Approximate Min-max Relations for Odd Cycles in Planar Graphs

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Abstract. We study the ratio between the minimum size of an odd cycle vertex transversal and the maximum size of a collection of vertex-disjoint odd cycles in a planar graph. We show that this ratio is at most 10. For the corresponding edge version of this problem, Král and Voss [7] recently proved that this ratio is at most 2; we also give a short proof of their result.

1 Introduction

A set of vertices of a graph G is an *odd cycle (vertex) transversal* (or *cover*) if its removal makes G bipartite. An *odd cycle (vertex) packing* in G is a collection of vertex-disjoint odd cycles in G. Let τ and ν respectively denote the minimum size of an odd cycle transversal and the maximum size of an odd cycle packing. Clearly we have $\nu \leq \tau$. Our main result is to show that $\tau \leq 10\nu$ for all planar graphs G. For general graphs, it is known that τ is not bounded by any function of ν. In other words, odd cycles do not satisfy the *Erdös-Pósa property.* In fact, there are graphs known as *Escher walls* [10] for which $\nu = 1$ and τ is arbitrarily large. In [10], Reed proved that the Erdös-Pósa property holds for odd cycles in graphs without Escher wall of height h, for any fixed $h \geq 3$. Since planar graphs do not contain any Escher wall, there exists a function f such that $\tau \leq f(\nu)$ for all planar graphs G. However, the function f implicit in [10] is huge. A similar type of result in this area applies to highly connected graphs G. Thomassen [15] proved that $\tau \leq 2\nu$ for $2^{3^{9\nu}}$-connected graphs. Reed and Rautenbach [9] generalised this to 576ν-connected graphs.

The minimum odd cycle transversal (cover) problem and maximum odd cycle packing problem can be formulated as integer programs whose linear program

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relaxations are duals. Letting V denote the vertex set of G and $\mathcal{O}$ denote the set of odd cycles in G, these dual LPs are:

$$\begin{aligned} &\textsc{Covering LP: } \min \sum_{v \in V} y_v \\ &\sum_{v: v \in C} y_v \geq 1 \ \forall C \in \mathcal{O} \\ &y_v \geq 0 \ \forall v \in V \end{aligned} \qquad \begin{aligned} &\textsc{Packing LP: } \max \sum_{C \in \mathcal{O}} x_C \\ &\sum_{C: v \in C} x_C \leq 1 \ \forall v \in V \\ &x_C \geq 0 \ \forall C \in \mathcal{O} \end{aligned}$$

Goemans and Williamson [5] gave a constant factor approximation algorithm for the minimum odd cycle transversal problem in planar graphs using the primal-dual method. In doing so they proved that the integrality gap of the covering LP is at most $\frac{9}{4}$. They conjecture that it is actually $\frac{3}{2}$. Our results show that the integrality gap of the packing LP is bounded by a constant. In addition, our structural result generates a polynomial time 10-approximation algorithm for the maximum odd cycle packing problem in planar graphs.

We remark that there are corresponding edge versions of these covering and packing problems. Recently, Král and Voss [7] showed that the minimum size of an odd cycle edge transversal in a planar graph is at most twice the maximum size of an odd cycle edge packing. We also give a short proof of their result.

We conclude this introductory section with an overview of the paper. At the heart of this work lies a connection between odd cycle transversals and T-joins in a certain auxiliary graph. In particular, T will correspond to the set of odd faces of G. In the vertex case the auxiliary graph we need to consider is the face-vertex incidence graph G^+. Specifically, we show that minimum vertex transversals correspond to T-joins in G^+ covering the least number of vertices of G. In the edge case the auxiliary graph is the dual graph G^*. Here minimum transversals correspond to T-joins in G^* with the least number of edges. This relationship was first used by Hadlock [6] to derive a polynomial time algorithm for the maximum cut problem in planar graphs. We present the necessary background on T-joins and T-cuts in Section 2.1 and describe the connection between odd cycle transversals and T-joins in Section 2.2 for the edge case and in Section 2.3 for the vertex case.

In Section 3, we present our proof that a minimum edge transversal has size at most twice the size of a maximum odd cycle edge packing. This proof, as with the proof of Král and Voss, requires the use of the Four Colour Theorem. We use it to show that any laminar 2-packing (defined below) of k odd cycles in G contains $\frac{1}{4}k$ edge-disjoint odd cycles. A result by Lovász on T-joins and T-cuts then guarantees the existence of a 2-packing whose size is twice the minimum size of an odd cycle edge transversal. This gives the result.

To prove our main result we begin by considering three special classes of planar graph. In Section 4.1 we consider graphs in which every pair of odd faces intersect. In Section 4.2 we consider 4-connected graphs such that some (possibly even) face intersects every odd face. Finally in Section 4.3 we examine graphs in which every pair of odd faces is "far" apart. In all of these classes of graph we show that $\tau \leq 2\nu$. The proof of our main result, given in Section 5, combines the techniques developed for these special cases. We let G be a minimum counterexample to the theorem. We take a minimum collection of faces of G, which we call *centers*, such that every odd face of G intersects some face of the collection. Then we show that G must be 4-connected with all its centers "far" apart. Next, using the techniques of Section 4.2 we find a "local" transversal around each center. At the same time, we find a packing of odd faces around each center. Since the centers are "far" apart the union of these packings is also a packing. We call this union the *local* packing. The results of Section 4.3 allow us to extend the "local" transversals to a transversal of the whole graph. Associated with this transversal we find a different packing of odd cycles, which we call the *global* packing. The size of the transversal we obtain is within a constant factor of the size of the largest of the local packing and the global packing. Our main result then follows.

2 T-Joins and Odd Cycle Transversals

In this section we show how minimum odd cycle edge and vertex transversals of a plane graph G relate to T-joins in the dual graph and face-vertex incidence graph of G, respectively. First, we give some background on T-joins along with two min-max results that we will need.

2.1 Background

Consider any graph H and set of vertices T in H. A *T-join* in H is a set of edges J such that T equals the set of odd degree vertices in the subgraph of H determined by J. There exists a T-join in H if and only if each connected component of H contains an even number of vertices of T. In particular, if H has a T-join then $|T|$ is even. A *T-cut* in H is a cut having an odd number of vertices of T on each side. In other words, whenever a set of vertices X contains an odd number of vertices of T, the cut $\delta(X) = \{xy \in E(H) : x \in X, y \notin X\}$ is a T-cut. The *length* of a T-join is the number of edges it contains. A *packing* of T-cuts is a collection of edge-disjoint T-cuts. Because every T-join intersects every T-cut, the minimum length of a T-join in H is at least the maximum size of a packing of T-cuts in H. In fact, equality holds for bipartite graphs, see Proposition 1 below.

Two sets X and Y of vertices of H are said to be *laminar* if either $X \subseteq Y$ or $Y \subseteq X$ or $X \cap Y = \varnothing$. The sets X and Y are *cross-free* when they are laminar or $X \cup Y = V(H)$. A collection of subsets of $V(H)$ is said to be *laminar* (resp. *cross-free*) if any two of its members are laminar (resp. cross-free). Consider a collection $\mathcal{F}$ of subsets of $V(H)$. Letting $\delta(\mathcal{F}) = \{\delta(X) : X \in \mathcal{F}\}$, the collection of cuts $\delta(\mathcal{F})$ is said to be *laminar* (resp. *cross-free*) whenever $\mathcal{F}$ is.

Proposition 1 (Seymour [13]). *Let H be a bipartite graph and let T be an even set of vertices of H. The minimum length of a T-join in H equals the* *maximum size of a packing of T-cuts in H. The maximum is attained by a cross-free collection of T-cuts.* □

The latter proposition implies the next, where a 2-*packing* of T-cuts is a collection of T-cuts such that each edge is contained in at most two T-cuts of the collection.

Proposition 2 (Lovász [8]). *Let H be a graph and T be an even set of vertices of H. The minimum length of a T-join in G equals half the maximum cardinality of a 2-packing of T-cuts in H. The maximum is attained by a cross-free collection of T-cuts.* □

The following observation will be useful in subsequent sections. Its proof is based on standard uncrossing techniques (see, e.g., Proposition 3.4 in [4] or Section 80.7b in [12]) and is not included here due to length restrictions.

Observation 1. *In Propositions 1 and 2, there exists an optimal collection of T-cuts which is laminar and consists only of inclusion-wise minimal T-cuts.*

2.2 Relating Edge Transversals to T-Joins in G^*

Hadlock [6] first noted the following correspondence between odd cycle edge transversals of G and T-joins in its dual graph. Below, G^* denotes the dual graph of G and T the set of odd faces of G, regarded as a subset of $V(G^*)$. We remind the reader that the parity of a face equals the parity of its boundary, counting bridges twice. Note that $|T|$ is always even.

Lemma 1 (Hadlock [6]). *A set of edges F is an odd cycle edge transversal of G if and only if $F^* = \{e^* : e \in F\}$ is a T-join in G^*. Hence, the minimum size of an odd cycle edge transversal of G equals the minimum length of a T-join in G^*.* □

2.3 Relating Vertex Transversals to T-Joins in G^+

We now show how odd vertex cycle transversals of G relate to T-joins in its face-vertex incidence graph. As above, T denotes the set of odd faces of G. The *face-vertex incidence graph* of G is the bipartite graph G^+ on the faces and vertices of G whose edges are the pairs fv, where f is a face of G and v is a vertex of G incident to f. The face-vertex incidence graph is planar because it can be drawn in the plane as follows. Keep all vertices of G as vertices of G^+ and add a new vertex v_f in each face f of G. Then link each new vertex v_f to the vertices of G which are incident to f by an arc whose interior is contained in f. Do this in such a way that two arcs never have a common interior point. The resulting drawing of G^+ is referred to as a *standard drawing.* The relationship between transversals of G and T-joins in the face-vertex incidence graph G^+ is as follows. Below, and henceforth, $F(G)$ denotes the face set of G.

Observation 2. *Let $\delta(X)$ be a T-cut in the face-vertex incidence graph G^+ and let R denote the subgraph of G determined by the edges incident to a face in X and to a face in $\bar{X}$. Then R is Eulerian and has an odd number of edges. Hence, R contains an odd cycle.*

Proof. Pick some vertex v of G. Let $e_1, \ldots, e_d$ denote the edges of G incident to v listed in clockwise order and, for $1 \leq i \leq d$, let f_i be the face of G incident to both e_i and e_{i+1} (we let $e_{d+1} = e_1$). Each face f_i belongs either to X or to $\bar{X}$. Because there is an even number of switches between X and $\bar{X}$ when one goes clockwise around v, the degree of v in R is even. In other words, R is Eulerian. So it can be decomposed into edge-disjoint cycles. Since X contains an odd number of vertices of T, that is, an odd number of odd faces of G, subgraph R has an odd number of edges. □

Lemma 2. *A subset W of $V(G)$ is a transversal of G if and only if the subgraph of the face-vertex incidence graph G^+ induced by $W \cup F(G)$ contains a T-join, that is, every component of the subgraph has an even number of vertices of T.*

Proof. We first prove the forward direction. Suppose, by contradiction, that some connected component X of the subgraph of G^+ induced on $W \cup F(G)$ contains an odd number of vertices of T. Then $\delta(X)$ is a T-cut in G^+. Consider the edges of G incident to a face in X and to a face in $\bar{X}$. These edges determine a subgraph R of G. Let e be an edge of R. None of the endpoints of e belongs to W because otherwise all the faces incident to this endpoint would be in X and e would not belong to R, a contradiction. Therefore, R is vertex-disjoint from W. By Observation 2, we know that R contains an odd cycle. So W is not a transversal, a contradiction.

To prove the backward direction, consider an odd cycle C and a T-join J in G^+ covering some vertices of W and no vertex of $G - W$. Let Y be the set of faces of G contained in C and let $X = Y \cup W$. Because C is odd, X contains an odd number of odd faces, that is, an odd number of elements of T. Because $|T|$ is even, there is an odd number of elements of T in $\bar{X}$ too. It follows that J contains a path P from an element of T in X to an element of T in $\bar{X}$. Let v be any vertex of G on P incident to a face in X and to a face in $\bar{X}$. Then v is a vertex of C covered by J. In other words, W intersects C. Therefore, W is a transversal. □

3 The Edge Case

In this section, we give a short proof that a minimum odd cycle edge transversal has size at most twice the size of a maximum packing of edge-disjoint odd cycles. This result was recently proved by Král and Voss [7]. Their proof is quite long (about 10 pages). Below, we give a concise proof. As with the proof of Král and Voss, our proof relies on the Four Colour Theorem [1, 11].

Theorem 3 (Král and Voss [7]). *The minimum size of an odd cycle edge transversal of G is at most twice the maximum size of an odd cycle edge packing in G.*

Proof. Let τ and ν respectively denote the minimum size of an odd cycle edge transversal of G and the maximum size of an odd cycle edge packing in G. The theorem trivially holds if $\nu = 0$. Assume that $\nu > 0$. By Lemma 1, the minimum size of a T-join in G^* equals τ. By Proposition 2 and Observation 1, there is a laminar family $\mathcal{F}$ of 2τ subsets of $V(G^*)$ such that $\delta(\mathcal{F}) = \{\delta(X) : X \in \mathcal{F}\}$ is a 2-packing of inclusion-wise minimal T-cuts in G^*. Without loss of generality, we can assume that the outer face o of G is odd and that no member of $\mathcal{F}$ contains o.

Let H denote the graph on $\mathcal{F}$ in which X and Y are adjacent whenever the corresponding T-cuts intersect. We claim that H is planar. The claim obviously implies the theorem because, by the Four Colour Theorem, H has a stable set of size at least $|V(H)|/4 = 2\tau/4 = \tau/2$. This implies the desired inequality $\tau \leq 2\nu$. In order to show that H is planar, it suffices to show that every block H' of H is planar. Let $\mathcal{F}'$ denote the vertex set of H'. Since $\mathcal{F}$ is laminar, $\mathcal{F}'$ is also laminar and the set $\mathcal{F}'$ partially ordered by inclusion is a forest, i.e., every point is covered by at most one point. Let X, Y and Z be three distinct elements of $\mathcal{F}'$. The following cannot occur: (i) $X \subseteq Y \subseteq Z$, (ii) $X \subseteq Y$ and $Y \cap Z = \varnothing$. Indeed, if (i) or (ii) holds then every X–Z path in H' intersects Y because $\delta(\mathcal{F})$ is a 2-packing. This contradicts our assumption that H' is a block of H. Then $\mathcal{F}'$ partially ordered by inclusion is either a forest of height 0 (that is, an antichain) or a tree of height 1. In both cases, it is easy to construct a planar drawing for H' from G. Each element of $\mathcal{F}'$ determines a cycle in the plane graph G. In the first case, we pick any point in the bounded face of each of these cycles and connect the points by an arc whenever there is an edge in H' between the two corresponding elements of $\mathcal{F}'$. This can be done in such a way that the resulting graph is planar. The second case is similar. □

4 Special Classes

In this section we show that the minimum size of an odd cycle (vertex) transversal is at most twice the maximum size of an odd cycle (vertex) packing for a collection of special classes of planar graphs. The techniques we develop here will then be applied in the next section to give our main result for general planar graphs. For technical reasons it will be useful to assume that G is signed. A *signed* graph is a graph whose edges are labeled *odd* ('$-$') or *even* ('$+$'). In a signed graph, a cycle (or more generally a subgraph) is said to be *odd* if it contains an odd number of odd edges and *even* otherwise. Similarly, a face of a plane signed graph is said to be *odd* if its boundary has an odd number of odd edges, counting bridges twice. Otherwise, the face is said to be *even.* A signed graph is said to be *balanced* if it has no odd cycle. Odd cycle transversals and odd cycle packings are defined as in the unsigned case. Henceforth, in order to avoid unnecessary repetitions, we abbreviate odd cycle vertex transversal and odd cycle vertex packing respectively as *transversal* and *packing*. We denote by $\tau(G)$ and $\nu(G)$ the minimum size of a transversal of G and the maximum size of a packing in G, respectively. We assume that G has no loops and no multiple edges, with one exception. We allow *odd digons*, i.e., subgraphs with two vertices and two edges between them, one of which is odd and the other even.

4.1 When All Odd Faces Mutually Intersect

The proofs of the following two results could not be included in this version of the article due to length restrictions, but will be included in the journal version. The second result will be used as a base case to prove our main approximate min-max result.

Lemma 3. *Let G be a 3-connected plane graph. Then every face of G is bounded by a cycle. For any two faces f and f' of G whose respective boundaries C and C' intersect, the following holds. Either C and C' share exactly one vertex, or two adjacent vertices and the edge between them.*

Proposition 3. *If every two odd faces of G have intersecting boundaries, then G has a transversal of size at most* 2.

4.2 When Some Face Intersects Every Odd Face

In this section, we consider the graphs that have some face whose boundary intersects the boundary of every odd face.

Proposition 4. *Assume G is* 4*-connected, simple, has at least five vertices and is such that the boundary of the outer face intersects the boundary of every odd face. Then the minimum size of a transversal of G is at most twice the maximum size of a packing in G.*

Proof. We assume that G is not balanced. Otherwise, the result trivially holds. The hypotheses severely restrict the way face boundaries intersect each other. Consider a vertex y not incident to the outer face. If there are two distinct vertices x, z such that each one of them is incident to the outer face and to a face incident to y, then x and z have to be adjacent. Indeed, there exists a polygon P in $\mathbb{R}^2$ intersecting G exactly in x, y and z. By the Jordan Curve Theorem, we know that all paths from a vertex of G in the bounded region of $\mathbb{R}^2 \setminus P$ to a vertex of G in the unbounded region of $\mathbb{R}^2 \setminus P$ go through x, y or z. If x and z are not adjacent, then the two neighbours of x on the boundary of the outer face lie in a different region of $\mathbb{R}^2 \setminus P$. Hence $X = \{x, y, z\}$ is a cutset of size 3 in G, a contradiction.

Now consider two distinct odd faces f and g different from the outer face. By Lemma 3, the boundaries of faces f and g intersect in a vertex or in a common edge. If the boundaries of f and g share a unique vertex y, then y is incident to the outer face unless the following occurs. There are vertices x and z incident to the outer face such that xy, xz and yz are edges, x is incident to f and z is incident to g Moreover, x is the only vertex incident to both f and the outer face, and z is the only vertex incident to both g and the outer face. We refer to the triangle on x, y and z as a *junctional triangle*. Note that junctional triangles can be even because G is signed. If the boundaries of f and g intersect in a common edge e then one of the endpoints of e is on the outer face and the other is not. Moreover, in that case f and g cannot both have a common incident edge with the outer face because otherwise $G = K_4$, contradicting the fact that G has at least five vertices.

Enumerate the vertices of the outer face in clockwise order as v_1, v_2, $\ldots$, v_n. For the sake of simplicity, let $v_0 = v_n$ and $v_{n+1} = v_1$. Let I be the set of indices i such that there is a junctional triangle containing the edge $v_i v_{i+1}$. For each $i \in I$, we let u_i be the vertex of the junctional triangle incident to the edge $e = v_i v_{i+1}$ and opposite to e, and we let w_i be any point in the interior of the edge e. For each odd face f different from the outer face, we define an arc A_f contained in the frontier of the outer face, as follows. If the boundary of f intersects the boundary of the outer face in an edge $v_i v_{i+1}$, then we let A_f be the edge $v_i v_{i+1}$. Otherwise, the boundary of f intersects the boundary of the outer face in a vertex v_i. If f is incident neither to u_{i-1} nor to u_i then we let A_f be the point $\{v_i\}$. If f is incident to u_{i-1} and not to u_i then we let A_f be the part of the edge $v_{i-1}v_i$ between w_{i-1} and v_i. If f is incident to u_i and not to u_{i-1} then we let A_f be the part of the edge $v_i v_{i+1}$ between v_i and w_i. Finally, if f is incident to both u_i and u_{i-1} then we let A_f be the arc linking w_{i-1} and w_i on the outer face and containing v_i. By construction, two odd faces f and g different from the outer face are incident to some common vertex if and only if their corresponding arcs A_f and A_g have a nonempty intersection.

Let H denote the graph whose vertices are the odd faces different from the outer face and whose edges are the pairs fg such that $A_f \cap A_g \neq \varnothing$. Then H is a circular arc graph. The maximum size of a packing in G is precisely equal to the maximum size of a stable set in H, that is, we have $\nu(G) = \alpha(H)$. We consider the following two cases.

Case 1. There is some point x on the boundary of the outer face that is not in any arc A_f. In this case, H is an interval graph. Let W be a minimum cardinality subset of $\{v_i : 1 \leq i \leq n\} \cup \{w_i : i \in I\}$ meeting all the arcs. By Dilworth's chain partitioning theorem [3], the complement of an interval graph H is perfect, hence we have $|W| = \alpha(H) = \nu(G)$. Now replace each $w_i \in W$ by v_i and v_{i+1}. Let W' be the resulting set of vertices of G. Then W' is a transversal of G of cardinality at most $2|W|$. In other words, we have $\tau(G) \leq 2\nu(G)$.

Case 2. The arcs A_f cover the whole boundary of the outer face. It follows that for each edge $e = v_i v_{i+1}$, the face f_i incident to e and different from the outer face is either an odd face or an even junctional triangle. If f_i is odd, then we let $g_i = f_i$. Otherwise, we let g_i be the odd face incident to $v_i u_i$. As above, u_i denotes the vertex of the junctional triangle incident to e which is opposite to e. Thus, every edge of the outer face has a corresponding odd face. Note that if the boundaries of g_i and g_j intersect then $i \in \{j-1, j+1\}$ or $i = j$. So if

n is even, then we have $\tau(G) \leq 2\nu(G)$ because $\{v_1, \ldots, v_n\}$ is a transversal of size n and $\{f_1, f_3, \ldots, f_{n-1}\}$ yields a packing of size $n/2$. Now assume that n is odd. Let H' be the graph whose vertices are the faces g_i and whose edges are the pairs $g_i g_j$ such that the boundary of g_i intersects that of g_j and $i \neq j$. By what precedes, we know that the graph H' is a subgraph of the odd cycle with vertex sequence $v_1, \ldots, v_n, v_1$. If H' is not connected, then it has a stable set of size $(n+1)/2$ and we get $\tau(G) \leq 2\nu(G)$ as before. For the rest of the proof, we assume that H' is connected. We claim that either all f_i's are odd faces or all f_i's are even junctional triangles. Otherwise, there is some index i such that f_i is an even junctional triangle and f_{i+1} is an odd face. By what precedes, the boundaries of g_i and g_{i+1} cannot intersect. So our claim holds.

If all f_i's are odd faces then consider vertex v_1. If $\{v_1, \ldots, v_n\} \setminus \{v_1\}$ is a transversal then we have $\tau(G) \leq n-1 \leq 2\nu(G)$ because H' has a stable set of size $(n-1)/2$. Otherwise, there is some odd face f incident to v_1 and to no other v_i. Then $\{f\} \cup \{f_2, f_4, \ldots, f_{n-1}\}$ yields a packing of size $(n+1)/2$. Hence, we have $\tau(G) \leq 2\nu(G)$. If all f_i's are even junctional triangles, then the odd faces of G are exactly the outer face and the faces g_i for $i = 1, \ldots, n$. It is easy to see that $\{u_1, v_3, v_4, \ldots, v_n\}$ is a transversal of size $n-1$. Because H' has a stable set of size $(n-1)/2$, we have $\tau(G) \leq 2\nu(G)$. This concludes the proof. □

We need a slight generalization of Proposition 4. Consider some face f of G, which we refer to as a *center*. The odd faces of G whose boundary intersects the boundary of the center are called the *targets* (around f). In particular, if f is odd then f is itself a target. A *local transversal* is a set W of vertices of G satisfying the following properties:

(i) every target is incident to some vertex of W;
(ii) at most one vertex of W is not incident to the center;
(iii) if $u \in W$ is not incident to the center, then u is incident to exactly two targets.

The proof of Proposition 4 in fact shows:

Lemma 4. *Assume G is 4-connected, simple and has at least five vertices. Let f be a face of G acting as center. Then the minimum size of a local transversal of G is at most twice the maximum number of boundary-disjoint targets in G.* □

4.3 When Odd Faces Are Disjoint

We begin this section by recasting the minimum transversal and the maximum packing problems entirely in terms of T-joins and T-cuts in the face-vertex incidence graph. This slight change of terminology simplifies the proofs and enables us to state our results with more generality. Let H denote any bipartite graph with bipartition $\{A, B\}$, and let T be any even subset of B. The *width* of a T-join in H is the number of vertices of A it covers. The *fringe* of a T-cut $\delta(X)$ in H is the set of vertices of A which have a neighbour in X and a neighbour in $\bar{X}$. Note that the minimum width of a T-join in H is at least the maximum number of fringe-disjoint T-cuts in H. This is due to the fact that every T-join covers some element in the fringe of every T-cut.

We now relate the above definitions to odd cycle vertex transversal and packing in plane signed graphs. Consider the case where H is the face-vertex incidence graph G^+ of the plane signed graph G, set A is the vertex set of G, set B is the face set of G, and set T is, as before, the set of odd faces of G. By Lemma 2, every T-join in H defines a transversal of G, namely, the vertices of A it covers. Reciprocally, to every transversal W there corresponds a T-join in H which covers some vertices of W and no vertex of $A \setminus W$. So the minimum width of a T-join in H equals the minimum size of a transversal of G. Furthermore, there is a correspondence between T-cuts in H and odd cycles in G. By Observation 2, every T-cut in H determines a Eulerian subgraph of G with an odd number of odd edges. This subgraph contains an odd cycle. The vertex set of the subgraph is the fringe of the T-cut. Reciprocally, every odd cycle in G determines a T-cut in H whose fringe is the vertex set of the cycle. Hence the maximum size of a collection of fringe-disjoint T-cuts in H equals the maximum size of a packing in G. We use the following notation: let ν denote the maximum size of a collection of fringe-disjoint T-cuts in H, let τ denote the minimum width of a T-join in H and let ℓ denote the minimum length of a T-join in H.

Proposition 5. *Let H be any bipartite graph with bipartition $\{A, B\}$ and let T denote any even subset of B. Assume that the shortest path distance $d_H(t, t')$ between any two distinct elements t and t' of T is at least $2c$ for some $c \geq 1$. Then we have*

$$\nu \geq \frac{1}{2}(\ell - |T| + 1) \geq \left(1 - \frac{1}{c}\right)\tau.$$

Proof. Let $\mathcal{F}$ denote a laminar collection of ℓ sets of faces and vertices of G such that $\delta(\mathcal{F}) = \{\delta(X) : X \in \mathcal{F}\}$ is a collection of edge-disjoint T-cuts in H. Such a laminar collection of T-cuts is guaranteed to exist by Proposition 1 and Observation 1.

We claim that whenever X, Y and Z are three distinct elements of $\mathcal{F}$ such that $X \subseteq Y \subseteq Z$ or $X \subseteq Y$ and $Y \cap Z = \varnothing$, then T-cuts $\delta(X)$ and $\delta(Z)$ are fringe-disjoint. It suffices to consider the first case. Suppose there exists an element $a \in A$ which belongs to the fringes of X and Z. In particular, a has a neighbor b in X and a neighbor b' in $\bar{Z}$. If $a \in Y$ then $ab' \in \delta(Y) \cap \delta(Z)$, a contradiction. If $a \in \bar{Y}$ then $ab \in \delta(X) \cap \delta(Y)$, a contradiction. So our claim holds.

The set $\mathcal{F}$ partially ordered by inclusion is a forest. Without loss of generality, we can assume that the leaves of this forest are singletons of the form $\{b\}$ for some $b \in T$. It follows that $\mathcal{F}$ has at most $|T|$ leaves. Note that the claim above implies that two nodes of the forest X and Y are fringe-disjoint unless X is the parent of Y, Y is the parent of X, X and Y are siblings or X and Y are roots.

Rank the children of each node of the forest $\mathcal{F}$ arbitrarily and order its roots arbitrarily also. Let $\mathcal{F}'$ denote the subset of $\mathcal{F}$ formed by all nodes which are ranked first in their respective ordering. Letting λ denote the number of leaves of $\mathcal{F}$, we claim that $\mathcal{F}'$ contains at least $|\mathcal{F}| - \lambda + 1 \geq |\mathcal{F}| - |T| + 1$ elements. Let

X be any node. We define a function $f : \mathcal{F} \to \mathcal{F}$ as follows. If X is a leaf then we set $f(X) = X$. Otherwise, we set $f(X) = f(Y)$, where Y is the first child of X. So $f(X)$ is the "first" leaf amongst the descendants of X. Observe that, for any leaf node Z the preimage of Z, under f, either contains exactly one node in $\mathcal{F}'$ or contains the highest ranked root node. Moreover, by construction, the preimages of any pair of leaves are disjoint. Thus, $\lambda = |\mathcal{F}| - |\mathcal{F}'| + 1$. Our second claim follows.

To obtain a packing of fringe-disjoint T-cuts, colour the elements of $\mathcal{F}'$ black or white in such a way that no parent and child have the same colour, i.e., whenever X is the parent of Y then X and Y have different colours. In other words, colour the subgraph of the Hasse diagram of $\mathcal{F}$ induced on $\mathcal{F}'$ with two colours. Let $\mathcal{F}''$ denote the biggest of the two colour classes. Then $\delta(\mathcal{F}'')$ is a collection of fringe-disjoint T-cuts of size at least $\frac{1}{2}(\ell - |T| + 1)$.

Note that every minimum length T-join in G can be thought of as a perfect matching on T whose edges have become edge-disjoint shortest paths in H. Hence ℓ is at least $\frac{|T|}{2} \cdot 2c$. Note also that τ is at most $\frac{\ell}{2}$ because the width of any T-join is at most half its length. It follows that we have

$$\nu \geq \frac{1}{2}(\ell - |T| + 1) \geq \frac{1}{2}(\ell - |T|) \geq \left(1 - \frac{1}{c}\right)\frac{\ell}{2} \geq \left(1 - \frac{1}{c}\right)\tau. \qquad \square$$

Corollary 1. *If the boundaries of the odd faces of G are pairwise disjoint then the minimum size of a transversal of G is at most twice the maximum size of a packing in G.*

Proof. This follows directly from Proposition 5 with $H = G^+$, $A = V(G)$, $B = F(G)$ and $c = 2$. $\square$

5 Combining the Local and Global Approaches

We are now ready to prove our result for general planar graphs. We will combine the local and global approaches we have described to give our main approximate min-max result. Towards this end, let $\rho(G)$ denote the minimum size of a collection of faces of G such that the boundary of every odd face of G intersects the boundary of some face in the collection. The following two lemmas are simple and we omit their proofs. Lemma 5 implies that $\rho(G') \leq \rho(G)$ for any subgraph G' of G.

Lemma 5. *Let G be a plane signed graph with $\rho(G) = r$, and let $f_1, \ldots, f_r$ be a collection of faces such that for every odd face f there is an index i such that the boundary of f_i intersects the boundary of f. Then we have $\rho(G - e) \leq r$ for each edge e. Moreover, we have $\rho(G - e) \leq r - 1$ if edge e is incident to f_i and f_j for some distinct indices i and j.* $\square$

Lemma 6. *Let G be a plane graph and X be a cutset of G with at most three vertices (we allow the case $X = \varnothing$). If G has no cutset with fewer than $|X|$ elements, then there exists a polygon $P \subset \mathbb{R}^2$ intersecting G only in vertices and such that X is precisely the intersection of P and G and each region of $\mathbb{R}^2 \setminus P$ contains a vertex of G.* □

The next lemma, combined with Lemma 6, will allow us to focus on 4-connected graphs G.

Lemma 7. *Let G be a plane signed graph, let $P \subset \mathbb{R}^2$ be a polygon intersecting G only in vertices, and let $X = P \cap V(G)$. Assume that each region R_1 and R_2 of $\mathbb{R}^2 \setminus P$ contains at least one vertex of G. Then X is a cutset in G. For $i = 1, 2$, let G_i be the part of G contained in the closure of region R_i. Then we have $\rho(G_1) + \rho(G_2) \leq \rho(G) + 2$.*

Proof. Let $\{f_1, \dots, f_r\}$ denote a collection of $r = \rho(G)$ faces of G such that the boundary of every odd face of G intersects the boundary of some face of the collection. Without loss of generality, we can assume that there are some indices r_1 and r_2 with $r_1 \leq r_2$ such that $f_1, \dots, f_{r_1}$ are contained in R_1 and incident to no vertex of X, and $f_{r_2}, \dots, f_r$ are contained in R_2 and incident to no vertex of X. For $i = 1, 2$, let g_i denote the face of G_i containing R_{2-i+1}. Then the boundary of every odd face of G_1 intersects the boundary of some face in $\{f_1, \dots, f_{r_1}\} \cup \{g_1\}$. Similarly, the boundary of every odd face of G_2 intersects the boundary of some face in $\{f_{r_2}, \dots, f_r\} \cup \{g_2\}$. The lemma follows. □

Theorem 4. *For every unbalanced plane signed graph G, we have $\tau(G) \leq 7\nu(G) + 3\rho(G) - 8$.*

Proof. Let G be a counterexample with $|V(G)|$ as small as possible, and let $\{f_1, \dots, f_r\}$ denote any minimum collection of faces of G such that the boundary of every odd face of G intersects the boundary of some face in the collection. Note that we have $r = \rho(G) \geq 1$. We claim: (1) G has a packing of size 2, no transversal of size at most 9, and G is simple; (2) G is 4-connected; (3) the shortest path distance $d_{G^+}(f_i, f_j)$ between f_i and f_j is at least 8 whenever $i \neq j$.

Proof of Claim (1). If G has no packing of size 2, then by Proposition 3, we have

$$\tau(G) \leq 2 = 7 + 3 - 8 \leq 7\nu(G) + 3\rho(G) - 8,$$

a contradiction. So G has a packing of size 2, that is, we have $\nu(G) \geq 2$. Now a similar argument shows that G has no transversal of size at most 9, that is, we have $\tau(G) > 9$. If G is not simple, then it has an odd digon. Letting x and y be the vertices of the digon and $X = \{x, y\}$, we have $\nu(G - X) \leq \nu(G) - 1$. By Lemma 5, we have also $\rho(G - X) \leq \rho(G)$. Therefore, because $G - X$ is unbalanced and has less vertices that G, we have

$$\begin{aligned}\tau(G) &\leq 2 + \tau(G - X)\\ &\leq 2 + 7\nu(G - X) + 3\rho(G - X) - 8\\ &\leq 7\nu(G) + 3\rho(G) + 2 - 7 - 8 \leq 7\nu(G) + 3\rho(G) - 8,\end{aligned}$$

a contradiction. So Claim (1) holds.

Proof of Claim (2). By the previous claim, G has at least 10 vertices. Therefore, to prove the present claim, it suffices to prove that G has no cutset of size 3. However, in order to use Lemma 6 we need to show that G is 3-connected; we leave this straightforward task to the reader. Now assume that G is 3-connected. Suppose that G has a cutset X consisting of three vertices x, y and z. Let $Y = \{y, z\}$. By Lemma 6, there exist induced subgraphs G_1 and G_2 of G and a polygon $P \subset \mathbb{R}^2$ determining two regions R_1 and R_2 in the plane such that P intersects G precisely in x, y and z, and G_i equals the restriction of G to the closure of region R_i, for $i = 1, 2$. It suffices to consider the following two cases. Indeed, if $G_1 - X$ and $G_2 - X$ are both balanced then G has a transversal of size at most 3, contradicting Claim (1).

Case 1. Neither $G_1 - X$ nor $G_2 - X$ is balanced. It follows that neither $G_1 - Y$ nor $G_2 - Y$ is balanced. If we have $\nu(G) \geq \nu(G_1 - Y) + \nu(G_2 - Y)$ then Lemma 7 implies

$$\begin{aligned}\tau(G) &\leq 2 + \tau(G_1 - Y) + \tau(G_2 - Y)\\ &\leq 2 + 7\nu(G_1 - Y) + 3\rho(G_1 - Y) - 8 + 7\nu(G_2 - Y) + 3\rho(G_2 - Y) - 8\\ &\leq 7\nu(G) + 3\rho(G) + 2 + 6 - 16 = 7\nu(G) + 3\rho(G) - 8,\end{aligned}$$

a contradiction. Else, we have $\nu(G) = \nu(G_1 - Y) + \nu(G_2 - Y) - 1$. It follows that every maximum packing of $G_1 - Y$ and every maximum packing of $G_2 - Y$ hit the vertex x. So we have $\nu(G_1 - X) = \nu(G_1 - Y) - 1$, $\nu(G_2 - X) = \nu(G_2 - Y) - 1$ and $\nu(G) = \nu(G_1 - X) + \nu(G_2 - X) + 1$. Therefore, we have

$$\begin{aligned}\tau(G) &\leq 3 + \tau(G_1 - X) + \tau(G_2 - X)\\ &\leq 3 + 7\nu(G_1 - X) + 3\rho(G_1 - X) - 8 + 7\nu(G_2 - X) + 3\rho(G_2 - X) - 8\\ &\leq 7\nu(G) + 3\rho(G) + 3 - 7 + 6 - 16 \leq 7\nu(G) + 3\rho(G) - 8,\end{aligned}$$

a contradiction.

Case 2. $G_1 - X$ is balanced and $G_2 - X$ is not balanced. If G_1 is not balanced, then we have $\nu(G) \geq \nu(G_2 - X) + 1$ and hence

$$\begin{aligned}\tau(G) &\leq 3 + \tau(G_2 - X) \leq 3 + 7\nu(G_2 - X) + 3\rho(G_2 - X) - 8\\ &\leq 7\nu(G) + 3\rho(G) + 3 - 7 - 8 \leq 7\nu(G) + 3\rho(G) - 8,\end{aligned}$$

a contradiction. Otherwise, G_1 is balanced. Consider the graph G_2' obtained from G_2 by adding a triangle on x, y and z to G_2'. We do not add an edge if it is already present in G_2. Since we can easily modify G to get a drawing of G_2', we can regard G_2' as a plane graph. Consider any two distinct vertices u, v in $X = \{x, y, z\}$. Because G_1 is balanced, all u–v paths in G_1 have the same parity. We let the parity of the edge uv in G_2' be the parity of all u–v paths in G_1. Note that we have $\tau(G) \leq \tau(G_2')$ and $\nu(G_2') \leq \nu(G)$. Moreover, we have $\rho(G_2') \leq \rho(G)$, as we now prove. Since G is 3-connected, there is a vertex t in $G_1 - X$ sending three independent paths to x, y and z in G_1. By Lemma 5, if we delete from G all edges which are contained in G_1 except those which belong to one of the three paths, the resulting graph G' satisfies $\rho(G') \leq \rho(G)$. Since the triangle on x, y, z determines an even face in G'_2, we have $\rho(G'_2) \leq \rho(G')$. Hence, we have $\rho(G'_2) \leq \rho(G)$, as claimed. It follows that we have

$$\tau(G) \leq \tau(G'_2) \leq 7\nu(G'_2) + 3\rho(G'_2) - 8 \leq 7\nu(G) + 3\rho(G) - 8,$$

a contradiction. In conclusion, Claim (2) holds.

Proof of Claim (3). Suppose that $d_{G^+}(f_i, f_j) \leq 6$ for some distinct indices i and j. Let X denote the set of vertices of G on a shortest path between f_i and f_j in G^+. So X contains at most three vertices. By Lemma 5, we have $\rho(G - X) \leq \rho(G) - 1$. Because $G - X$ is not balanced, we have

$$\tau(G) \leq 3 + \tau(G - X) \leq 3 + 7\nu(G - X) + 3\rho(G - X) - 8 \leq 7\nu(G) + 3\rho(G) - 8,$$

a contradiction. So Claim (3) holds.

Now we would like to apply Lemma 4 around each face in the collection $\{f_1, \ldots, f_r\}$. So each face f_i will perform as a center. The targets around f_i are the odd faces of G whose boundary intersects the boundary of f. By Claim (3), whenever g is a target around f_i and g' is a target around f_j with $i \neq j$, the boundaries of g and g' are disjoint. By Lemma 4, for each center f_i there exists a packing of odd cycles $\mathcal{C}_i$ formed by target boundaries, and a local transversal W_i whose size is at most twice the size of packing $\mathcal{C}_i$. Let $\mathcal{C}_{\text{local}}$ denote the union of packings $\mathcal{C}_1, \ldots, \mathcal{C}_r$. Then $\mathcal{C}_{\text{local}}$ is a packing.

Now let $H = G^+$, let $A = V(G)$ and let $B = F(G)$. Consider the graph $\tilde{H}$ obtained from H by contracting, for $1 \leq i \leq r$, all vertices of H at distance at most 2 from f_i to a single vertex $\tilde{f}_i$. Note that $\tilde{H}$ is still bipartite, with bipartition $\{\tilde{A}, \tilde{B}\}$, where

$$\tilde{A} = A \setminus \{a \in A : d_H(a, f_i) \leq 2 \text{ for some } i \text{ with } 1 \leq i \leq r\},$$
$$\tilde{B} = B \setminus \{b \in B : d_H(b, f_i) \leq 2 \text{ for some } i \text{ with } 1 \leq i \leq r\} \cup \{\tilde{f}_i : 1 \leq i \leq r\}.$$

Let $\tilde{T}$ denote the set of those $\tilde{f}_i$'s that correspond to centers f_i which have an odd number of targets around them. So $\tilde{T}$ is an even subset of $\tilde{B}$. Let $\tilde{J}$ denote a minimum length $\tilde{T}$-join in $\tilde{H}$. Then, by Proposition 5, there is a collection of fringe-disjoint $\tilde{T}$-cuts $\delta(\tilde{\mathcal{F}})$ in $\tilde{H}$ such that

$$|\delta(\tilde{\mathcal{F}})| \geq \frac{1}{2}(|\tilde{J}| - |\tilde{T}| + 1) \geq \frac{1}{2}(|\tilde{J}| - r + 1) \Rightarrow |\tilde{J}| \leq 2|\delta(\tilde{\mathcal{F}})| + r - 1.$$

This collection of fringe-disjoint $\tilde{T}$-cuts yields a packing of odd cycles $\mathcal{C}_{\text{global}}$ in G, of the same size. The $\tilde{T}$-join $\tilde{J}$ defines a set of edges J_{global} in $H = G^+$, as follows. Every edge of $\tilde{J}$ that belongs to H is kept as it is. Every other edge of $\tilde{J}$ is of the form $v\tilde{f}_i$ and is replaced by any shortest path between v and f_i in H. Because we have $d_{\tilde{H}}(\tilde{f}_i, \tilde{f}_j) \geq 4$ whenever $i \neq j$ and because $\tilde{J}$ is the edge-disjoint union of shortest paths between pairs of vertices of $\tilde{T}$, the length of J_{global} is at most twice the length of $\tilde{J}$.

For each local transversal W_i, let J_i denote the set of edges vf of the face-vertex incidence graph G^+ such that $v \in W_i$ and f is a target around f_i incident to v. Let J_{local} denote the union of $J_1, \ldots, J_r$. The union of J_{local} and J_{global} contains a T-join, say J. Because the width of J is at most the width of J_{local} plus the width of J_{global} and because the width of a T-join is at most half of its length, the width of J is at most

$$\sum_{i=1}^{r} |W_i| + \frac{1}{2}|J_{\text{global}}| \leq 2|\mathcal{C}_{\text{local}}| + 2|\mathcal{C}_{\text{global}}| + r - 1 \leq 4\nu(G) + \rho(G) - 1.$$

By Claim (1), we have $\nu(G) \geq 2$. Therefore, we have

$$\tau(G) \leq 4\nu(G) + \rho(G) - 1 \leq 7\nu(G) + 3\rho(G) - 8,$$

a contradiction. This concludes the proof of the theorem. □

Because $\rho(G)$ is at most the size of any inclusion-wise maximal collection of boundary-disjoint odd faces in G, which is in turn at most $\nu(G)$, we obtain our main result from Theorem 4.

Corollary 2. *For every plane signed graph G, we have $\tau(G) \leq 10\nu(G)$.* □

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