Ballot theorems, old and new

L. Addario-Berry[*] B.A. Reed[†]

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"There is a big difference between a fair game and a game it's wise to play."
-Bertrand (1887b).

1 A brief history of ballot theorems

1.1 Discrete time ballot theorems

We begin by sketching the development of the classical ballot theorem as it first appeared in the Comptes Rendus de l'Academie des Sciences. The statement that is fairly called the first Ballot Theorem was due to Bertrand:

Theorem 1 (Bertrand (1887c)). *We suppose that two candidates have been submitted to a vote in which the number of voters is μ. Candidate A obtains n votes and is elected; candidate B obtains $m = \mu - n$ votes. We ask for the probability that during the counting of the votes, the number of votes for A is at all times greater than the number of votes for B. This probability is $(2n - \mu)/\mu = (n - m)/(n + m)$.*

Bertrand's "proof" of this theorem consists only of the observation that if $P_{n,m}$ counts the number of "favourable" voting records in which A obtains n votes, B obtains m votes and A always leads during counting of the votes, then

$$P_{n+1,m+1} = P_{n+1,m} + P_{n,m+1},$$

the two terms on the right-hand side corresponding to whether the last vote counted is for candidate B or candidate A, respectively. This "proof" can be easily formalized as

[*]Department of Statistics, University of Oxford, UK.
[†]School of Computer Science, McGill University, Canada *and* Projet Mascotte, I3S (CNRS/UNSA)-INRIA, Sophia Antipolis, France.

follows. We first note that the binomial coefficient $B_{n,m} = (n+m)!/n!m!$ counts the total number of possible voting records in which A receives n votes and B receives m, Thus, the theorem equivalently states that for any $n \geq m$, $B_{n,m} - P_{n,m}$, which we denote by $\Delta_{n,m}$, equals $2mB_{n,m}/(n+m)$. This certainly holds in the case $m = 0$ as $B_{n,0} = 1 = P_{n,0}$, and in the case $m = n$, as $P_{n,n} = 0$. The binomial coefficients also satisfy the recurrence $B_{n+1,m+1} = B_{n+1,m} + B_{n,m+1}$, thus so does the difference $\Delta_{n,m}$. By induction,

$$\begin{aligned}\Delta_{n+1,m+1} &= \Delta_{n+1,m} + \Delta_{n,m+1} \\ &= \frac{2m}{n+m+1}B_{n+1,m} + \frac{2(m+1)}{n+m+1}B_{n,m+1} = \frac{2(m+1)}{n+m+2}B_{n+1,m+1},\end{aligned}$$

as is easily checked; it is likely that this is the proof Bertrand had in mind.

After Bertrand announced his result, there was a brief flurry of research into ballot theorems and coin-tossing games by the probabilists at the Academie des Sciences. The first formal proof of Bertrand's Ballot Theorem was due to André and appeared only weeks later (André, 1887). André exhibited a bijection between unfavourable voting records starting with a vote for A and unfavourable voting records starting with a vote for B. As the latter number is clearly $B_{n,m-1}$, this immediately establishes that $B_{n,m} - P_{n,m} = 2B_{n,m-1} = 2mB_{n,m}/(n+m)$.

A little later, Barbier (1887) asserted but did not prove the following generalization of the classical Ballot Theorem: if $n > km$ for some integer k, then the probability candidate A always has more than k-times as many votes as B is precisely $(n-km)/(n+m)$. In response to the work of André and Barbier, Bertrand had this to say:

> "Though I proposed this curious question as an exercise in reason and calculation, in fact it is of great importance. It is linked to the important question of duration of games, previously considered by Huygens, [de] Moivre, Laplace, Lagrange, and Ampere. The problem is this: a gambler plays a game of chance in which in each round he wagers $\frac{1}{n}$'th of his initial fortune. What is the probability he is eventually ruined and that he spends his last coin in the $(n+2\mu)$'th round?" (Bertrand, 1887a)

He notes that by considering the rounds in reverse order and applying Theorem 1 it is clear that the probability that ruin occurs in the $(n+2\mu)$'th round is nothing but $\frac{n}{n+2\mu}\binom{n+2\mu}{\mu}2^{-(2\mu+n)}$. By informal but basic computations, he then derives that the probability ruin occurs *before* the $(n+2\mu)$'th round is approximately $1 - \frac{\sqrt{2/\pi}n}{\sqrt{n+2\mu}}$, so for this probability to be large, μ must be large compared to n^2. (Bertrand might have added Pascal, Fermat, and the Bernoullis (Hald, 1990, pp. 226-228) to his list of notable mathematicians who had considered the game of ruin; (Balakrishnan, 1997, pp. 98-114) gives an overview of prior work on ruin with an eye to its connections to the ballot theorem.)

Later in the same year, he proved that in a *fair game* (a game in which, at each step, the average change in fortunes is nil) where at each step, one coin changes hands, the expected number of rounds before ruin is infinite. He did so using the fact that by the above formula, the probability of ruin in the t'th round (for t large) is of the order $1/t^{3/2}$, so the expected time to ruin behaves as the sum of $1/t^{1/2}$, which is divergent. He also stated that in a fair game in which player A starts with a dollars and player B starts with b dollars, the expected time until the game ends (until one is ruined) is precisely ab (Bertrand, 1887b). This fact is easily proved by letting $e_{a,b}$ denote the expected time until the game ends and using the recurrence $e_{a,b} = 1 + (e_{a-1,b} + e_{a,b-1})/2$ (with boundary conditions $e_{a+b,0} = e_{0,a+b} = 0$). Expanding on Bertrand's work, Rouché provided an alternate proof of the above formula for the probability of ruin (Rouché, 1888a). He also provided an exact formula for the expected number of rounds in a biased game in which player A has a dollars and bets a_0 dollars each round, player B has b dollars and bets b_0 dollars each round, and in each round player A wins with probability p satisfying $a_0 p > b_0(1-p)$ (Rouché, 1888b).

All the above questions and results can be restated in terms of a *random walk* on the set of integers $\mathbb{Z}$. For example, let $S_0 = 0$ and, for $i \geq 0$, $S_{i+1} = S_i \pm 1$, each with probability $1/2$ and independently of the other steps - this is called a *symmetric simple random walk*. (For the remainder of this section, we will phrase our discussion in terms of random walks instead of votes, with $X_{i+1} = S_{i+1} - S_i$ constituting a step of the random walk.) Then Theorem 1 simply states that given that $S_t = h > 0$, the probability that $S_i > 0$ for all $i = 1, 2, \ldots, t$ (i.e. the random walk is favourable for A) is precisely h/t. Furthermore, the time to ruin when player A has a dollars and player B has b dollars is the *exit time* of the random walk S from the interval $[a, -b]$. The research sketched above constitutes the first detailed examination of the properties of a random walk $S_0, S_1, \ldots, S_n$ conditioned on the value S_n, and the use of such information to study the asymptotic properties of such a walk.

In 1923, Aeppli proved Barbier's generalized ballot theorem by an argument similar to that used by André's. This proof is presented in Balakrishnan (1997, pp.101-102), where it is also observed that Barbier's theorem can be proved using Bertrand's original recurrence in the same fashion as above. A simple and elegant technique was used by Dvoretzky and Motzkin (1947) to prove Barbier's theorem; we use it to prove Bertrand's theorem as an example of its application, as it highlights an interesting perspective on ballot-style results.

We think of $\mathcal{X} = (X_1, \ldots, X_{n+m}, X_1)$ as being arranged clockwise around a cycle (so that $X_{n+m+1} = X_1$). There are precisely $n+m$ walks corresponding to this set, obtained by choosing a first step X_i, so to establish Bertrand's theorem it suffices to show that however $X_1, \ldots, X_{n+m}$ are chosen such that $S_n = n - m > 0$, precisely $n - m$ of the walks $X_{i+1}, \ldots, X_{n+m}, X_1, \ldots, X_i$ are favourable for A. Let $S_{ij} = X_{i+1} + \ldots + X_j$ (this sum includes X_{n+m} if $i < j$). We say that $X_i, \ldots, X_j$ is a *bad run* if $S_{ij} = 0$ and $S_{i'j} < 0$ for all $i' \in \{i+1, \ldots, j\}$ (this set includes $n+m$ if $i > j$). In words, this condition states that i is the first index for which the reversed walk starting with X_j and ending with X_{i+1} is nonnegative. It is immediate that if two bad runs intersect then one is contained in the other, so the maximal bad runs are pairwise disjoint. (An example of a random walk and its bad runs is shown in Figure 1).

If $X_i = 1$ and $S_{ij} = 0$ for some j then X_i begins a bad run, and since $S_n = \sum_{i=1}^{n} X_i > 0$, if

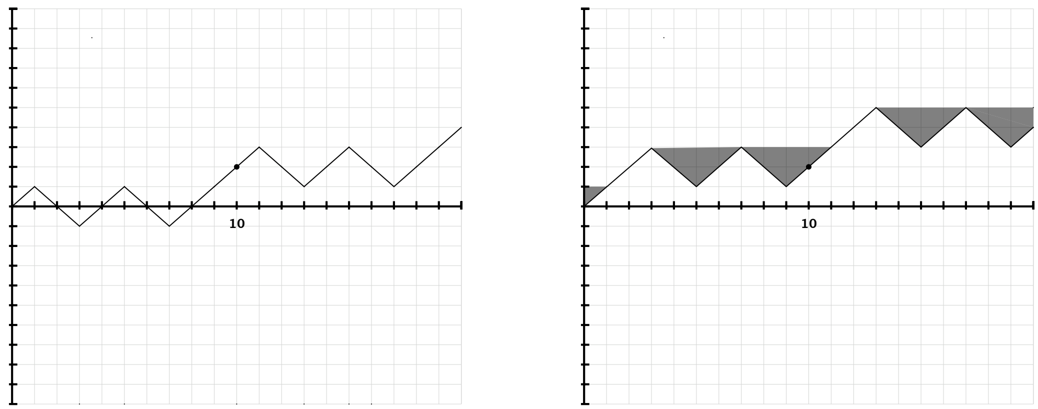

Figure 1: On the left appears the random walk corresponding to the voting sequence $(1, -1, -1, 1, 1, -1, -1, 1, 1, 1)$, doubled to indicate the cyclic nature of the argument. On the right is the reversal of the random walk; the maximal bad runs are shaded grey.

$X_i = -1$ then X_i ends a bad run. As $S_{ij} = 0$ for a maximal bad run and $X_i = 1$ for every X_i not in a bad run, there must be precisely $n - m$ values of i for which X_i is not in a bad run. If the walk starting with X_i is favourable for A then for all $i \neq j$, S_{ij} is positive, so X_i is not in a bad run. Conversely, if X_i is not in a bad run then $X_i = 1$ and for all $j \neq i$, $S_{ij} > 0$, so the walk starting with X_i is favourable for A. Thus there are precisely $(n - m)$ favourable walks corresponding to $\mathcal{X}$, which is what we set out to prove.

With this technique, the proof of Barbier's theorem requires nothing more than letting the positive steps have value $1/k$ instead of 1. This proof is notable as it is the first time the idea of cyclic permutations was applied to prove a ballot-style result. This "rotation principle" is closely related to the *strong Markov property* of the random walk: for any integer $t \geq 0$, the random walk $S_t - S_t, S_{t+1} - S_t, S_{t+2} - S_t, \ldots$ has identical behavior to the walk S_0, S_1, S_2 and is independent of $S_0, S_1, \ldots, S_t$. (Informally, if we have examined the behavior of the walk up to time S, we may think of *restarting* the random walk at time t, starting from a height of S_t; this will be important in the generalized ballot theorems to be presented later in the paper.) This proof can be rewritten in terms of *lattice paths* by letting votes for A be unit steps in the positive x-direction and votes for B be unit steps in the positive y-direction. When conceived of in this manner, this proof immediately yields several natural generalizations (Dvoretzky and Motzkin, 1947; Grossman, 1950; Mohanty, 1966).

Starting in 1962, Lajos Takács proved a sequence of increasingly general ballot-style results and statements about the distribution of the maxima when the ballot is viewed as a random walk (Takács, 1962a,b,c, 1963, 1964a,b, 1967). We highlight two of these theorems below; we have not chosen the most general statements possible, but rather theorems which we believe capture key properties of ballot-style results.

A family of random variables $X_i, \ldots, X_n$ is *interchangeable* if for all $(r_1, \ldots, r_n) \in \mathbb{R}^n$ and all permutations σ of $\{1,\ldots,n\}$, $\mathbf{P}\{X_i \le r_i \forall 1 \le i \le n\} = \mathbf{P}\left\{X_i \le r_{\sigma(i)} \forall 1 \le i \le n\right\}$. We say $X_1,\ldots,X_n$ is *cyclically interchangeable* if this equality holds for all *cyclic* permutations σ. A family of interchangeable random variables is cyclically interchangeable, but the converse is not always true. The first theorem states:

Theorem 2. *Suppose that $X_1,\ldots,X_n$ are integer-valued, cyclically interchangeable random variables with maximum value 1, and for $1 \le i \le n$, let $S_i = X_1 + \ldots + X_i$. Then for any integer $0 \le k \le n$,*

$$\mathbf{P}\{S_i > 0 \ \forall 1 \le i \le n | S_n = k\} = \frac{k}{n}.$$

This theorem was proved independently by Tanner (1961) and Dwass (1962) – we note that it can also be proved by Dvoretzky and Motzkin's approach. (As a point of historical curiosity, Takács' proof of this result in the special case of interchangeable random variables, and Dwass' proof of the more general result above, appeared in the same issue of Annals of Mathematical Statistics.) Theorem 2 and the "bad run" proof of Barbier's ballot theorem both suggest that the notion of cyclic interchangeability or something similar may lie at the heart of all ballot-style results.

Theorem 3 (Takács (1967), p. 12)**.** *Let $X_1, X_2, \ldots$ be an infinite sequence of iid integer random variables with mean μ and maximum value 1 and for any $i \ge 1$, let $S_i = X_1 + \ldots + X_i$. Then*

$$\mathbf{P}\{S_n > 0 \text{ for } n = 1,2,\ldots\} = \begin{cases} \mu & \text{if } \mu > 0, \\ 0 & \text{if } \mu \le 0. \end{cases}$$

The proof of Theorem 3 proceeds by applying Theorem 2 to finite subsequences $X_1, X_2, \ldots, X_n$, so this theorem also seems to be based on cyclic interchangeability. Takács has generalized these theorems even further, proving similar statements for step functions with countably many discontinuities and in many cases finding the exact distribution of $\max_{i=1}^n (S_i - i)$.

(Takács originally stated his results in terms of non-negative integer random variables – his original formulation results if we consider the variables $(1 - X_1), (1 - X_2), \ldots$ and the corresponding random walk.) Theorem 3 implies the following classical result about the probability of ever returning to zero in a biased simple random walk:

Theorem 4 (Feller (1968), p. 274)**.** *In a biased simple random walk $0 = R_0, R_1, \ldots$ in which $\mathbf{P}\{R_{i+1} - R_i = 1\} = p > 1/2$, $\mathbf{P}\{R_{i+1} - R_i = -1\} = 1 - p$, the probability that there is no $n \ge 1$ for which $R_n = 0$ is $2p - 1$.*

Proof. Observe that the expected value of $R_i - R_{i-1}$ is $2p - 1 > 0$, so if $R_1 = -1$ then with probability 1, $R_i = 0$ for some $i \ge 2$. Thus,

$$\mathbf{P}\{R_n \ne 0 \text{ for all } n \ge 1\} = \mathbf{P}\{R_n > 0 \text{ for all } n \ge 1\}.$$

The latter probability is equal to $2p - 1$ by Theorem 3. $\square$

We close this section by presenting the beautiful "reflection principle" proof of Bertrand's theorem. We think of representing the symmetric simple random walk as a sequence of points $(0, S_0), (1, S_1), \ldots, (n, S_n)$ and connecting neighbouring points. If $S_1 = 1$ and the walk is unfavourable, then letting k be the smallest value for which $S_k = 0$ and "reflecting" the random walk $S_0, \ldots, S_k$ in the x-axis yields a walk from $(0, 0)$ to (n, t) whose first step is negative – this is shown in Figure 2. This yields a bijection between walks that are unfavourable for A and start with a positive step, and walks that are unfavourable for A and start with a negative step. As all walks starting with a negative step are unfavourable for A, all that remains is rote calculation. This proof is often incorrectly attributed to André (1887), who established the same bijection in a different way - its true origins remain unknown.

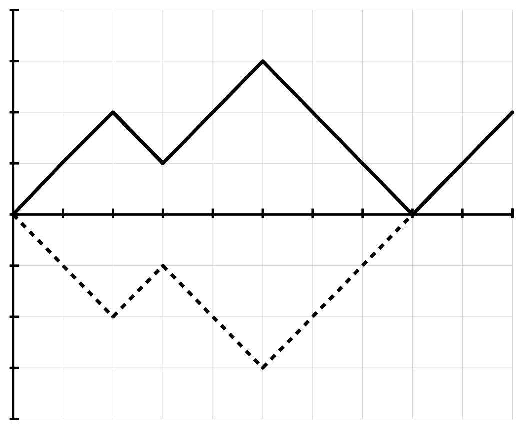

Figure 2: The dashed line is the reflection of the random walk from (0,0) to the first visit of the x-axis.

1.2 Continuous time ballot theorems

The theorems which follow are natural given the results presented in Section 1.1; however, their statements require slightly more preliminaries. A *stochastic process* is simply a nonempty set of real numbers T and a collection of random variables $\{X_t, t \in T\}$ defined on some probability space. The collection of random variables $\{X_1, \ldots, X_n\}$ seen in Section 1.1 is an example of a stochastic process for which $T = \{1, 2, \ldots, n\}$. In this section we are concerned with stochastic processes for which $T = [0, r]$ for some $0 < r < \infty$ or else $T = [0, \infty)$.

A stochastic process $\{X_t, 0 \leq t \leq r\}$ has *(cyclically) interchangeable increments* if for all $n = 2, 3, \ldots$, the finite collection of random variables $\{X_{rt/n} - X_{r(t-1)/n}, t = 1, 2, \ldots, n\}$ is (cyclically) intechangeable. A process $\{X_t, t \geq 0\}$ has *interchangeable increments* if for all $r > 0$ and $n > 0$, $\{X_{rt/n} - X_{r(t-1)/n}, t = 1, 2, \ldots, n\}$ is interchangeable, and is *stationary* if this latter collection is composed of independent identically distributed random variables. As in the discrete case, these are natural sorts of prerequisites for a ballot-style theorem to apply.

There is an unfortunate technical restriction which applies to all the ballot-style results we will see in this section. The stochastic process $\{X_t, t \in T\}$ is said to be *separable* if there are almost-everywhere-unique measurable functions X^+, X_- such that almost surely $X_- \leq X_t \leq X^+$ for all $t \in T$, and there are countable subsets S_-, S^+ of T such that almost surely $X^+ = \sup_{t \in S^+} X_t$ and $X_- = \inf_{t \in S_-} X_t$. The results of this section only hold for separable stochastic processes. In defense of the results, we note that there are nonseparable stochastic processes $\{X_t, 0 \leq t \leq r\}$ for which $\sup\{X_t - t, 0 \leq t \leq r\}$ is non-measurable. As the distribution of this random variable is one of the key issues with which we are concerned, the assumption of separability is natural and in some sense necessary in order for the results to be meaningful. Moreover, in very general settings it is possible to construct a separable stochastic process $\{Y_t | t \in T\}$ such that for all $t \in T$, Y_t and X_t are almost surely equal (see, e.g., Gikhman and Skorokhod, 1969, Sec.IV.2); in this case it can be fairly said that the assumption of separability is no loss.

The following theorem is the first example of a continuous-time ballot theorem. A *sample function* of a stochastic process is a function $x_\omega : T \to \mathbb{R}$ given by fixing some $\omega \in \Omega$ and letting $x_\omega(t) = X_t(\omega)$.

Theorem 5 (Takács (1965a)). *If $\{X_t, 0 \leq t \leq r\}$ is a separable stochastic process with cyclically interchangeable increments whose sample functions are almost surely nondecreasing step functions, then*

$$\mathbf{P}\left\{X_t - X_0 \leq t \text{ for } 0 \leq t \leq r | X_r - X_0 = s\right\} = \begin{cases} \frac{t-s}{t} & \text{if } 0 \leq s \leq t, \\ 0 & \text{otherwise.} \end{cases}$$

This theorem is a natural continuous equivalent of Theorem 2 of Section 1.1; it can be used to prove a theorem in the vein of Theorem 3 which applies to stochastic processes $\{X_t, t \geq 0\}$. Takács' other ballot-style results for continuous stochastic processes are also essentially step-by-step extensions of his results from the discrete setting; see Takács (1965a,b, 1967, 1970b).

In 1957, Baxter and Donsker derived a double integral representation for $\sup\{X_t - t, t \geq 0\}$ when this process has stationary independent increments. Their proof proceeds by analyzing the zeros of a complex-valued function associated to the process. They are able to use their representation to explicitly derive its distribution when the process is a Gaussian process, a coin-tossing process, or a Poisson process. This result was rediscovered by Takács (1970a), who also derived the joint distribution of X_r and $\sup\{X_t - t, 0 \leq t \leq r\}$ for r finite, using a generating function approach. Though these results are clearly related to the continuous ballot theorems, they are not as elegant, and neither their statements nor their proofs bring to mind the ballot theorem. It seems that considering separable stationary processes in their full generality does not impose enough structure for it to be possible to prove these results via straightforward generalization of the discrete equivalents.

A beautiful perspective on the ballot theorem appears by considering *random measures* instead of stochastic processes. Given an almost surely nondecreasing separable stochastic process $\{X_t, 0 \leq t \leq r\}$, fixing any element ω of the underlying probability space Ω yields a sample function x_ω. By our assumptions on the stochastic process, almost every sample function x_ω yields a measure μ_ω on $[0, r]$, where $\mu_\omega[0, b] = x_\omega(b) - x_\omega(a)$. This allows us to define a "random" measure μ on $[0, r]$; μ is a function with domain Ω, $\mu(\omega) = \mu_\omega$, and for almost all $\omega \in \Omega$, $\mu(\omega)$ is a measure on $[0, r]$. If x_ω is a nondecreasing step function, then μ_ω has countable support, so is *singular* with respect to the Lebesgue measure (i.e. the set of points which have positive μ_ω-measure has *Lebesgue* measure 0); if this holds for almost all ω then μ is an "almost surely singular" random measure.

We have just seen an example of a random measure; we now turn to a more precise definition. Given a probability space $\mathcal{S} = (\Omega, \Sigma, \mathbf{P})$, a random measure on a possibly infinite interval $T \subset \mathbb{R}$ is a function μ with domain $\Omega \times T$ satisfying that for all $r \in T$, $\mu(\cdot, r)$ is a random variable in $\mathcal{S}$, and for almost all $\omega \in \Omega$, $\mu(\omega, \cdot)$ is a measure on T. A random measure μ is almost surely singular if for almost all $\omega \in \Omega$, $\mu(\omega, \cdot)$ is a measure on T singular with respect to the Lebesgue measure. (This definition hides some technicality; in particular, for the definition to be useful it is key that the set of ω for which μ is singular is itself a measurable set! See Kallenberg (1999) for details.) A random measure μ on $\mathbb{R}^+$, say, is stationary if for all t, letting $X_{t,i} = \mu(\cdot, (i+1)/t) - \mu(\cdot, i/t)$, the family $\{X_{t,i} | i \in \mathbb{N}\}$ is composed of independent identically distributed random variables; stationarity for finite intervals is defined similarly.

This perspective can be used to generalize Theorem 5. Konstantopoulos (1995) has done so, as well as providing a beautiful proof using a continuous analog of the reflection principle. The most powerful theorem along these lines to date is due to Kallenberg. To a given stationary random measure μ defined on $T \subseteq \mathbb{R}^+$ we associate a random variable I called the *sample intensity* of μ. (Intuitively, I is the random average number of points in an arbitrary measurable set $B \subset T$ of positive finite measure, normalized by the measure of B. For a formal definition, see (Kallenberg, 2003, p. 189).)

Theorem 6 (Kallenberg (1999))**.** *Let μ be an almost surely singular, stationary random measure on $T = \mathbb{R}^+$ or $T = (0, 1]$ with sample intensity I and let $X_t = \mu(\cdot, t) - \mu(\cdot, 0)$ for $t \in T$. Then there exists a uniform $[0, 1]$ random variable U independent from I such that*

$$\sup_{t \in T} \frac{X_t}{t} = \frac{I}{U} \qquad \textit{almost surely.}$$

It turns out that if $T = (0, 1]$ then conditional upon the event that $X_1 = m$, $I = m$ almost surely. It follows that in this case $\mathbf{P}\left\{\sup_{t \in T} \frac{X_t}{t} \leq 1 | X_1\right\} = \max\{1 - X_1, 0\}$. Similarly, if $T = \mathbb{R}^+$ and $\frac{X_t}{t} \to m$ almost surely as $t \to \infty$, then $I = m$ almost surely, so in this case $\mathbf{P}\left\{\sup_{t \in T} \frac{X_t}{t} \leq 1\right\} = \max\{1 - m, 0\}$. This theorem can thus be seen to include continuous generalizations of both Theorem 2 and Theorem 3.

Kallenberg has also proved the following as a corollary of Theorem 6 (this is a slight reformulation of his original statement, which applied to infinite sequences):

Theorem 7. *If X is a real random variable with maximum value 1 and $\{X_1, X_2, \dots, X_n\}$ are iid copies of X with corresponding partial sums $\{0 = S_0, S_1, \dots, S_n\}$, then*

$$\mathbf{P}\left\{S_i > 0 \forall 1 \leq i \leq n | S_n\right\} \geq \frac{S_n}{n}.$$

It is worth comparing this theorem with Theorem 2; the theorems are almost identical, but Theorem 7 relaxes the integrality restriction at the cost of replacing the equality of Theorem 2 with an inequality.

1.3 Outline

To date, Theorem 7 is the only ballot-style result which has been proved for random walks that may take non-integer values. Paraphrasing Harry Kesten (1993), the goal of our research is to move towards making ballot theorems part of "the general theory of random walks" – part of the body of results that hold for *all* random walks (with independent identically distributed steps), regardless of the precise distribution of their steps. We succeed in proving ballot-style theorems that hold for a broad class of random walks, including all random walks that can be renormalized to converge in distribution to a normal random variable. A *truly* general ballot theorem, however, remains beyond our grasp.

In Section 2 we discuss in what sense existing ballot theorems such as those presented in Section 1 are optimal, and what sorts of "general ballot theorems" it makes sense to search for in light of this optimality. In Section 3 we demonstrate our approach in a restricted setting and prove a weakening of our main result. This allows us to highlight the key ideas behind our general ballot theorems without too much notational and technical burden. In Section 4, we sketch the main ideas required to strengthen the presented result. Finally, in Section 5 we address the limits of our approach and suggest some avenues for future research.

2 General ballot theorems

The aim of our research is to prove analogs of the discrete-time ballot theorems of Section 1 for more general random variables. The Theorems of Section 1.1 all have two restrictions: (1) they apply only to integer-valued random variables, and (2) they apply only to random variables that are bounded from one or both sides. (In the continuous-time setting, the restriction that the stochastic processes are almost surely integer-valued, increasing step functions is of the same flavour.) In this section we investigate what ballot-style theorems can be proved when such restrictions are removed.

The restrictions (1) and (2) are necessary for the results of Section 1.1 to hold. Suppose, for example, that we relax the condition of Theorem 2 requiring that the variables are bounded from above by $+1$. If X takes every value in $\mathbb{N}$ with positive probability, then $\mathbf{P}\{S_i > 0 \forall 1 \leq i \leq n | S_n = n\} < 1$, so the conclusion of the theorem fails to hold. For a more explicit example, let X be any random variable taking values $\pm 1, \pm 4$ and define the corresponding cyclically interchangeable sequence and random walk. For $S_3 = 2$ to occur, we must have $\{X_1, X_2, X_3\} = \{4, -1, -1\}$. In this case, for $S_i > 0$, $i = 1, 2, 3$ to occur, X_1 must equal 4. By cyclic interchangeability, this occurs with probability 1/3, and not 2/3, as Theorem 2 would suggest. This shows that the boundedness condition (2) is required. If we relax the integrality condition (1), we can construct a similar example where the conclusions of Theorem 2 do not hold.

Since the results of Section 1.1 can not be directly generalized to a broader class of random variables, we seek conditions on the distribution of X so that the bounds of that section have the correct order, i.e., so that $\mathbf{P}\{S_i > 0 \ \forall \ 1 \leq i \leq n | S_n = k\} = \Theta(k/n)$. (When we consider random variables that are not necessarily integer-valued, the right conditioning will in fact be on an event such as $\{k \leq S_n < k + 1\}$ or something similar.) How close we can come to this conclusion will depend on what restrictions on X we are willing to accept. It turns out that a statement of this flavour holds for the *mean zero* random walk $S_n^0 = S_n - n\mathbf{E}X$ as long as there is a sequence $\{a_n\}_{n \geq 0}$ for which $(S_n - n\mathbf{E}X)/a_n$ converges to a non-degenerate normal distribution (in this case, we say that X is in the *range of attraction of the normal distribution* and write $X \in \mathcal{D}$; for example, the classical central limit theorem states that if $\mathbf{E}\{X^2\} < \infty$ then we may take $a_n = \sqrt{n}$ for all n.) For the purposes of this expository article, however, we shall impose a slightly stronger condition than that stated above.

From this point on, we restrict our attention to sums of mean zero random variables. We note this condition is in some sense necessary in order for the results we are hoping for to hold. If $\mathbf{E}X \neq 0$ – say $\mathbf{E}X > 0$ – then it is possible that X is non-negative, so the only way for $S_n = 0$ to occur is that $X_1 = \ldots = X_n = 0$, and so $\mathbf{P}\{S_i > 0 \ \forall \ 1 \leq i \leq n | S_n = 0\} = 0$, and not $\Theta(1/n)$ as we would hope from the results of Section 1.

3 Ballot theorems for closely fought elections

One of the most basic questions a ballot theorem can be said to answer is: given that an election resulted in a *tie*, what is the probability that one of the candidates had the lead at every point aside from the very beginning and the very end. In the language of random walks, the question is: given that $S_n = 0$, what is the probability that S does not return to 0 or change sign between time 0 and time n? Erik Sparre Andersen has studied the conditional behavior of random walks given that $S_n = 0$ in great detail, in particular deriving beautiful results on the distribution of the maximum, the minimum, and the amount of time spent above zero. Much of the next five paragraphs can be found in Andersen (1953), for example, in slightly altered terminology.

We call the event that S_n does not return to zero or change sign before time n, $Lead_n$. We can easily bound $\mathbf{P}\{Lead_n | S_n = 0\}$ using the fact that $X_1, \ldots, X_n$ are interchangeable. If we condition on the multiset of outcomes $\{X_1, \ldots, X_n\} = \{x_{\sigma(1)}, \ldots, x_{\sigma(n)}\}$, and then choose a uniformly random cyclic permutation σ and a uniform element i of $\{1, \ldots, n\}$, then the interchangeability of $X_1, \ldots, X_n$ implies that $(x_{\sigma(i)}, \ldots, x_{\sigma(n)}, x_{\sigma(1)}, \ldots, x_{\sigma(i-1)})$ has the same distribution as if we had sampled directly from $(X_1, \ldots, X_n)$.

Letting $s_j = \sum_{k=1}^{j-1} x_{\sigma(k)}$, in order for $Lead_n$ to occur given that $S_n = 0$, it must be the case that s_i is either the unique maximum or the unique minimum among $\{s_1, \ldots, s_n\}$. The probability that this occurs is at most $2/n$ as it is exactly $2/n$ if there are unique maxima and minima, and less if either the maximum or minimum is not unique. Therefore,

$$\mathbf{P}\{Lead_n | S_n = 0\} \leq \frac{2}{n}. \tag{1}$$

On the other hand, the sequence certainly has *some* maximum (resp. minimum) s_i, and if $X_1 = x_i$ then S_j is always non-positive (resp. non-negative). Denoting this event by $Nonpos_n$ (resp. $Nonneg_n$), we therefore have

$$\mathbf{P}\{Nonpos_n | S_n = 0\} \geq \frac{1}{n} \quad \text{and} \quad \mathbf{P}\{Nonneg_n | S_n = 0\} \geq \frac{1}{n} \tag{2}$$

If $S_n = 0$ then the $(n-1)$ renormalized random variables given by $X'_i = X_{i+1} + X_1/(n-1)$ satisfy $(n-1)S'_{n-1} = (n-1)\sum_{i=1}^{n-1} X'_i = (n-1)\sum_{i=1}^{n} X_i = 0$. If $X_1 > 0$ and none of the *renormalized* partial sums are negative, then $Lead_n$ occurs. The renormalized random variables are still interchangeable (see Andersen (1953, Lemma 2) for a proof of this easy fact), so we may apply the second bound of (2) to obtain

$$\mathbf{P}\{Lead_n | S_n = 0, X_1 > 0\} \geq \frac{1}{n-1}.$$

An identical argument yields the same bound for $\mathbf{P}\{Lead_n | S_n = 0, X_1 < 0\}$, and combining these bounds yields

$$\begin{aligned} \mathbf{P}\{Lead_n | S_n = 0\} &\geq \mathbf{P}\{Lead_n | S_n = 0, X_1 \neq 0\} \mathbf{P}\{X_1 \neq 0 | S_n = 0\} \\ &\geq \frac{1 - \mathbf{P}\{X_1 = 0 | S_n = 0\}}{n-1}. \end{aligned}$$

As long as $\mathbf{P}\{X_1 = 0 | S_n = 0\} < 1$, this yields that $\mathbf{P}\{Lead_n | S_n = 0\} \geq \alpha/n$ for some $\alpha > 0$. By interchangeability, it is easy to see that $\mathbf{P}\{X_1 = 0 | S_n = 0\}$ is bounded uniformly away from 1 for large n, as long as $S_n = 0$ does not imply that $X_1 = \ldots = X_n = 0$ almost surely. (Note, however, that there *are* cases where $\mathbf{P}\{X_1 = 0 | S_n = 0\} = 1$, for example if the X_i only take values in the non-negative integers and in the negative multiples of $\sqrt{2}$.)

Sparre Andersen's approach gives a necessary and sufficient, though not terribly explicit, condition for $\mathbf{P}\{Lead_n | S_n = 0\} = \Theta(1/n)$ to hold. Philosophically, in order to make ballot theorems part of the "general theory of random walks", we would like necessary and sufficient conditions on the distribution of X_1 for $\mathbf{P}\{Lead_n | S_n = k\} = \Theta(k/n)$ for *all* $k = O(n)$. Even more generally, we may ask: what are sufficient conditions on the structure of a multiset S

of n numbers to ensure that if the elements of the multiset sum to k, then in a uniformly random permutation of the set, all partial sums are positive with probability of order k/n? In the remainder of the section, we focus our attention on sets S whose elements are sampled independently from a mean-zero probability distribution, i.e., they are the steps of a mean-zero random walk. (We remark that it is possible to apply parts of our analysis to sets S that do not obey this restriction, but we will not pursue such an investigation here.) We will derive *sufficient* conditions for such bounds to hold in the case that $k = O(\sqrt{n})$; it turns out that for our approach to succeed it suffices that the step size X is in the range of attraction of the normal distribution, though our best result requires slightly stronger moment conditions on X than those of the classical central limit theorem.

Before stating our generalized ballot theorems, we need one additional definition. We say a variable X has *period* $d > 0$ if dX is an integer random variable and d is the smallest positive real number for which this holds; in this case X is called a *lattice* random variable, otherwise X is *non-lattice*. We can prove the following:

Theorem 8. *Suppose X satisfies $\mathbf{E}X = 0$, $\mathbf{Var}\{X\} > 0$, $\mathbf{E}\{X^{2+\alpha}\} < \infty$ for some $\alpha > 0$, and X is non-lattice. Then for any fixed $A > 0$, given independent random variables $X_1, X_2, \ldots$ distributed as X with associated partial sums $S_i = \sum_{j=1}^{i} X_j$, for all k such that $0 \le k = O(\sqrt{n})$,*

$$\mathbf{P}\{k \le S_n \le k + A, S_i > 0 \;\forall\; 0 < i < n\} = \Theta\left(\frac{k+1}{n^{3/2}}\right).$$

Theorem 9. *Suppose X satisfies $\mathbf{E}X = 0$, $\mathbf{Var}\{X\} > 0$, $\mathbf{E}\{X^{2+\alpha}\} < \infty$ for some $\alpha > 0$, and X is a lattice random variable with period d. Then given independent random variables $X_1, X_2, \ldots$ distributed as X with associated partial sums $S_i = \sum_{j=1}^{i} X_j$, for all k such that $0 \le k = O(\sqrt{n})$ and such that k is a multiple of $1/d$,*

$$\mathbf{P}\{S_n = k, S_i > 0 \;\forall\; 0 < i < n\} = \Theta\left(\frac{k+1}{n^{3/2}}\right).$$

From these theorems, we may derive "true" (conditional) ballot theorems as corollaries, at least in the case that $k = O(\sqrt{n})$. The following result was proved by Stone (1965), and is the tip of an iceberg of related results. Let Φ be the density function a $\mathcal{N}(0, 1)$ random variable.

Theorem 10. *Suppose S_n is a sum of independent, identically distributed random variables distributed as X with $\mathbf{E}X = 0$, and there is a constant a such that $S_n/a\sqrt{n}$ converges to a $\mathcal{N}(0, 1)$ random variable. If X is non-lattice let B be any bounded set; then for any $h \in B$ and $x \in R$*

$$\mathbf{P}\{|S_n - x| \le h/2\} = \frac{h\Phi(x/a\sqrt{n})}{a\sqrt{n}} + o(1/\sqrt{n}).$$

Furthermore, if X is a lattice random variable with period d, then for any $x \in \{n/d \mid n \in \mathbb{Z}\}$,

$$\mathbf{P}\{S_n = x\} = \frac{\Phi(x/a\sqrt{n})}{a\sqrt{n}} + o(1/\sqrt{n}).$$

In both cases, $\sqrt{n}o(1/\sqrt{n}) \to 0$ as $n \to \infty$ uniformly over all $x \in \mathbb{R}$ and $h \in B$.

Together with Theorem 8 this immediately yields:

Corollary 11. *Under the conditions of Theorem 8,*

$$\mathbf{P}\left\{S_i > 0 \ \forall\ 0 < i < n | k \leq S_n \leq k + A\right\} = \Theta\left(\frac{k+1}{n}\right).$$

Similarly, combining Theorem 9 with Theorem 10, we have

Corollary 12. *Under the conditions of Theorem 9,*

$$\mathbf{P}\left\{S_i > 0 \ \forall\ 0 < i < n | S_n = k\right\} = \Theta\left(\frac{k+1}{n}\right).$$

As we remarked above, the approach we are about to sketch can also be used to prove a ballot theorem under the weaker restriction that X is in the range of attraction of the normal distribution, at the cost of replacing the bound $\Theta\left(\frac{k+1}{n}\right)$ by the bound $\frac{k+1}{n^{1+o(1)}}$; for the sake of brevity and clarity we will not discuss the rather minor modifications to our approach that are needed to handle this case. Furthermore, for the purposes of this expository article, we shall not prove Theorems 8 or 9 in their full generality or strength, instead restricting our attention to a special case which allows us to highlight the key elements of our proofs. Finally, we shall provide a detailed explanation of only the upper bound, after which we shall briefly discuss our approach to the lower bound. We will prove:

Theorem 13. *Suppose X satisfies $\mathbf{E}X = 0$, $\mathbf{Var}\left\{X\right\} > 0$, $|X| < C$, and X is non-lattice. Then for any fixed $A > 0$, given independent random variables $X_1, X_2, \ldots$, distributed as X with associated partial sums $S_i = \sum_{j=1}^{i} X_j$, for all $0 \leq k = o(\sqrt{n/\log n})$,*

$$\mathbf{P}\left\{k \leq S_n \leq k + A, S_i > 0 \ \forall\ 0 < i < n\right\} = O\left(\frac{(k+1)\log n}{n^{3/2}}\right).$$

Of course, a conditional ballot theorem that is correspondingly weaker than Corollary 11 follows by combining Theorem 13 with Theorem 10. We remark that in cases where Theorem 7 applies, it provides a lower bound on $\mathbf{P}\left\{k \leq S_n \leq k + A, S_i > 0 \ \forall\ 0 < i < n\right\}$ of the same order as the upper bound of Theorem 13. *From this point forward, X will always be a random variable satisfying the conditions in Theorem 13, and $X_1, X_2, \ldots,$ will be independent copies of X with corresponding partial sums $S_1, S_2, \ldots$.*

To begin providing an intuition of our approach, we first remark that if $S_i > 0 \ \forall 0 < i < n$ is to occur, then for any r, letting T be the first time $t \geq 1$ that $S_t > r$ or $S_t \leq 0$, we have either $S_T > 0$ or $T > n$. (We will end up choosing the value r so that $T = o(n)$ except with negligibly small probability, so to bound the previous probability we shall essentially need to bound the probability that $S_T > 0$, i.e., that the walk "stays positive". We will see shortly that Wald's identity implies that $\mathbf{P}\{S_T > 0\} = O(1/r)$.

We may impose a similar constraint on the "other end" of the random walk S, by letting S' be the *negative reversed* random walk given by $S'_0 = 0$, and for $i > 0$, $S'_{i+1} = S'_i - X_{n-i}$ (it will be useful to think of S'_i as being defined even for $i > n$, which we may do by letting $X_0, X_{-1}, \ldots$ be independent copies of X). If $S_i > 0\ \forall 0 < i < n$ and $k \leq S_n \leq k + C$ are to occur, then letting T' be the first time t that $S'_t \leq -(k+A)$ or $S'_t > r - (k+A)$, it must be the case that either $S'_{T'} > 0$ or $T' > n$. (Again, we will choose r so that $T' = o(n)$ with extremely high probability.)

Finally, in order for $k \leq S_n \leq k + A$ to occur, the two ends of the random walk must "match up". We may make this mathematically precise by noting that as long as $T < n - T'$, we may write S_n as $S_T + (S_{n-T'} - S_T) - S'_{T'}$, and may thus write the condition $k \leq S_n \leq k + A$ as

$$k + S'_{T'} - S_T \leq (S_{n-T'} - S_T) \leq k + A + S'_{T'} - S_T.$$

If $T + T'$ is at most $n/2$, say, then $S_{n-T'} - S_T$ is the sum of at least $n/2$ random variables. In this case, the classical central limit theorem suggests that $S_{n-T'} - S_T$ should "spread itself out" over a range of order $\sqrt{n}$, and essentially this fact will allow us to show that the two ends "meet up" with probability $O(1/\sqrt{n})$.

3.1 Staying positive

To begin formalizing the above sketch, let us first turn to bounds on the probabilities of the events $S_T > 0$ and $S'_{T'} > 0$.

Lemma 14. *Fix $r > 0$ and $s \geq 0$, and let $T_{r,s}$ be the first time $t > 0$ that either $S_t > r$ or $S_t \leq -s$. Then $\mathbf{P}\left\{S_{T_{r,s}} > 0\right\} \leq (s+C)/(r+s+C)$.*

Proof. We first remark that $\mathbf{E}T_{r,s}$ is finite; this is a standard result that can be found in, e.g., (Feller, 1968, Chapter 14.4), and we shall also rederive this result a little later. Thus, by Wald's identity, we have that $\mathbf{E}S_{T_{r,s}} = \mathbf{E}T_{r,s}\mathbf{E}X_1 = 0$, and letting Pos_r denote the event $\{S_{T_{r,s}} > 0\}$; we may therefore write

$$0 = \mathbf{E}S_{T_{r,s}} = \mathbf{E}\left\{S_{T_{r,s}} | Pos_r\right\}\mathbf{P}\left\{Pos_r\right\} + \mathbf{E}\left\{S_{T_{r,s}} | \overline{Pos_r}\right\}\mathbf{P}\left\{\overline{Pos_r}\right\}. \tag{3}$$

By definition $\mathbf{E}\{S_T | Pos_r\} \geq r$, and by our assumption that X has absolute value at most C, we have $\mathbf{E}\left\{S_T | \overline{Pos_r}\right\} \geq -(s+C)$. Therefore

$$0 \geq r\mathbf{P}\{Pos_r\} - (s+C)\mathbf{P}\left\{\overline{Pos_r}\right\} = r\mathbf{P}\{Pos_r\} - (s+C)(1 - \mathbf{P}\{Pos_r\}),$$

and rearranging the latter inequality yields that $\mathbf{P}\{Pos_r\} \leq (s+C)/(r+s+C)$. □

As an aside, we note that may easily derive a lower bound of the same order for $\mathbf{P}\{Pos_r\}$ in a similar fashion; we first observe that $\mathbf{E}\left\{S_{T_{r,s}}|Pos_r\right\} < r+C$. Similarly, $\mathbf{E}\left\{S_{T_{r,s}}|\overline{Pos_r}\right\} \leq -s$, and using the fact that X has zero mean and positive variance, it is also easy to see that there is $\epsilon > 0$ such that in fact $\mathbf{E}\left\{S_{T_{r,s}}|\overline{Pos_r}\right\} \leq -\max\{\epsilon, s\}$. Combining (3) these two bounds, we thus have

$$0 < (r+C)\mathbf{P}\{Pos_r\} - \max\{\epsilon, s\}\mathbf{P}\left\{\overline{Pos_r}\right\} = (r+C)\mathbf{P}\{Pos_r\} - \max\{\epsilon, s\}(1-\mathbf{P}\{Pos_r\}),$$

so $\mathbf{P}\{Pos_r\} \geq \max\{\epsilon, s\}/(r+C+\max\{\epsilon, s\})$. Lemma 14 immediately yields the bounds we require for $\mathbf{P}\{S_T > 0\}$ and $\mathbf{P}\{S'_{T'} > 0\}$; next we show that for a suitable choice of r, with extremely high probability, both T and T' are $o(n)$.

3.2 The time to exit a strip.

For $r \geq 0$, we consider the first time t for which $|S_t| \geq r$, denoting this time T_r. We prove

Lemma 15. *There is $B > 0$ such that for all $r \geq 1$, $\mathbf{E}T_r \leq Br^2$ and for all integers $k \geq 1$, $\mathbf{P}\{T_r \geq kBr^2\} \leq 1/2^k$.*

This is an easy consequence of a classical result on how "spread out" sums of independent identically distributed random variables become (which we will also use later when bounding the probability that the two ends of the random walk "match up"). The version we present can be found in Kesten (1972):

Theorem 16. *For any family of independent identically distributed real random variables $X_1, X_2, \ldots$ with positive, possibly infinite variance and associated partial sums $S_1, S_2, \ldots$, there is a constant c depending only on the distribution of X_1 such that for all n,*

$$\sup_{x\in\mathbb{R}} \mathbf{P}\{x \leq S_n \leq x+1\} \leq c/\sqrt{n}.$$

Proof of Lemma 15. Observe that the expectation bound follows directly from the probability bound, since if the probability bound holds then we have

$$\mathbf{E}T_r \leq \sum_{j=0}^{\infty} \mathbf{P}\{T_r \geq j\} \leq \sum_{i=0}^{\infty} \lceil Br^2 \rceil \mathbf{P}\left\{T_r > i\lceil Br^2\rceil\right\} \leq \sum_{i=0}^{\infty} \frac{\lceil Br^2\rceil}{2^i} = 2\lceil Br^2\rceil,$$

which establishes the expectation bound with a slightly changed value of B. It thus remains to prove the probability bound. By Theorem 16, there is $c > 0$ (and we can and will assume $c > 1$) such that

$$\begin{aligned} \mathbf{P}\left\{|S_{\lceil 128c^2r^2\rceil}| \leq 2r\right\} &\leq \sum_{i=\lfloor -2r\rfloor}^{\lfloor 2r\rfloor} \mathbf{P}\left\{i \leq S_{\lceil 128c^2r^2\rceil} \leq i+1\right\} \\ &\leq (4r+1)\frac{c}{\sqrt{\lceil 128c^2r^2\rceil}} < \frac{1}{2}, \end{aligned} \tag{4}$$

the last inequality holding as $c > 1$ and $r > 1$. Let $t^* = \lceil 128c^2r^2 \rceil$ - then $\mathbf{P}\{T_r > t^*\} \leq 1/2$. We use this fact to show that for any positive integer k, $\mathbf{P}\{T_r > kt^*\} \leq 1/2^k$, which will establish the claim with $B = 128c^2 + 1$, for example. We proceed by induction on k, having just proved the claim for $k = 1$. We have

$$\begin{aligned}
\mathbf{P}\{T_r > (k+1)t^*\} &= \mathbf{P}\{T_r > (k+1)t^* \cap T > kt\} \\
&= \mathbf{P}\{T_r > (k+1)t^* | T_r > kt^*\} \mathbf{P}\{T_r > kt\} \\
&= \frac{1}{2^k} \cdot \mathbf{P}\{T_r > (k+1)t^* | T_r > kt^*\},
\end{aligned}$$

by induction. It remains to show that $\mathbf{P}\{T_r > (k+1)t^* | T_r > kt^*\} \leq 1/2$. If $T_r > kt^*$ then by the strong Markov property we may think of restarting the random walk at time kt^*. Whatever the value of S_{kt^*}, if the restarted random walk exits $[-2r, 2r]$ then the original random walk exits $[-r, r]$, so this inequality holds by (4). This proves the lemma. $\square$

This bound on the time to exit a strip is the last ingredient we need; we now turn to the proof of Theorem 13.

3.3 Proof of Theorem 13

Fix $A > 0$ as in the statement of the theorem. For $r \geq 1$ we denote by T_r the first time t that $|S_t| \geq r$. We let S' be the negative reversed random walk given by $S'_0 = 0$, and for $i > 0$, $S'_{i+1} = S'_i - X_{n-i}$ (again as above, we define S'_i for $i > n$ by letting $X_0, X_{-1}, \ldots$ be independent copies of X), and let T'_r be the first time t that $|S'_t| \geq r$. We choose B such that for all $r \geq 1$ and and for all integers $k \geq 1$, $\mathbf{P}\{T_r \geq kBr^2\} \leq 1/2^k$ and $\mathbf{P}\{T'_r \geq kBr^2\} \leq 1/2^k$ – such a choice exists by Lemma 15.

Choose $r^* = \lfloor \sqrt{n/9B\log n} \rfloor$ – then with $k = \lceil 2\log n \rceil < 2\log n + 1$, it is the case that

$$kB(r^*)^2 \leq \frac{kBn}{9B\log n} < \frac{(2\log n + 1)n}{9\log n} < \frac{n}{4},$$

so $\mathbf{P}\{T_{r^*} \geq n/4\} \leq 1/2^k \leq 1/n^2$, and similarly $\mathbf{P}\{T'_{r^*} \geq n/4\} \leq 1/n^2$.

Next let T be the first time t that $S_t > r^*$ or $S_t \leq 0$, and let T' be the first time t that $S'_t > r^* - (k + A)$ or $S'_t \leq -(k + A)$. It is immediate that $T < T_{r^*}$. Furthermore, since $k = o(\sqrt{n/\log n})$, $(k+A) < r^*$ for n large enough, so $r^* > r^* - (k+A) > 0 > -(k+A) > -r^*$; it follows that $T' < T'_{r^*}$. These two inequalities, combined with the bounds for T_{r^*} and T'_{r^*}, yield

$$\mathbf{P}\left\{T \geq \frac{n}{4}\right\} \leq \frac{1}{n^2} \qquad \text{and} \qquad \mathbf{P}\left\{T' \geq \frac{n}{4}\right\} \leq \frac{1}{n^2} \tag{5}$$

Let E be the event that $k \leq S_n \leq k + A$, and $S_i > 0$ for all $0 < i < n$ – we aim to show that $\mathbf{P}\{E\} = O((k+1)\log n/n^{3/2})$. In order that E occur, it is necessary that either $T \geq n/4$ or

$T' \geq n/4$ (we denote the union of these two events by D), or that the following three events occur (these events control the behavior of the beginning, end, and middle of the random walk, respectively):

E_1: $S_T > 0$ and $T < n/4$,

E_2: $S'_{T'} > 0$ and $T' < n/4$,

E_3: letting $\Delta = S'_{\lfloor n/4 \rfloor} - S_{\lfloor n/4 \rfloor}$, we have $k + \Delta \leq S_{n-\lfloor n/4 \rfloor} - S_{\lfloor n/4 \rfloor} \leq k + \Delta + A$

It follows that

$$\mathbf{P}\{E\} \leq \mathbf{P}\{D\} + \mathbf{P}\{E_1, E_2, E_3\}.$$

Furthermore, $\mathbf{P}\{D\} \leq \mathbf{P}\{T \geq n/4\} + \mathbf{P}\{T' \geq n/4\} \leq 2/n^2$ by (5), so to show that $\mathbf{P}\{E\} = O(\log n/n^{3/2})$, it suffices to show that $\mathbf{P}\{E_1, E_2, E_3\} = O(\log n/n^{3/2})$; we now demonstrate that this latter bound holds, which will complete the proof.

The events E_1 and E_2 are independent, as E_1 is determined by the random variables $X_1, \ldots, X_{\lfloor n/4 \rfloor}$, and E_2 is determined by the random variables $X_{n-\lfloor n/4 \rfloor+1}, \ldots, X_n$. Furthermore, in the notation of Lemma 14, T is an event of the form $T_{r,s}$ with $r = r^*$, $s = 0$; it follows that $\mathbf{P}\{S_T > 0\} \leq C/(r^* + C)$. Since S' has step size $-X$ and $|-X| < C$, we may also apply Lemma 14 to the walk S' with the choice $r = r^* - k + C$, $s = k + C$, to obtain the bound $\mathbf{P}\{S'_{T'} > 0\} \leq (k + 2C)/(r + k + 2C)$. Therefore

$$\begin{aligned}
\mathbf{P}\{E_1, E_2, E_3\} &= \mathbf{P}\{E_3 | E_1, E_2\} \mathbf{P}\{E_1\} \mathbf{P}\{E_2\} \\
&\leq \mathbf{P}\{E_3 | E_1, E_2\} \mathbf{P}\{S_T > 0\} \mathbf{P}\{S'_{T'} > 0\} \\
&\leq \mathbf{P}\{E_3 | E_1, E_2\} \cdot \frac{C(k + 2C)}{r^*(r^* + k + 2C)} < \mathbf{P}\{E_3 | E_1, E_2\} \cdot \frac{2C^2(k+1)}{(r^*)^2} \qquad (6)
\end{aligned}$$

To bound $\mathbf{P}\{E_3 | E_1, E_2\}$, we observe that

$$\begin{aligned}
\mathbf{P}\{E_3 | E_1, E_2\} &\leq \sup_{x \in \mathbb{R}} \mathbf{P}\{E_3 | E_1, E_2, \Delta = x\} \\
&= \sup_{x \in \mathbb{R}} \mathbf{P}\left\{k + x \leq S_{n-\lfloor n/4 \rfloor} - S_{\lfloor n/4 \rfloor} \leq k + x + A | E_1, E_2, \Delta = x\right\}. \qquad (7)
\end{aligned}$$

Furthermore, the event that $k + x \leq S_{n-\lfloor n/4 \rfloor} - S_{\lfloor n/4 \rfloor} \leq k + x + A$ is independent from E_1, E_2, and from the event that $\Delta = x$, as the former event is determined by the random variables $X_{\lfloor n/4 \rfloor+1}, \ldots, X_{n-\lfloor n/4 \rfloor}$, and the latter events are determined by the random variables $X_1, \ldots, X_{\lfloor n/4 \rfloor}, X_{n-\lfloor n/4 \rfloor+1}, \ldots, X_n$. It follows from this independence, (7), and the strong Markov property that

$$\begin{aligned}
\mathbf{P}\{E_3 | E_1, E_2\} &\leq \sup_{x \in \mathbb{R}} \mathbf{P}\left\{k + x \leq S_{n-\lfloor n/4 \rfloor} - S_{\lfloor n/4 \rfloor} \leq k + x + A\right\} \\
&= \sup_{x \in \mathbb{R}} \mathbf{P}\left\{k + x \leq S_{n-2\lfloor n/4 \rfloor} \leq k + x + A\right\}. \\
&\leq (A+1) \sup_{x \in \mathbb{R}} \mathbf{P}\left\{k + x \leq S_{n-2\lfloor n/4 \rfloor} \leq k + x + 1\right\}, \qquad (8)
\end{aligned}$$

the last inequality holding by a union bound. By Theorem 16, there is $c > 0$ depending only on X, such that

$$\sup_{x\in\mathbb{R}} \mathbf{P}\left\{x \le S_{n-2\lfloor n/4\rfloor} \le x+1\right\} \le \frac{c}{\sqrt{n-2\lfloor n/4\rfloor}} \le \frac{\sqrt{2}c}{\sqrt{n}},$$

and it follows from this fact and from (8) that

$$\mathbf{P}\left\{E_3|E_1,E_2\right\} \le \frac{\sqrt{2}c(A+1)}{\sqrt{n}}.$$

Combining this bound with (6) yields

$$\mathbf{P}\left\{E_1,E_1,E_3\right\} \le \frac{\sqrt{2}c(A+1)}{\sqrt{n}} \cdot \frac{2C^2(k+1)}{(r^*)^2} = \frac{2\sqrt{2}c(A+1)C^2(k+1)}{(r^*)^2\sqrt{n}}.$$

Since $r^* = \lfloor\sqrt{n/9B\log n}\rfloor$, $(r^*)^2 \ge n/10B\log n$ for n large enough, so letting $a = 2\sqrt{2}(A+1)cC^2 \cdot 10B = O(1)$, we have

$$\mathbf{P}\left\{E_1,E_1,E_3\right\} \quad < \quad \frac{a(k+1)\log n}{n^{3/2}} = O\left(\frac{(k+1)\log n}{n^{3/2}}\right), \tag{9}$$

as claimed.

4 Strengthening Theorem 13

There are two key ingredients needed to move from the upper bound in Theorem 13 to the stronger and more general upper bound in Theorem 8. The first concerns stopping times $T_{r,s}$ of the form seen in Lemma 14. Without the assumption that the step size X is bounded, we have no *a priori* bound on $\mathbf{E}\left\{S_{T_{r,s}}|S_{T_{r,s}} > r\right\}$ or on $\mathbf{E}\left\{S_{T_{r,s}}|S_{T_{r,s}} \le -s\right\}$, so we can not straightforwardly apply Wald's identity to bound $\mathbf{P}\left\{S_{T_{r,s}} > 0\right\}$ as we did above.

Griffin and McConnell (1992) have proven bounds on $\mathbf{E}\left\{|S_{T_{r,r}}| - r\right\}$ (a quantity they call the *overshoot* at r), for random walks with step size X in the domain of attraction of the normal distribution; their results are the best possible in the setting they consider. Their bounds do not directly imply the bounds we need, but we are able to use their results to obtain such bounds using a bootstrapping technique we refer to as a "doubling argument". The key idea behind this argument can be seen by considering a symmetric simple random walk S and a stopping time $T_{3k,0}$, for some positive integer k. Let T be the first time $t > 0$ that $|S_t| = k$. If the event $S_{T_{3k,0}} > 0$ is to occur, it must be the case that $S_T = k$. Next, let T' be the first time $t > T$ that $|S_t - S_T| \ge 2k$ – if $S_{T_{3k,0}} > 0$ is to occur, it must also be the case that $S_{T'} - S_T = 2k$. By the independence of disjoint sections of the random walk, it follows that

$$\mathbf{P}\left\{S_{T_{3k,0}} > 0\right\} \le \mathbf{P}\left\{S_T = k\right\}\mathbf{P}\left\{S_{T'} - S_T = 2k\right\}.$$

In the notation of Lemma 14, T is a stopping time of the form $T_{k,k}$, and T_2 is a stopping time of the form $T_{2k,2k}$, so we have

$$\mathbf{P}\left\{S_{T_{3k,0}} > 0\right\} \leq \mathbf{P}\left\{S_{T_{k,k}} > 0\right\} \mathbf{P}\left\{S_{T_{2k,2k}} > 0\right\}.$$

Furthermore, for a general random walk S we can use the Griffin and McConnell's bounds on the overshoot together with the approach of Lemma 14 to prove bounds $\mathbf{P}\left\{S_{T_{k,k}} > 0\right\}$. In general, we consider a sequence of stopping times $T_1, T_2, \ldots$, where T_{i+1} is the first time after T_i that $|S_{T_{i+1}} - S_{T_i}| \geq 2^i$, and apply Griffin and McConnell's results to bound the probability that the random walk goes positive at each step. By applying their results to such a sequence of stopping times, we are able to ensure that the error in our bounds resulting from the "overshoot" does not accumulate, and thereby prove the stronger bounds we require.

The second difficulty we must overcome is due to the fact that in order to remove the superfluous $\log n$ factor in the bound of Theorem 13, we need to replace the stopping time T_{r^*} (with $r^* = O(\sqrt{n/\log n})$) by a stopping time $T_{r'}$ (with $r' = \Theta(\sqrt{n})$). However, for such a value r', $\mathbf{E}T_{r'} = \Theta(n)$, and our upper tail bounds on $T_{r'}$ are not strong enough to ensure that $T_{r'} \leq \lfloor n/4 \rfloor$ with sufficiently high probability.

To deal with this problem, we apply the ballot theorem inductively. Instead of stopping the walk at a stopping time $T_{r'}$, we stop the walk *deterministically* at time $t_1 = \lfloor n/4 \rfloor$. In order that $S_n > 0$ for all $0 < i < n$ occur, it must be the case that *either* $T_{r'} \leq t_1$ and $S_{T_{r'}} > 0$, *or* there is $0 < k \leq r' - C$ such that $k \leq S_{t_1} \leq k + A$ and additionally, $S_i > 0$ for all $0 < i < t_1$. We bound the probability of the former event using our strengthening of Lemma 14, and bound the probability of the latter event by inductively applying the ballot theorem. Of course, an identical analysis applies to the negative reversed random walk S', and allows us to strengthen our control of the end of the random walk correspondingly.

Finally, we give some idea of our lower bound. We fix some value r' of order $\Theta(\sqrt{n})$; paralleling the proof of Theorem 13, we let T be the first time t that $S_t > r'$ or $S_t \leq 0$, and let T' be the first time t that $S'_t > r' - k$ or $S'_t \leq -k$. In order that $k \leq S_n \leq k + A$, and $S_i > 0$ for all $0 < i < n$, it suffices that the following three events occur (these events control the behavior of the beginning, end, and middle of the random walk, respectively):

- E_1: $S_T > 0$, $S_T \leq 2r'$, and $T < n/4$,
- E_2: $S'_{T'} > 0$, $S'_{T'} \leq 2r' - k$, and $T' < n/4$,
- E_3: letting $\Delta = S'_{T'} - S_T$, we have $k + \Delta \leq S_{n-T'} - S_T \leq k + \Delta + A$ and $S_i - S_T > -r'$ for all $T < i < n - T'$.

Using an approach very similar to that of Theorem 13, we are able to show that in fact $\mathbf{P}\{E_1\} = \Theta(1/r')$ and that $\mathbf{P}\{E_2\} = \Theta((k+1)/r')$. The two key observations of that allow us to prove a lower bound on $\mathbf{P}\{E_3\}$ are the following:

- Given that E_1 and E_2 occur, $k + \Delta = O(\sqrt{n})$, and so it is not hard to show using Theorem 10 that $\mathbf{P}\{k + \Delta \le S_{n-T'} - S_T \le k + \Delta + A | E_1, E_2\} = \Theta(1/\sqrt{n})$.

- Since $n - T - T' = O(n)$, we expect the random walk $S_T, S_{T+1}, \ldots, S_{n-T'}$ to have a spread of order $O(\sqrt{n})$. Since $r' = \Theta(\sqrt{n})$, it easy to see (again using Theorem 10, or by the classical central limit theorem) that $S_i - S_T > -r'$ for all $T < i < n - T'$ with probability $\Omega(1)$.

Based on these two observations, we trust that the reader will find it plausible that given E_1 and E_2, the intersection of the events in E_3 occurs with probability $\Theta(1/\sqrt{n})$; in this case, combining our bounds much as in Theorem 13 yields a lower bound on $\mathbf{P}\{E_1, E_2, E_3\}$ of order $(k+1)/(r')^2\sqrt{n} = \Theta((k+1)/n^{3/2})$.

5 Conclusion

In writing this survey, we hoped to convince the reader the theory of ballots is not only rich and beautiful, in-and-of itself, but is also very much alive. Our new results are far from conclusive in terms of when ballot-style behavior can be expected of sums of independent random variables, and more generally of permutations of sets of real numbers. In the final paragraphs, we highlight some of the questions that remain unanswered.

The results of Section 3 are unsatisfactory in that they only yield "true" (conditional) ballot theorems when $S_n = O(\sqrt{n})$. Ideally, we would like such results to hold *whatever* the range of S_n. Two key weaknesses of our approach are that it (a) relies on estimates for $\mathbf{P}\{x \le S_n \le x + c\}$ that are based on the central limit theorem, and these estimates are not good enough when S_n is not $O(\sqrt{n})$, and (b) relies on bounds on the "overshoot" that only hold when the step size X is in the range of attraction of the normal distribution, Kesten and Maller (1994) and, independently, Griffin and McConnell (1994), have derived necessary and sufficient conditions in order that $\mathbf{P}\left\{S_{T_{r,r}}\right\} \to 1/2$ as $r \to \infty$; in particular they show that for any $\alpha < 2$, there are distributions with $\mathbf{E}\{X^\alpha\} = \infty$ for which $\mathbf{P}\left\{S_{T_{r,r}}\right\} \to 1$. Therefore, we can not expect to use a doubling argument in this case, which seriously undermines our approach.

As we touched upon at various points in the paper, aspects of our technique seem as though they should work for analyzing more general random permutations of sets of real numbers. Since Andersen observed the connection between conditioned random walks and random permutations (Andersen, 1953, 1954), and Spitzer (1956) pointed out the full generality of Andersen's observations, just about every result on conditioned random walks has been approached from the permutation-theoretic perspective sooner or later. There is no reason our results should not benefit from such an approach.

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