from i to j is $\frac{1}{d(i)}$. i.e. we exit a vertex via a uniformly chosen edge. Formally, we have defined the entries of an n by n transition matrix P for the chain (where $p_{i,j} = 0$ if ij is not an edge) and the distribution of the last point in a t step walk from initial distribution x_0 is $x_0 P^t$.

This chain (is ergodic and therefore) has a limit distribution precisely if G is connected and non-bipartite. In this case, the limit distribution π satisfies $\pi_i = \frac{d(i)}{2|E(G)|}$ where $d(i)$, the degree of i , is the number of edges of G incident to i (we extend this notation to sets letting $d(S)$ be the sum of the degrees of the vertices in S). Furthermore, the chain is reversible, as $\pi_i p_{i,j} = \pi_j p_{j,i} = \frac{1}{2|E(G)|}$.

We are interested in the mixing time of this chain for various random graphs on V_n . In this setting, we say that an event occurs asymptotically almost surely (a.a.s.) if its probability tends to 1 as $n \rightarrow \infty$.

We consider the mixing time convergence with respect to the total variation distance d_{TV} between two probability distributions on V_n defined as:

$$d_{TV}(p^{(1)}, p^{(2)}) = \max_{A \subseteq V_n} |p^{(1)}(A) - p^{(2)}(A)|.$$

Thus the mixing time of the chain is

$$T_{mix}(G) = \sup_{x_0} \min\{t : d_{TV}(x_0 P^t, \pi) < 1/e\}.$$

It is easy to prove that $\min\{t : d_{TV}(x_0 P^t, \pi) < (2/e)^t\} \leq IT_{mix}$. So, T_{mix} measures not only how long it takes to get to within $1/e$ of π , but also bounds how long it takes to get arbitrarily close to π . Thus, it measures the rate at which the Markov chain mixes.

If we choose a graph G uniformly at random from all graphs on V_n then a.a.s.

$$\forall i \in V_n \quad \left|d(i) - \frac{n}{2}\right| \leq \sqrt{n} \ln n$$

and

$$\forall i \neq j \quad \left| |N(i) \cap N(j)| - \frac{n}{4} \right| \leq \sqrt{n} \ln n,$$

where $N(i)$ is the neighbourhood of vertex i in G . It follows easily that a.a.s. for every j $\pi(j) = \frac{1}{n} + o(\frac{1}{n})$ and, by counting the number of paths of length 2 from i using the inequalities above, that a.a.s.

$$\forall i \neq j, \quad P_i^2(j) = \frac{1}{n} + o\left(\frac{1}{n}\right).$$

So a.a.s. $T_{mix}(G) = 2$, which is also the diameter of G .

We shall consider the random graph $G_{n,p}$ on V_n where each edge is present independently with probability p , and hence the expected degree of a vertex is $p(n-1)$. In what follows we always couple the use of p and d by insisting that $d = pn$; this is essentially the expected degree and makes our formulas a little easier than if we used d to represent the actual expected degree.

In [13], Hildebrand determined the mixing time of a simple random walk on a $G_{n,p}$ random graph for $p \geq \frac{\ln^2 n}{n}$, proving that a.a.s. it is of order $\frac{\ln n}{\ln d}$. Note that such a random graph is a.a.s. connected and that this is its diameter (see for example [6] Corollary 10.12(i)).

The situation for $p \leq \frac{\ln n}{n}$ is more problematic, as the local structure of the graph begins to play a role. For one thing, if $\ln n - d = \omega(1)$ then $G_{n,p}$ almost surely has vertices of degree zero and hence is not connected (see for example Theorem 7.3 in [6]). However, if $p > \frac{1+\epsilon}{n}$ for some $\epsilon > 0$ then a.a.s. the largest component $H_{n,p}$ has order $\Omega(n)$ vertices whereas the second largest component has $O(\ln n)$ vertices. So, for small p , we consider the simple random walk on the giant component $H_{n,p}$ of $G_{n,p}$. A problem of similar flavour was studied by Benjamini and Mossel in [4]. In this work, the authors consider a supercritical percolation process on \mathbb{Z}^d , for $d \geq 2$, and they determined asymptotically the mixing time of a random walk on the infinite cluster but inside $\{-n, \dots, n\}^d$, as $n \rightarrow \infty$.

A second type of local structure which comes into play at this point are the vertices of $H_{n,p}$ which are far away from any vertex of degree exceeding two. An easy second moment argument, given at the end of the article, shows that:

Lemma 1.1. *For every $\epsilon > 0$, if $\frac{1+\epsilon}{n} < p < \frac{\ln n}{5n}$, then a.a.s. $H_{n,p}$ contains paths having at least $\frac{\ln n}{4d}$ vertices, all of whose interior vertices have degree two.*

Given such a path Q with $2l + 1$ vertices, we can label these vertices with consecutive integers so that the midpoint $x = x_Q$ is labelled 0. Now, if we start our random walk at x , then we can mimic the initial part of the walk, until we touch an endpoint of Q , by the standard random walk on the integers (where we go from i to $i - 1$ or $i + 1$ with equal probability) starting at 0. It is well known (see for example [8] pp. 349) that we expect to take l^2 steps before we have seen an integer with absolute value l in this walk. Thus if x also denotes the distribution on V_n that gives unit mass to x , it follows easily that $xP^l(Q) > \frac{1}{2}$ for $t < \frac{1}{10} \left(\frac{\ln n}{8d}\right)^2$ and so $T_{mix} > \frac{1}{10} \left(\frac{\ln n}{8d}\right)^2$.

For $p \leq \frac{\sqrt{\ln n \ln \ln n}}{n}$ this bound is larger than the diameter of $H_{n,p}$ and is tight up to a multiplicative factor. For p which exceed $\frac{\sqrt{\ln n \ln \ln n}}{n}$ but are $O\left(\frac{\ln n}{n}\right)$, it is the diameter which correctly approximates T_{mix} as these paths are too small to exert much influence. So, we can determine the first order term in the mixing time precisely.

In doing so, we actually consider a slightly different definition of the mixing time: for $t > 0$ we let T_t be uniformly distributed on $\{0, \dots, t - 1\}$ and we set

$$T'_{mix}(G) = \sup_{x_0} \min\{t : d_{TV}(x_0 P^{T_t}, \pi) < 1/e\}.$$

This is within a constant factor of many other mixing times (its definition is inspired by the properties of the uniform averaging rule - see e.g. [17] Theorem 4.22 or [18] Theorem 5.4 for further details). It is at most a constant factor larger than T_{mix} . For the modified chain in which we stay in the current state with probability $1/2$ and take a step with probability $1/2$ in each iteration, T_{mix} is at most a constant factor larger than this mixing time. We have shown that actually for the chains we are considering, a.a.s. T'_{mix} is within a constant factor of T_{mix} even without this modification. This result is a consequence of a more general result which will appear in a separate paper [12]. In particular, it answers Problem 17 in Section 4.3.3 in [1].

In this article we show:

Theorem 1.2. For every $\epsilon > 0$, and $p = p(n)$ with $\frac{1+\epsilon}{n} < p = O\left(\frac{\sqrt{\ln n}}{n}\right)$ we have that a.a.s.

$$T'_{mix}(G_{n,p}) = \Theta\left(\left(\frac{\ln n}{d}\right)^2\right).$$

A similar result concerning the $G_{n,m}$ model with constant average degree was proved independently by Benjamini et al. in [3]. We remark that as lower bounds on T_{mix} are also lower bounds on T'_{mix} (up to a constant factor); our results are tight. Also in [20] Nachmias and Peres determined the mixing time when p is close to the critical point. They show that for $p = \frac{1+\lambda n^{-1/3}}{n}$, the mixing time of a lazy random walk on $H_{n,p}$ in a.a.s. $O(n)$.

We close this introductory section presenting some tools which we use later in our proofs.

For any set S of states of a Markov chain we define $Q(S)$ to be the probability we leave S when we are in the steady state; so $Q(S) = \sum_{i \in S, j \notin S} \pi(i)p_{i,j}$ and $\frac{Q(S)}{\pi(S)}$ is the probability that we leave S given we are in it. Thus, $\frac{\pi(S)}{Q(S)}$ is the expected length of a sojourn in S when we are in the steady state. It follows easily, see e.g. [10], that the mixing time of any Markov chain is at least $\max_{\{S:0 < \pi(S) \leq \frac{1}{2}\}} \frac{\pi(S)}{10Q(S)}$.

As in [15], we define the conductance of S , denoted $\Phi(S)$ to be $\frac{Q(S)}{\pi(S)\pi(V_n \setminus S)}$ and the conductance $\Phi = \Phi(G)$ of G to be $\max_{\{S:0 < \pi(S) < 1\}} \Phi(S)$. We note that for reversible chains, $\Phi(S) = \Phi(V_n \setminus S)$, so

$$T_{mix} \geq \frac{1}{10\Phi}.$$

Moreover, Jerrum and Sinclair proved in [15] that the mixing time of an irreducible, aperiodic and reversible Markov chain satisfies:

$$T'_{mix} \leq \frac{C}{\Phi^2} \log \pi_{min}^{-1}, \quad (1)$$

for some constant C . Thus, the mixing time of a reversible Markov chain is approximately determined by its conductance.

In [10], the authors of this article, treading a path blazed by Lovász and Kannan in [16], proved a strengthening of this result which can be used to tie down the mixing time of many Markov chains more precisely.

For $p > \pi_{min}$ we let $\Phi(p)$ be the minimum conductance of a connected set S with $\frac{p}{2} \leq \pi(S) \leq p$ (if there is no such a set we define $\Phi(p) = 1$). In [10], we prove:

Theorem 1.3. For an irreducible, reversible and aperiodic Markov chain on V_n we have

$$T'_{mix} \leq C \sum_{j=1}^{\lceil \log \pi_{min}^{-1} \rceil} \Phi^{-2}(2^{-j}),$$

for some constant C that does not depend on the chain.

Remark. The above sum can be approximated within a constant factor by the integral $\int_{\pi_{min}/2}^{1/2} \frac{dx}{x\Phi^2(x)}$. So if we bound $\Phi(p)$ by Φ , we obtain (1). However, the bound of Theorem 1.3 is often tighter. Such is the case for the chain considered in this article.

We apply Theorem 1.3 to deal with the mixing time for small p . To do so, we need to bound from above the conductance of connected sets of various sizes.

This completes our preliminary remarks. In the next section, we discuss the precise results on the conductance of $H_{n,p}$ that we need to prove Theorem 1.2 and show that they do indeed imply this result. In Sections 3-4, we prove these results. In Section 5, we prove Lemma 1.1 which shows that the bounds of Theorem 1.2 are tight. We close this section with the statement of Chernoff's bound (see for example inequalities (2.5), (2.6) p. 42 in [14]), which we will use at various points in our proofs to derive concentration bounds: for any $t > 0$, if X is a binomially distributed random variable:

$$\mathbb{P}[|X - \mathbb{E}[X]| > t] \leq 2e^{-\frac{1}{2}t^2 / (\mathbb{E}[X] + t)}. \tag{2}$$

2. THE EVOLUTION OF THE CONDUCTANCE

We focus now on the conductance of the connected subsets of $H_{n,p}$. To this end, we let e^* be the number of edges of $H_{n,p}$ and recall that for non-bipartite $H_{n,p}$, for every vertex v , $\pi(v) = \frac{d(v)}{2e^*}$.

Letting $e(S) = |\{\{i,j\} \in E : i,j \in S\}|$ and $e^{out}(S) = |\{\{i,j\} : i \in S, j \notin S\}|$, we have that if $H_{n,p}$ is non-bipartite: $\pi(S) = \frac{d(S)}{2e^*} = \frac{e^{out}(S) + 2e(S)}{2e^*}$ and $Q(S) = \frac{e^{out}(S)}{2e^*}$. Thus for such nonbipartite $H_{n,p}$, $\Phi(S) = \frac{e^{out}(S)}{(2e(S) + e^{out}(S))\pi(V_n \setminus S)}$ which is within a factor of 2 of $\frac{e^{out}(S)}{2e(S) + e^{out}(S)}$ and so to bound conductance we need to know about the behaviour of these two variables.

The advantages of focussing on these variables individually, rather than on conductance itself are two-fold. The first is that they are defined for all $H_{n,p}$ not just for non-bipartite ones. The second is that we can quickly see that it is $e^{out}(S)$ which gives us difficulty. Indeed, standard concentration results easily yield:

Lemma 2.1. *There exists an absolute constant l such that for any p with $\frac{1}{n} < p < \frac{2 \ln n}{n}$, a.a.s. every connected set S in $G_{n,p}$ satisfies $e(S) \leq ld|S|$. Furthermore, if $|S| = o((\ln n)^2)$ then $e(S) \leq 2|S|$.*

The easy proof of Lemma 2.1 is given in the next section. The behaviour of $e^{out}(S)$ is more problematic. However, more careful counting arguments allow us to show:

Lemma 2.2. *There exist constants $\epsilon > 0, c > 0$ and $d_0 > 1$ such that for every $p = p(n)$ satisfying $\frac{d_0}{n} \leq p \leq \frac{2 \ln n}{n}$, a.a.s. every connected subset S of $H_{n,p}$ with $|S| \geq \frac{c \ln n}{d}$, and $d(S) \leq \frac{d(H_{n,p})}{2}$ satisfies $e^{out}(S) \geq \epsilon d|S|$.*

Lemma 2.3. *For any two constants c_1 and c_2 with $1 < c_1 < c_2$ there exist $\epsilon, A > 0$ such that a.a.s. for $d = d(n)$ lying between c_1 and c_2 , every connected subset S of $H_{n,p}$ with $|S| \geq A \ln n$ and $d(S) \leq \frac{d(H_{n,p})}{2}$ satisfies $e^{out}(S) \geq \epsilon|S|$.*

Since $H_{n,p}$ is connected we know that $e^{out}(S)$ is at least one for all strict subsets of $V(H_{n,p})$. Armed with this fact and Lemmas 2.1, 2.2, 2.3, we can now give the:

Proof of Theorem 1.2. It is well known that $H_{n,p}$ is a.a.s non-bipartite. So we assume this to be the case and apply Theorem 1.3, to obtain:

$$T'_{mix}(H_{n,p}) \leq C \sum_{j=1}^{\lceil \log \pi_{min}^{-1} \rceil} \Phi^{-2}(2^{-j}).$$

We now bound the sum $\sum_{j=1}^{\lceil \log \pi_{min}^{-1} \rceil} \Phi^{-2}(2^{-j})$.

Applying Lemma 2.1 with $S = V(H_{n,p})$, we have that $e^* = O(dn)$. Since $H_{n,p}$ is connected, π_{min} is at least $\frac{1}{e^*} = \Omega(\frac{1}{dn})$. So $\log \pi_{min}^{-1} = O(\ln n)$. We claim that we can apply Lemmas 2.1, 2.2, and 2.3 to show that there exists absolute constants r and c such that if $\Phi^{-1}(S) > r$ then $\pi(S) \leq \frac{c \ln n}{d^2 n}$. So letting I be the set of j such that 2^{-j} lies between π_{min} and $\frac{c \ln n}{d^2 n}$, our sum is bounded by $O(\ln n) + \sum_{j \in I} \Phi^{-2}(2^{-j})$.

But, $\Phi(S) \geq \frac{1}{e^* \pi(S)}$, for such S because $H_{n,p}$ is connected. So, since $e^* = O(dn)$, $\Phi^{-2}(2^{-j}) = O(d^2 n^2 2^{-2j})$ uniformly for $j \in I$. Since this is a geometric sum, it is of the same order as its largest term, which is $O\left(\left(\frac{\ln n}{d}\right)^2\right)$, as required.

It remains to prove our claim.

Recalling how we have expressed Φ in terms of e^{out} and e , and combining Lemmas 2.1, 2.2, and 2.3 for $c_1 = 1 + \delta, c_2 = d_0$ we have:

For any $\delta > 0$, there is a c and an r such that for any $p = p(n)$ which is at most $\frac{\ln n}{n}$ and at least $\frac{1+\delta}{n}$, a.a.s any connected set S with at least $\frac{c \ln n}{d}$ vertices and $\pi(S) \leq \frac{1}{2}$ satisfies $\Phi^{-1}(S) \leq r$ (note that when d is less than d_0 , it can be treated as a constant).

But since $\Phi(S) > \frac{1}{3}$ if $e^{out}(S) > e(S)$, the second half of Lemma 2.1 tells us that a.a.s. every such small set whose conductance is small satisfies $d(S) = 2e(S) + e^{out}(S) \leq 3e(S) \leq 6|S|$ and so $\pi(S) = O\left(\frac{|S|}{dn}\right)$. Combining this with our bound on the size of S proves the claim and the theorem. ■

Lemma 2.2 is easy to prove using the Chernoff's bound. Lemma 2.3 is much more difficult. To prove it we consider the core of $G_{n,p}$ which is the maximal subgraph of $G_{n,p}$ all of whose vertices have degree at least 2. We prove some results about the expansion properties of the core and then translate them into results on $H_{n,p}$ which imply Lemma 2.3. In doing so, we condition on the degree sequence of the core and we use the configuration model for graphs with a given degree sequence. We discuss this model in Section 4.1 but readers may consult [14] Section 9.1 for further details.

In the next section we present some well-known properties of $G_{n,p}$ and prove Lemmas 2.1, 2.2 and the following, of which we shall have need.

Lemma 2.4. *For every fixed $d > 1$ there exists $L > 0$ such that if $p = \frac{d}{n}$ then a.a.s. every connected set $S \subseteq V_n$ in $G_{n,p}$ with $n \geq |S| \geq \ln n$ has $d(S) \leq L|S|$.*

Now Lemma 1.1 guarantees that a.a.s. $G_{n,p}$ contains paths which are connected sets with conductance between $\frac{1}{\lfloor \frac{\ln n}{d} \rfloor}$ and $\frac{2}{\lfloor \frac{\ln n}{d} \rfloor}$. Since $\frac{x}{m+x}$ is nondecreasing in x , Lemma 3.1 below tells us that every set S of order $k \leq A_p \frac{\ln n}{d}$ has conductance at least $\frac{1}{2k+1} = \Omega\left(\frac{d}{\ln n}\right)$. On the other hand Lemmas 2.2 and 2.3 tell us that the minimum conductance a set of order at least $A_p \ln n/d$ can have is a.a.s. $\Omega(1)$. So, if $\Phi(H_{n,p})$ denotes the minimum conductance of a subset of $H_{n,p}$ we have:

Theorem 2.5. *Whenever $d = d(n) > 1 + \Theta(1)$, a.a.s,*

$$\Phi(H_{n,p}) = \Theta\left(\min\left\{\frac{d}{\ln n}, 1\right\}\right).$$

3. SOME SIMPLE FACTS ABOUT $G_{n,p}$

To prove Lemma 2.1, we simply combine the following two lemmas.

Lemma 3.1. *For every $p = p(n)$ with $\frac{1}{n} < p \leq \frac{2\ln n}{n}$, a.a.s. every connected set $S \subseteq V_n$ of $G_{n,p}$ with $|S| \leq \frac{n}{60d^2}$ satisfies $|S| - 1 \leq e(S) \leq 2|S|$.*

Proof. The expected number of sets of $2k$ edges spanning a set of k vertices is $\binom{n}{k} \binom{\binom{k}{2}}{2k} p^{2k} \leq \left(\frac{ne}{k}\right)^k \frac{e^{2k} k^{4k}}{k^{2k} n^{2k}} = \left(\frac{e^3 kd^2}{n}\right)^k$. For $k \leq \frac{n}{60d^2}$ this is less than 2^{-k} . For k less than $\frac{\sqrt{n}}{e^3 d^2}$, it is less than \sqrt{n}^{-k} . The result follows by the first moment method. ■

Lemma 3.2. *There exists a constant l such that for any $d = d(n)$ between 1 and $2 \ln n$, and $p = \frac{d}{n}$, a.a.s. every set S with $|S| \geq \frac{n}{60d^2}$ satisfies $e(S) \leq ld|S|$.*

Proof. If e_k denotes the number of edges on a set of k vertices, then $\mathbb{E}[e_k] = p \binom{k}{2} \leq \frac{dk}{2}$. As e_k is binomially distributed, we can use Chernoff’s bound (2) to show that

$$\mathbb{P}[e_k > \mathbb{E}[e_k] + t] \leq 2 \exp\left(-\frac{1}{2} \frac{t^2}{\mathbb{E}[e_k] + t}\right).$$

In particular,

$$\begin{aligned} \mathbb{P}[e_k > ldk] &\leq \mathbb{P}[e_k > \mathbb{E}[e_k] + (l - 1/2)dk] \leq 2 \exp\left(-\frac{1}{2} \frac{(l - 1/2)^2 (dk)^2}{ldk}\right) \\ &\leq 2 \exp\left(-\frac{ldk}{4}\right), \end{aligned}$$

for l sufficiently large. Therefore the expected number of sets with k vertices and at least ldk edges is at most $2 \binom{n}{k} e^{-\frac{ldk}{4}} \leq 2 \left(\frac{ne}{k} e^{-\frac{ld}{4}}\right)^k \leq 2(60ed^2 e^{-\frac{ld}{4}})^k$ and

$$2 \sum_{k \geq \frac{\ln n}{cd^2}} \left(60ed^2 e^{-\frac{ld}{4}}\right)^k = o(1),$$

for l a sufficiently large constant. ■

We will need the following simple fact in the proof of Lemma 2.2. It will also be useful later. It is an immediate consequence of, for example, Chernoff’s bound.

Lemma 3.3. *For $p > \frac{1}{n}$, a.a.s $G_{n,p}$ has $(\frac{1}{2} + o(1))pn^2$ edges.*

Proof of lemma 2.2. We specify our choice of $\epsilon < \frac{1}{6}$ below. So we can assume $e^{out}(S) = e^{out}(V - S)$ is at most $\frac{dn}{6}$ as otherwise we are done. Since $d(S) \leq \frac{d(H_{n,p})}{2}$, by Lemma 3.3, $d(V - S) \geq \frac{dn}{3}$ and so $e(V - S) \geq \frac{dn}{12}$. So, we are done by Lemma 2.1 for S with $|S| \geq n - \frac{n}{12}$.

Key to our analysis is the fact that there are k^{k-2} labelled trees on k vertices. Thus the expected number of trees of $\mathcal{G}_{n,p}$ with k vertices is

$$\binom{n}{k} k^{k-2} \left(\frac{d}{n}\right)^{k-1} \leq n(ed)^k. \tag{3}$$

Having fixed both a set S of $k \leq n - \frac{n}{12l}$ vertices and the set of edges within S , the expected value of $e^{\text{out}}(S)$ is $dk(1 - k/n)$ which is at least $dk/12l$. Since $e^{\text{out}}(S)$ is binomially distributed, Chernoff's bound (2) yields

$$\mathbb{P}[e^{\text{out}}(S) < \mathbb{E}[e^{\text{out}}(S)] - t] \leq 2 \exp\left(-\frac{1}{2} \frac{t^2}{\mathbb{E}[e^{\text{out}}(S)] + t}\right).$$

Setting $t = dk/24l$, we obtain:

$$\mathbb{P}\left[e^{\text{out}}(S) < \frac{dk}{24l}\right] \leq 2 \exp\left(-\frac{1}{2} \frac{dk}{1200l^2}\right). \tag{4}$$

Combining (3) and (4), we obtain that for $k \leq n(1 - 1/12l)$, the expected number of connected sets S of $G_{n,d/n}$ with $|S| = k$ and $e^{\text{out}}(S) \leq dk/24l$ is at most

$$2n(ed)^k e^{-dk/2400l^2}.$$

If we choose d_0 such that $ede^{-d/2400l^2} < e^{-d/4800l^2}$ for every $d > d_0$, and set $c = 14400l^2$ then

$$2n \sum_{k \geq \frac{c \ln n}{d}} \left(e^{1-d/2400l^2} d\right)^k \leq 2n \sum_{k \geq \frac{c \ln n}{d}} \left(e^{-d/4800l^2}\right)^k = O\left(\frac{1}{n^2}\right) = o(1). \quad \blacksquare$$

Proof of lemma 2.4. Since $d(S) = 2e(S) + e^{\text{out}}(S)$ and we have an upper bound on $e(S)$ by Lemmas 3.1 and 3.2, it suffices to bound $e^{\text{out}}(S)$. For a set S with k vertices $\mathbb{E}[e^{\text{out}}(S)] = pk(n - k) = dk(1 - \frac{k}{n}) \leq dk$. By Chernoff's bound (2)

$$\begin{aligned} \mathbb{P}[e^{\text{out}}(S) > ldk] &\leq \mathbb{P}[e^{\text{out}}(S) > \mathbb{E}[e^{\text{out}}(S)] + (l-1)dk] \leq 2 \exp\left(-\frac{((l-1)dk)^2}{2ldk}\right) \\ &\leq 2 \exp\left(-\frac{ldk}{8}\right). \end{aligned}$$

Hence the expected number of connected sets S with k vertices and $e^{\text{out}}(S)$ which is at least ldk is at most

$$2n \left(ed e^{-\frac{ld}{8}}\right)^k.$$

Choosing $l > 80$ so that $ede^{-ld/8} < e^{-3}$ we obtain:

$$2n \sum_{k=\lceil \ln n \rceil}^n \left(ed e^{-\frac{ld}{8}}\right)^k \leq 2n^2 e^{-3 \ln n} = o(1),$$

and this concludes the proof of the lemma. \blacksquare

4. THE CORE OF THE PROOF

In this section we obtain a lower bound on $e^{\text{out}}(S)$ for any sufficiently large connected subset of $H_{n,p}$, for any d bounded between two constants. To do so, we mainly investigate the expanding properties of the core of the biggest component of a $G_{n,p}$ random graph, which we will denote by $C(H_{n,p})$. This is the maximal subgraph of $H_{n,p}$ having minimum degree at least 2. We let $N = |V(C(H_{n,p}))|$ and $M = |E(C(H_{n,p}))|$ and recall (see [21]) that a.a.s. both M and N are $\Theta(n)$, for the d within the range we are interested. The following lemma bounds below $e^{\text{out}}(S)$ for any connected S sufficiently large which is a subset of the vertex-set of the core.

Lemma 4.1. *For every $d = d(n)$ with $1 < c_1 < d < c_2$ for two constants c_1 and c_2 , and $\lambda \in (0, 1)$ there exist constants $\alpha_0 = \alpha_0(d, \lambda) > 0$ and $C_1 = C_1(d, \lambda)$ such that a.a.s. every connected $S \subseteq V(C(H_{n,p}))$ with $C_1 \ln n \leq |S| \leq \lambda N$ is joined to $C(H_{n,p}) - S$ by at least $\alpha_0|S|$ edges.*

The proof of this lemma is postponed until the next subsection. It is reasonably easy because the fact that the core has minimum degree two gives it strong expansion properties.

We are interested in the expanding properties of connected subsets of $H_{n,p}$ rather than its core. That is we would like to bound below $e^{\text{out}}(S)$ in the case where S is an arbitrary sufficiently large connected subset of $V(H_{n,p})$, using Lemma 4.1. To this end, we note that the components of $H_{n,p} - C(H_{n,p})$ are trees. Each such tree has a unique vertex that is adjacent to a vertex of the core; we say that the tree is rooted at that specific vertex. We call these trees decorations of the core. To apply the above lemma we need to prove that any sufficiently large connected subset of vertices of $H_{n,p}$ has at least a certain proportion of its vertices belonging to the core.

We show that this is indeed the case in the following lemma, which bounds the probability that a sufficiently big connected set of vertices of the core has at least l times more vertices in the decorations dangling from its vertices.

Lemma 4.2. *For every $d = d(n)$ with $1 < c_1 < d < c_2$ for some constants c_1 and c_2 , there are constants $\chi, l > 1$ such that a.a.s. every connected subset S of $H_{n,p}$ with $\chi \ln n \leq |S| \leq N$ is such that the number of vertices belonging to $C(H_{n,p})$ is at least $|S|/l$.*

Proof. Clearly, if we choose a counterexample S so as to minimize the number of decorations which it intersects but does not contain, then it partially contains at most one decoration, i.e., it suffices to prove that for some $\chi > 0$, for any sufficiently large integer t , the expected number of trees of $G_{n,p}$ with $tk \geq \chi \ln n$ vertices, such that $(t - 1)k$ of these vertices are incident to no edges off the tree and induce a forest tends to 0 as $n \rightarrow \infty$. We can see that the expected number of such trees is bounded above by

$$\binom{n}{tk} \binom{tk}{k} (tk)^{tk-2} p^{tk-1} (1-p)^{(n-tk)(tk-k) + \binom{tk-k}{2} - (tk-k)}. \tag{5}$$

If $tk < \epsilon n$, then using (3) we can see that for any given t and k this is bounded above by

$$nd^{tk} e^{tk} \binom{tk}{k} (1-p)^{ntk(1-1/t)(1-\epsilon)}.$$

This is at most $nz_d^{tk} \binom{tk}{k} e^{c_2k(1-\epsilon)}$ for $z_d = e^{1+\ln d-d(1-\epsilon)}$. We note that if ϵ is chosen to be sufficiently small, then $z_d < z_{c_1} < 1$. By making t large enough we can make $\binom{tk}{k} e^{c_2k(1-\epsilon)} \leq \left(\frac{tk}{k}\right)^k e^{c_2k(1-\epsilon)} = ((te)^{2/t} e^{2c_2(1-\epsilon)/t})^{tk/2}$ smaller than $(1/z_{c_1})^{tk/2}$. Thus, we see that the expected number of such trees having at most ϵn vertices is at most

$$nz_{c_1}^\chi \ln n/2.$$

Setting $\chi = 4/\ln(1/z_{c_1})$ clearly suffices.

If $tk \geq \epsilon n$, we need to use more accurate bounds. Since S lies in $H_{n,p}$ its number of vertices is a.s. at most λn for some $\lambda = \lambda(d) < 1$. Then writing $tk = \alpha n$, (5) is bounded above by

$$O(n) \left(\frac{1}{\alpha^\alpha (1-\alpha)^{1-\alpha}}\right)^n \binom{tk}{k} (\alpha n)^{\alpha n} \left(\frac{d}{n}\right)^{\alpha n} e^{-nd\alpha(1-\alpha)(1-1/t) - nd\alpha^2/2 + 3d\alpha/2},$$

uniformly for any $\alpha \in [\epsilon, \lambda]$. In turn, this is bounded above by

$$O(n) \left(\frac{\alpha n}{\alpha n/t}\right) e^{n(h(\alpha,d) + c_2/t)},$$

where $h(\alpha, d) = -(1-\alpha) \ln(1-\alpha) + \alpha \ln d - d\alpha(1-\alpha) - d\alpha^2/2$. Observe that $h(0, d) = 0$ and, moreover, $\partial h/\partial \alpha = \ln(1-\alpha) + 1 + \ln d - d + d\alpha = 1 + \ln(d(1-\alpha)) - d(1-\alpha) < 0$, for any $\alpha \in (0, 1)$ except from $\alpha = (d-1)/d$ where it is equal to 0. Thus $h(\alpha, d) < 0$, for any $\alpha \in (0, 1)$, and therefore for each $d > 1$ there exists $c_d > 0$ such that $h(\alpha, d) < -c_d$, for every $\alpha \in [\epsilon, \lambda]$. Taking t sufficiently large so that $\left(\frac{\alpha n}{\alpha n/t}\right) e^{nc_2/t} < \left(\frac{\alpha n t e}{\alpha n}\right)^{\alpha n/t} e^{nc_2/t} = (et)^{\alpha n/t} e^{nc_2/t} < e^{nc_d/2}$, it turns out that (5) is no more than

$$O(n) e^{-nc_d/2},$$

whenever $\epsilon n \leq tk \leq \lambda n$. The result follows. ■

In order to prove Lemma 2.3, we will need another result of a similar flavor.

A *dangling tree* is a rooted tree T of $G_{n,p}$ all of whose non-root nodes are incident only to the edges of T .

Lemma 4.3. *For every $d = d(n)$ with $1 < c_1 < d < c_2$ for some constants c_1 and c_2 there is a constant I such that setting $p = \frac{d}{n}$, a.s. for every $i \geq I$, the number of nodes of $G_{n,p}$ in dangling trees of size i or greater is less than $\frac{n}{i^3}$.*

Proof. Using (3) we can see that for any given $j = O(\ln n)$, the expected number of dangling trees of size j is bounded above by:

$$nd^j e^j (1-p)^{(j-1)(n-j)} = O(nd^j e^j e^{-dj}).$$

As in the proof of the last lemma, this is falling exponentially quickly as j increases. It follows that a.s every dangling tree has size at most $\frac{10}{c_1-1} \ln n$. Furthermore, a straightforward second moment calculation shows that asymptotically almost surely for every j , the number of dangling trees of size j is at most twice its expected value.

Indeed, let us assume that S_j and S_k be two subsets of vertices with $|S_j| = j$, $|S_k| = k$, where $j, k \leq \frac{10}{c_1-1} \ln n$. If $|S_j \cap S_k| \geq 2$, then they cannot both induce dangling trees. Thus we

only need to work with pairs of subsets that they intersect in at most 1 vertex. This vertex must necessarily be the root of the corresponding dangling trees. If S is a subset of vertices and $v \in S$, we denote by $S(v) \in \text{Dang}$ the event that a dangling tree is induced on S whose root is v . If $|S_j \cap S_k| \leq 1$ with $v \in S_j$ and $v' \in S_k$, we have

$$\mathbb{P}[S_k(v') \in \text{Dang} | S_j(v) \in \text{Dang}] = \frac{\mathbb{P}[S_k(v') \in \text{Dang}]}{(1-p)^{(j-1)(k-1)}} = \mathbb{P}[S_k(v') \in \text{Dang}](1+o(1)),$$

since $j, k = O(\ln n)$ and therefore $(1-p)^{(j-1)(k-1)} = (1-d/n)^{O(\ln^2 n)} = 1 - o(1)$. So the expected number of ordered pairs of subsets of vertices that form dangling trees is

$$(1+o(1)) \sum_{j=1}^{\frac{10 \ln n}{c_1-1}} \sum_{k=1}^{\frac{10 \ln n}{c_1-1}} \left(\binom{n}{j} \binom{n-j}{k} jk + \binom{n}{j} \binom{n-j}{k-1} j \right) \mathbb{P}[S_j(v) \in \text{Dang}] \mathbb{P}[S_k(v') \in \text{Dang}].$$

But $\binom{n-j}{k} = \binom{n}{k}(1-o(1))$ and $\binom{n-j}{k-1} = o\left(\binom{n}{k}\right)$. Thus the above quantity is

$$(1+o(1)) \sum_{j=1}^{\frac{10 \ln n}{c_1-1}} \sum_{k=1}^{\frac{10 \ln n}{c_1-1}} \binom{n}{j} \binom{n}{k} jk \mathbb{P}[S_j(v) \in \text{Dang}] \mathbb{P}[S_k(v') \in \text{Dang}].$$

In other words, if D denotes the number of dangling trees in $G_{n,p}$, we have $\mathbb{E}[D^2] = (1+o(1))\mathbb{E}^2[D]$. Therefore $\text{Var}[D] = o(\mathbb{E}^2(D))$. Now Chebyshev's inequality concludes the proof of the lemma. ■

With these three results in hand, we can give the

Proof of Lemma 2.3. To obtain a lower bound on $e^{\text{out}}(S)$ where S is a sufficiently large subset of $V(H_{n,p})$, we want to use Lemma 4.2 to argue that at least $|S|/l$ of its vertices belong to the core and then apply Lemma 4.1 to deduce that S has at least $\alpha_0|S|/l$ edges to \bar{S} . Of course such an approach is valid whenever S contains a proportion of vertices of the core that is bounded away from 1. As we are interested in bounding the conductance of sets having $\pi(S) \leq 1/2$, it might be the case that S contains almost all of the core or even all of it. In particular, this might be the case when d is close to 1 (see [21] for precise bounds on the proportion of vertices of $H_{n,p}$ belonging to the core). Hence, in this case we must argue in a different way as we shall see below.

First we need the following:

Claim 4.4. *Suppose that p is as in Lemma 4.2. Let T denote a collection of vertex-disjoint trees which are induced in $H_{n,p}$ and let t be the total number of vertices they contain. There exist constants $\chi, l > 0$, such that every such T with all its vertices, except one per tree, incident only to edges within T and $t \geq \chi \ln n$ has a.a.s. $|T| > t/l$.*

Proof. The proof of this claim is essentially the same as that of Lemma 4.2, except that in (5) we replace the factor $(tk)^{tk-2}$ by $k(tk)^{tk-k-1}$, which is the number of forests we can build on tk vertices, with k particular vertices belonging to different trees. We omit the details. ■

By the above remarks, to complete the proof of Lemma 2.3 we need only prove for S satisfying $|S \cap V(C(H_{n,p}))| > (1 - \tau_0)N$, for a constant $\tau_0 = \tau_0(d)$ to be specified later.

We note that since S is connected, for any vertex w in $V(C(H_{n,p}) - S)$ the union of the decorations attached at w is disjoint from S . Let $e_C(\bar{S})$ denote the number of edges in $V(C(H_{n,p}) - S)$ along with the attached decorations.

Furthermore, any component of $H_{n,p} - S$ disjoint from the core is a dangling tree rooted at a vertex which has a neighbour in S . Let t_S be the total number of vertices involved in these trees. If T_S is the number of these trees, then $e^{out}(S) \geq T_S$. Recall that e^* is the number of edges of $H_{n,p}$. We show that $T_S \geq e^*/4l$, where l is the constant that appears in the above claim.

Indeed, since $\pi(V(H_{n,p}) - S) \geq 1/2$, we have

$$e^* \leq d(V(H_{n,p}) - S) \leq 2t_S + e^{out}(S) + 2e_C(\bar{S}). \quad (6)$$

We may assume that $e^{out}(S) \leq e^*/4$, as otherwise we are done. Since $V(C(H_{n,p}) - S)$ contains at most $\tau_0 N$ vertices, Lemma 4.3 implies that $2e_C(\bar{S}) \leq e^*/4$, provided that we choose τ_0 small enough. Plugging these bounds into (6), we deduce that $t_S \geq e^*/4$. Claim 4.4 yields the bound on T_S and the proof of the lemma is now complete. ■

4.1. The Expanding Properties of $C(H_{n,p})$

In this section we investigate the expanding properties of the core of the giant component of a $G_{n,d/n}$ random graph, where d is bounded between two constants. We are aiming towards the proof of Lemma 4.1.

We actually consider the core of $G_{n,p}$ rather than the core of $H_{n,p}$ because it is known that $C(\mathcal{G}_{n,p})$ conditioned on its degree sequence is uniformly distributed over all graphs having this degree sequence. This follows from Proposition 2.1(b) in [22] that describes the distribution of the graph that remains from the stripping-off process conditioned on its degree sequence. It is also known (again see [22]) that a.s. the difference between these two graphs consists of a set of $O(\ln n)$ cycles each with at most $O(\ln n)$ vertices. So it is enough to prove the variant of Lemma 4.1 obtained by replacing the core of $H_{n,p}$ by the core of $G_{n,p}$. In doing so, we will use the configuration model for random graphs with a given degree sequence. This was introduced in two different versions by Bender and Canfield in [2] and by Bollobás in [5] (see also [6]). We will use the second version.

Suppose we want to analyze a uniformly random graph G_N on $V_N = \{1, 2, \dots, N\}$ with a given degree sequence $\{d_1, \dots, d_N\}$. For each $i \in V_N$ we take d_i copies of i , thus forming a set P_N of $2M$ points. A perfect matching on P_N corresponds to a multigraph on V_N where an edge between copies of i and j yields an edge from i to j . Note that this may create loops or multiple edges. We consider the random multigraph G'_N that comes out of a uniformly random perfect matching on P_N . As shown by McKay and Wormald in [19], under certain technical conditions, results proven in this model can be transferred back to a uniformly chosen simple graph on the given degree sequence.

The key to our lemma is the following, which is very easy to prove by considering this correspondence.

Proposition 4.5. *For each integer $N \geq 1$ let (d_1, \dots, d_N) be a degree sequence on the set $V_N = \{1, \dots, N\}$ and let G_N be a random graph having this degree sequence. Assume that*

1. *For each N and for $i = 1, \dots, N$ we have $d_i \geq 2$.*
2. *$2M = \sum_{i=1}^N d_i \leq \sum_{i=1}^N d_i^2 \leq CN$, for some $C > 0$.*

3. $\max\{d_i\}_{1 \leq i \leq N} \leq \ln N$, for N sufficiently large.

Then uniformly for every set of vertices S with $d(S) = \sum_{i \in S} d_i$ being an even number, we have

$$\mathbb{P}[\text{There are no edges between } S, V_N \setminus S \text{ in } G_N] = O\left(\binom{M}{d(S)/2}^{-1}\right).$$

(In fact the constant involved in the $O(1)$ term depends only on C .)

Proof. Given a degree sequence and a set S of vertices satisfying the conditions of the lemma, we generate the perfect matching on P_N , which yields G'_N , one edge at a time.

Exposing the edge out of the first vertex of S , we see that the probability it is joined to a vertex of S is $\frac{d(S)-1}{2M-1}$. Given that this edge stays within S , we take a third vertex of S and note that the probability that it is joined to a vertex in S is $\frac{d(S)-3}{2M-3}$. More generally:

$$\begin{aligned} \mathbb{P}[\text{There are no edges between } S, V_N \setminus S \text{ in } G'_N] &= \\ &= \frac{d(S)-1}{2M-1} \frac{d(S)-3}{2M-3} \cdots \frac{1}{2M-d(S)+1} \\ &= \frac{d(S)!}{(d(S)/2)! 2^{d(S)/2}} \frac{(2M-d(S))!}{(M-d(S)/2)! 2^{M-d(S)/2}} \frac{M! 2^M}{(2M)!} \\ &= \binom{M}{d(S)/2} / \binom{2M}{d(S)} \leq \binom{M}{d(S)/2}^{-1}. \end{aligned}$$

Assumptions (2) and (3) along with the main theorem in [19] transfer this bound to the space of a G_N random graph, and we are done. ■

For each $n \geq 1$, and any $C, c, \varepsilon > 0, B$, we define $E_n = E_n(C, c, \varepsilon)$ to be the set of graphs on V_n such that their core is non-empty, it has N vertices and M edges, where $N \geq cn$ and $N/M \leq 1 - \varepsilon$, maximum degree at most $\ln N$ and moreover, the sum of the squares of the degrees in the core is no more than CN (see condition (2) in the previous proposition). To prove Lemma 4.1 we will condition on the event E_n for a specific choice of C, c and ε :

Remark. There exist C, c, ε such that for every $1 < c_1 < d < c_2$ the event E_n occurs a.a.s. (see [6], [21]).

Now we are ready to proceed with the proof of Lemma 4.1:

Proof of Lemma 4.1. In fact we shall prove that every $S \subseteq V(C(\mathcal{G}_{n,p}))$ with $G[S]$ being connected and $\lambda N \geq |S| \geq C_1 \ln n$ has more than $\alpha_0 |S|$ edges joining it with $V(C(\mathcal{G}_{n,p})) \setminus S$, for some α_0, C_1 which will be specified during our proof. For the time being we assume that $C_1 \geq 1$, with the prospect of using Lemma 2.4.

Now for any integer s between $C_1 \ln n$ and λN , let $X_s = X_s(d, \alpha_0)$ be the number of subsets S of $V(C(\mathcal{G}_{n,p}))$ with $|S| = s$, $d(S) \leq Ls$, $e(S) \geq s - 1$ and having at most $\alpha_0 |S|$ edges joining it with $V(C(\mathcal{G}_{n,p})) \setminus S$. Considering such sets is sufficient, since by Lemma 2.4 a.a.s. every connected set of vertices S with $|S| \geq C_1 \ln n$ has total degree no more than $L|S|$. We will condition on the event E_n and, more specifically, we will show that $\mathbb{E}[X_s | E_n] = o(1/n)$ uniformly for every s between $C_1 \ln n$ and λN . To do so, we shall estimate the conditional

expectation of X_s given a degree sequence of the core of a graph in E_n (in the conditional probability space of the event E_n), which we denote by $\tilde{\mathbb{E}}[X_s]$. Proving that $\tilde{\mathbb{E}}[X_s] = o(1/n)$, i.e. that the random variable $\tilde{\mathbb{E}}[X_s]$ is bounded above by a function that is $o(1/n)$, uniformly for any degree sequence of the core of a graph in E_n will be sufficient. Then the lemma will follow since E_n occurs a.a.s., if we choose C, c and ε as in the above remark.

We proceed with the estimation of $\tilde{\mathbb{E}}[X_s]$, where $C_1 \ln n \leq s \leq \lambda N$. For any $\alpha \leq \alpha_0$, if the set S has total degree t , then we can choose the edges that will be adjacent to $V(C(\mathcal{G}_{n,p})) \setminus S$ in at most $\binom{t}{\alpha s}$ ways. Having specified these elements of the total degree, now the available total degree in S is equal to $t' = t - \alpha s$. Then the probability (i.e. the conditional expectation of the indicator random variable that is equal to 1 on the event) that every other edge is not adjacent to $V(C(\mathcal{G}_{n,p})) \setminus S$ is by Proposition 4.5 $O\left(\binom{M}{t'/2}^{-1}\right)$, uniformly over all sets S as above. This is the case because the premises of Proposition 4.5 are satisfied, by the definition of E_n .

Since $s - 1 \leq t'/2 \leq M - N + s$, the above bound is $O(1) \max\left\{\binom{M}{s-1}^{-1}, \binom{M}{N-s}^{-1}\right\}$. Note that by Stirling's formula we have

$$\begin{aligned} \frac{\binom{N}{s}}{\binom{M}{s-1}} &= \frac{(s-1)! N! (M-s+1)!}{s! M! (N-s)!} = O(1) \frac{1}{s} \frac{N^N (M-(s-1))^{M-(s-1)}}{M^M (N-s)^{N-s}} \\ &= O(1) \frac{1}{s} N^s \left(1 - \frac{s}{N}\right)^{-N+s} \frac{M^{M-s+1}}{M^M} \left(1 - \frac{s-1}{M}\right)^{M-(s-1)} \\ &= O(1) \frac{M}{s} \left(\frac{N}{M}\right)^s \exp\left(s - \frac{s^2}{N} - s + 1 + \frac{s^2}{M}\right) = O(1) \frac{M}{s} \left(\frac{N}{M}\right)^s. \end{aligned} \tag{7}$$

Since $\binom{N}{s} = \binom{N}{N-s}$ and $\binom{M}{s} = \frac{M-s+1}{s} \binom{M}{s-1}$, applying (7) with $N - s$ in place of s yields:

$$\frac{\binom{N}{s}}{\binom{M}{N-s}} = O(1) \frac{M}{M - N + s + 1} \left(\frac{N}{M}\right)^{N-s}.$$

Provided $\frac{N}{M} \leq (1 - \epsilon)$, both these bounds are $O(Me^{-\beta s})$ for some $\beta = \beta(\epsilon, \lambda)$.

We are now ready to bound $\tilde{\mathbb{E}}[X_s]$, for any s as above. Let $\mathbf{D} = \mathbf{D}(s, \alpha_0) = \{\alpha : \alpha s \in \mathbb{N}, \alpha \leq \alpha_0\}$ and $T(s, \alpha) = \{t \in \mathbb{N} : 2(s-1) + \alpha s \leq t \leq \min\{2(M-N+s), Ls\}, t - \alpha s \text{ is even}\}$. We now apply (7), and its corollary, to obtain:

$$\begin{aligned} \tilde{\mathbb{E}}[X_s] &= O(1) \binom{N}{s} \sum_{\alpha \in \mathbf{D}(s, \alpha_0)} \sum_{t \in T(s, \alpha)} \binom{M}{(t - \alpha s)/2}^{-1} \binom{t}{\alpha s} \\ &= O(1) \sum_{\alpha \in \mathbf{D}(s, \alpha_0)} \sum_{t \in T(s, \alpha)} \left(\frac{te}{\alpha s}\right)^{\alpha s} Me^{-\beta s}, \end{aligned}$$

since t is bounded by Ls , for small enough α , $\left(\frac{te}{\alpha s}\right)^{\alpha s} \leq e^{\frac{\beta s}{2}}$, and each term in the above sum is at most $Me^{-\frac{\beta s}{2}}$. Further the sum has $O(s^2)$ terms. So, for $a_0 > 0$ sufficiently small and $s \geq C_1 \ln n$, where $C_1 = C_1(d)$ is a sufficiently large constant the sum is $o\left(\frac{1}{n}\right)$. ■

5. LONG INDUCED PATHS

In this section we give the

Proof of Lemma 1.1. We focus on the set \mathcal{S} of those paths of $G_{n,p}$ all of whose internal vertices have degree two but whose endpoints have degree greater than 2. Note that some of these paths are edges. We let \mathcal{S}' be the multiset consisting of the interiors of these paths. \mathcal{S}' is a multiset because it may (indeed a.a.s does) contain many empty paths.

Let $\mathcal{P} = |\mathcal{S}'|$ and let \mathcal{P}_i be the number of paths in \mathcal{S}' that have exactly i vertices. Let us assume now that $3 \leq i \leq n$. If P is a permutation of an i -subset of vertices of V_n , then we have

$$\mathbb{E}[\mathcal{P}_i] = \binom{n}{i} \frac{i!}{2} \mathbb{P}[P \in \mathcal{S}],$$

where

$$\mathbb{P}[P \in \mathcal{S}] = p^{i-1} (1-p)^{(i-2)(n-i) + \binom{i}{2} - i} \hat{p},$$

with \hat{p} being the probability that the two endpoints of P have degree at least 2 in $V_n \setminus I(P)$, where $I(P)$ is the set of interior vertices of P . (To see the exponent of $1-p$ in the above product note that $(i-2)(n-i)$ is the number of potential edges between the interior of P and the vertices outside P , which must not be present, and also all the edges between the vertices of P except from those that belong to the path and possibly the edge between its endpoints must not be present as well; there are $\binom{i}{2} - (i-1) - 1$ of them.) Observe first that

$$\mathbb{E}[\mathcal{P}_i] \leq \frac{n!}{(n-i)!} p^{i-1} (1-p)^{(i-2)(n-i)}$$

and therefore the expected number of such paths with more than $\frac{10 \ln n}{d}$ vertices is $o(1)$. Hence a.a.s. there are no such paths of length at least $\frac{10 \ln n}{d}$. So we may assume that $i \leq \frac{10 \ln n}{d}$. For a lower bound observe that if v_1 and v_2 are the endpoints of P and $d(v_1)$, $d(v_2)$ are their degrees in $V_n \setminus I(P)$, then

$$\hat{p} = \mathbb{P}[d(v_1), d(v_2) \geq 2] \geq \mathbb{P}[d(v_1) \geq 2] \mathbb{P}[d(v_2) \geq 2] = \mathbb{P}^2[d(v_1) \geq 2] \quad (8)$$

by the FKG inequality (see Theorem 2.12 in [14]). But

$$\begin{aligned} \mathbb{P}[d(v_1) \geq 2] &= 1 - (1-p)^{n-i+1} - (n-i+1)p(1-p)^{n-i} \geq 1 - (1-p)^{n-i+1} \\ &\times (1 + np(1-p)^{-1}) = 1 - e^{-d}(1+d)(1+o(1)) \geq 1 - \frac{2.1}{e}. \end{aligned} \quad (9)$$

The last inequality follows from the fact that $e^{-d}(1+d)$ is a decreasing function of d and $d \geq 1$. Hence

$$\hat{p} \geq \left(1 - \frac{2.1}{e}\right)^2 > \frac{1}{100}. \quad (10)$$

Also since $i \leq \frac{10 \ln n}{d}$, we have $(1-p)^{\binom{i}{2} - i} = 1 - o(1)$. So

$$\mathbb{E}[\mathcal{P}_i] \geq \frac{1}{200} \frac{n!}{(n-i)!} p^{i-1} (1-p)^{in}.$$

In particular, it follows that the expected number of paths of length $\frac{\ln n}{4d}$ in \mathcal{S} is $\Omega(n^{3/4})$. Now, a second moment calculation will show that a.a.s there will be at least \sqrt{n} paths of length $\frac{\ln n}{4d}$ in \mathcal{S} .

Let S, S' be two sets of vertices with $|S|, |S'| = \frac{\ln n}{4d}$. Note that they cannot both form paths that belong to \mathcal{S} , if they intersect in more than two vertices. Moreover, if they intersect at all, the common vertices must be endvertices of the paths that S and S' form.

For simplicity we set $\ell = \frac{\ln n}{4d}$, and let us consider first the case where $S \cap S' = \emptyset$. Let $P(S)$ (and $P(S')$ respectively) be the set of all possible path arrangements of the vertices in S (and in S' , respectively). Let $P \in P(S)$ and $P' \in P(S')$ and assume that $v_1, v_2 \in S$ are the endvertices of P and $v'_1, v'_2 \in S'$ are the endvertices of P' . Also we let $I(P)$ and $I(P')$ be the interior vertices of P and P' , respectively. We will show that $\mathbb{P}[P' \in \mathcal{S} | P \in \mathcal{S}] = \mathbb{P}[P' \in \mathcal{S}](1 + o(1))$. Firstly, we show that

$$\mathbb{P}[d(v'_1), d(v'_2) \geq 2 | P \in \mathcal{S}] = \hat{p}(1 + o(1)). \quad (11)$$

If we let $d(v'_1)$ and $d(v'_2)$ be the degrees of v'_1 and v'_2 in $V_n \setminus I(P')$, then conditioning on whether or not $v'_1 \not\sim v'_2$ ¹ and $v'_1, v'_2 \not\sim v_1, v_2$ we obtain:

$$\begin{aligned} \mathbb{P}[d(v'_1), d(v'_2) \geq 2 | P \in \mathcal{S}] &= (1 - (1 - p)^{n - |S| - |S'| + 1} - (n - |S| - |S'| + 1) \\ &\quad \times p(1 - p)^{n - |S| - |S'|})^2 \times \mathbb{P}[v'_1, v'_2 \not\sim v_1, v_2, v'_1 \not\sim v'_2 | S \in \mathcal{S}] + o(1) \\ &= (1 - e^{-d}(1 + d))^2 + o(1). \end{aligned}$$

But (8) and (9) imply that $\hat{p} \geq (1 - e^{-d}(1 + d))^2 + o(1)$. We will show that this lower bound is tight. Conditioning on whether or not $v'_1 \sim v'_2$, we deduce that

$$\begin{aligned} \hat{p} &= \mathbb{P}[d(v'_1), d(v'_2) \geq 2] \\ &= (1 - (1 - p)^{n - \ell + 1} - (n - \ell + 1)p(1 - p)^{n - \ell})^2 \mathbb{P}[v'_1 \not\sim v'_2] + o(1) \\ &= (1 - e^{-d}(1 + d))^2 + o(1), \end{aligned}$$

which yields (11). Moreover, if we condition on $P \in \mathcal{S}$, then the $(|S| - 2)(|S'| - 2)$ edges between $I(P)$ and $I(P')$ are not present. Therefore,

$$\mathbb{P}[P' \in \mathcal{S} | P \in \mathcal{S}] = \frac{\mathbb{P}[P' \in \mathcal{S}]}{(1 - p)^{(|S| - 2)(|S'| - 2)}}(1 + o(1)) = \mathbb{P}[P' \in \mathcal{S}](1 + o(1)),$$

since $\ell = O(\ln n)$ and therefore $(1 - p)^{(|S| - 2)(|S'| - 2)} = (1 - p)^{(\ell - 2)^2} = 1 - o(1)$. If $|S \cap S'| = i$, where $i = 1, 2$, then in the same vein one can show that

$$\begin{aligned} \mathbb{P}[P, P' \in \mathcal{S}] &= \frac{1 + o(1)}{(1 - e^{-d}(1 + d))^i + o(1)} \mathbb{P}[P \in \mathcal{S}] \mathbb{P}[P' \in \mathcal{S}] \\ &= \frac{1 + o(1)}{(1 - e^{-d}(1 + d))^i + o(1)} \mathbb{P}^2[P \in \mathcal{S}]. \end{aligned}$$

By (9) and (10), $\frac{1}{(1 - e^{-d}(1 + d))^i + o(1)} < 100$. Thus

$$\begin{aligned} \mathbb{E}[\mathcal{P}_\ell^2] &\leq \left(\binom{n}{\ell} \binom{n - \ell}{\ell} \left(\frac{\ell!}{2} \right)^2 \mathbb{P}^2[P \in \mathcal{S}] + 100 \binom{n}{\ell} \binom{n - \ell}{\ell - 1} \frac{\ell!}{2} (\ell - 1)! \mathbb{P}^2[P \in \mathcal{S}] \right. \\ &\quad \left. + 100 \binom{n}{\ell} \binom{n - \ell}{\ell - 2} \frac{\ell!}{2} (\ell - 2)! \mathbb{P}^2[P \in \mathcal{S}] \right) (1 + o(1)). \end{aligned}$$

¹The symbol \sim denotes the adjacency of two vertices.

But since $\ell = O(\ln n)$ we have $\binom{n-\ell}{\ell} = \binom{n}{\ell}(1 - o(1))$ and $\binom{n-\ell}{\ell-1}, \binom{n-\ell}{\ell-2} = o\left(\binom{n}{\ell}\right)$. Therefore, $\mathbb{E}[\mathcal{P}_\ell^2] = \mathbb{E}^2[\mathcal{P}_\ell](1 + o(1))$, which implies that $\text{Var}[\mathcal{P}_\ell] = o(\mathbb{E}^2[\mathcal{P}_\ell])$. Hence applying Chebyshev's inequality, we have

$$\mathbb{P}[\mathcal{P}_\ell \leq \sqrt{n}] = \mathbb{P}[\mathcal{P}_\ell \leq \mathbb{E}[\mathcal{P}_\ell] + \sqrt{n} - \mathbb{E}[\mathcal{P}_\ell]] \leq \frac{o(\mathbb{E}^2[\mathcal{P}_\ell])}{(\mathbb{E}[\mathcal{P}_\ell] - \sqrt{n})^2} = o(1),$$

as $\mathbb{E}[\mathcal{P}_\ell] = \Omega(n^{3/4}) = \omega(\sqrt{n})$.

Now, we can construct an auxiliary multi-graph $G'_{n,p}$ from $G_{n,p}$ by replacing each path in \mathcal{S} by an edge with the same endpoints (so we delete the vertices on the interior of these paths). By our bounds on the size of the paths in \mathcal{S} , we know that a.a.s $G'_{n,p}$ has exactly one component of size $\Omega\left(\frac{n}{\ln n}\right)$ and this corresponds to $H_{n,p}$.

We would like to say that given \mathcal{S} and $G'_{n,p}$ we can generate $G_{n,p}$ by taking a uniform bijection between the elements of \mathcal{S}' and the edges of $G'_{n,p}$ whose endpoints have degrees bigger than 2. This may not be true, because $G'_{n,p}$ may have multiple edges, and we cannot map two edges of $G'_{n,p}$ to single edge paths in \mathcal{S} .

For the moment, we assume that it is true and that $G_{n,p}$ (and therefore $G'_{n,p}$) has $O(n \ln n)$ edges, there are at least \sqrt{n} paths in \mathcal{S} having $\frac{\ln n}{4d}$ vertices, and the component of $G'_{n,p}$ corresponding to $H_{n,p}$ has $\Omega\left(\frac{n}{\ln n}\right)$ edges. In a random bijection from \mathcal{S}' into the edge set of $G'_{n,p}$ the probability that a particular interior in \mathcal{S}' having $\frac{\ln n}{4d} - 2$ vertices is mapped to an edge of the component of $G'_{n,p}$ that corresponds to $H_{n,p}$ is $\Omega\left(\frac{n/\ln n}{n \ln n}\right) = \Omega\left(\frac{1}{\ln^2 n}\right)$. Since there are at least \sqrt{n} such paths, it turns out that the probability that no path of \mathcal{S} in $H_{n,p}$ has $\frac{\ln n}{4d}$ vertices is at most $\left(1 - \Omega\left(\frac{1}{\ln^2 n}\right)\right)^{\sqrt{n}} = o(1)$. But we have seen that these three conditions a.a.s hold, so we are done if $G'_{n,p}$ is a simple graph.

If $G'_{n,p}$ is not simple, instead of considering \mathcal{S}' we consider the sub(multi)set \mathcal{S}^* consisting of the interiors of those paths of \mathcal{S} whose endpoints are not joined by another path of \mathcal{S} . We let $G^*_{n,p}$ be the graph obtained by replacing the paths of \mathcal{S}^* by edges. Now, given $G^*_{n,p}$ and \mathcal{S}^* , we obtain $G_{n,p}$ by taking a uniform bijection between the paths of \mathcal{S}^* and the edges of $G^*_{n,p}$ whose endpoints both have degree at least three and are not joined by a path all of whose internal vertices have degree two. An easy first moment calculation shows that $|\mathcal{S}| - |\mathcal{S}^*|$ is a.a.s $o(|\mathcal{S}|)$ so we can apply the above argument to \mathcal{S}^* to prove our result. We omit the details. \blacksquare

6. CONCLUDING REMARKS

We have investigated some geometric properties of the giant component of supercritical random graphs. In particular, we analysed the edge expansion of connected subsets of various sizes, concluding that it is the sets of order $\frac{\ln n}{d}$ that have small expansion. As a consequence, these sets delay the mixing of a random walk. However, as the average degree grows, these shrink and the mixing time is determined by a global parameter which is the diameter. This was shown in [13], for average degree at least $\ln^2 n$. In a forthcoming article [11], we give a more detailed analysis of this situation for degrees between $\sqrt{\ln n}$ and $\ln^2 n$. In particular, for $d = \Omega(\sqrt{\ln n \ln \ln n})$ we prove that for some constant $c > 0$ a.a.s.

$$\left| T'_{\text{mix}}(G_{n,p}) - \frac{c \ln n}{\ln d} \right| = O\left(\left(\frac{\ln n}{d}\right)^2\right).$$

This result improves on the work of Hildebrand [13] who had determined the mixing time up to a multiplicative factor for $p = \Omega\left(\frac{\ln^2 n}{n}\right)$. Its proof relies on proving inductively that the number of nodes at the j -th level of the breadth-first search tree from an arbitrary vertex i is concentrated around $d(i)d^{j-1}$ provided this is $o(n)$. [11] relies on arguments which were developed in [23] to obtain results on the diameter of $G_{n,p}$ for $d = \omega(1)$, strengthening results in [7, 9].

In the present article, our analysis focuses on the worst case scenario where the random walk begins within one of the sets that have small conductance. In particular, if the walk starts within one of the induced paths which have length of order $\frac{\ln n}{d}$ the bound of Theorem 1.2 is attained. Thus a natural question for further study would be the analysis of the mixing time of a random walk whose initial states/vertices do not lie within connected sets of small conductance. In this case we believe that the mixing time is of order equal to the diameter of the giant component.

Another question for further study is the analysis of the mixing time of a random walk on a random d -regular graph with $d \geq 3$ (note that such a random graph is a.a.s. connected). One characteristic of such a random graph is the lack of local obstructions; in other words, it is a.a.s. an expander graph. In this case we expect that it will be the diameter of the graph that determines the mixing time. In particular, we believe that the mixing time is of order $\ln n / \ln d$. In fact, as it was proved in [13], when $d = \ln^c n$, for $c \geq 2$, the mixing time is a.a.s. $(1 \pm \varepsilon) \frac{\ln n}{\ln d}$ for any $\varepsilon > 0$.

Finally, the extension of the results of this article to $G_{n,m}$ random graphs would seem a natural next step, though it appears that it requires slightly different tools. This is evident from the techniques and the proofs that were used in [3] for the study of a random walk on the giant component of a supercritical $G_{n,m}$ random graph with constant average degree.

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