COMBINATORICA
Bolyai Society – Springer-Verlag

COMBINATORICA **21** (2) (2001) 267–278

THE ERDŐS–PÓSA PROPERTY FOR ODD CYCLES IN HIGHLY CONNECTED GRAPHS

DIETER RAUTENBACH*, BRUCE REED[†‡]

Dedicated to the memory of Paul Erdős

Received November 17, 1999

In [9] Thomassen proved that a $2^{3^{9k}}$-connected graph either contains k vertex disjoint odd cycles or an odd cycle cover containing at most $2k-2$ vertices, i.e. he showed that the Erdős–Pósa property holds for odd cycles in highly connected graphs. In this paper, we will show that the above statement is still valid for $576k$-connected graphs which is essentially best possible.

1. Introduction

A family $\mathcal{F}$ of graphs is said to have the *Erdős–Pósa property*, if for every integer k there is an integer $f(k,\mathcal{F})$ such that every graph G either contains k vertex disjoint subgraphs each isomorphic to a graph in $\mathcal{F}$ or a set C of at most $f(k,\mathcal{F})$ vertices such that G–C has no subgraph isomorphic to a graph in $\mathcal{F}$. The term *Erdős–Pósa property* arose because in [3] Erdős and Pósa proved that the family of cycles has this property. The family of odd cycles does not have the Erdős–Pósa property, as we now show. For a graph G an *odd cycle cover* is a set of vertices $C\subseteq V(G)$ such that $G-C$ is bipartite.

An elementary wall of height eight is depicted in Figure 1. An *elementary wall* of height h for $h\geq 3$ is similar. It consists of h levels each containing h bricks, where a brick is a cycle of length six. A *wall* of height h is obtained

Mathematics Subject Classification (2000): 05C40, 05C70, 05C99

* This work was supported by a post-doctoral DONET grant.

† This work was supported by an NSF-CNRS collaborative research grant.

‡ This work was performed while both authors were visiting the LIRMM, Université de Montpellier II, France.

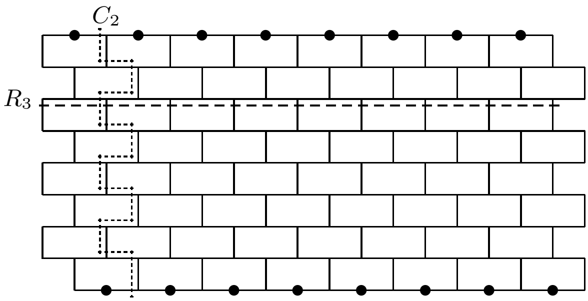

Fig. 1. An elementary wall of height 8

from an elementary wall of height h by subdividing some of the edges, i.e. replacing the edges with internally vertex disjoint paths with the same end-points (see Fig. 2).

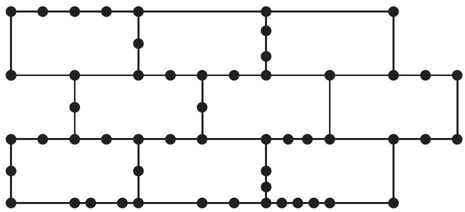

Fig. 2. A wall of height 3

An *Escher wall of height* h consists of a wall of height h: W, and h vertex disjoint paths $P_1, \ldots, P_h$ such that:

(i) Each P_i has both endpoints on W but is otherwise disjoint from W.
(ii) One endpoint of P_i is in the ith brick of the top row of bricks of W, the other is in the $(h+1-i)$th brick of the bottom row of W. Furthermore, both of these vertices are in only one brick of W.
(iii) W is bipartite but for each i, $W \cup P_i$ contains an odd cycle.

See Figure 3 for an example.

We remark that, as pointed out by Lovász and Schrijver [5], an Escher wall W of height h contains neither 2 vertex disjoint odd cycles nor an odd cycle cover with fewer than h vertices (the first fact follows from the fact that for any Escher wall $W, P_1, \ldots, P_k$, the planar embedding of W can be extended to an embedding of the Escher wall in the projective plane so that every odd cycle is non-null homotopic by routing the P_i through a cross-cap;

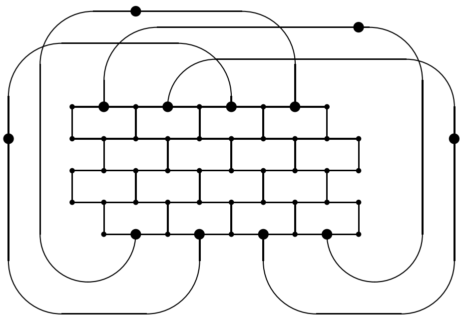

Fig. 3. An Escher wall of height 4

the fact that there is no odd cycle cover of size $h-1$ follows from the fact that if X is a set of $h-1$ vertices then X fails to intersect some path along the top of a level of bricks of W and then, as is easily verified, there is some i such that P_i is disjoint from X and both endpoints of P_i are connected to this row in $W-X$. Hence the two endpoints of this P_i are connected by a path Q in $W-X$ and then P_i+Q is an odd cycle in $G-X$). This shows that the Erdős–Pósa property does not hold for odd cycles (in fact, it holds for the cycles of length $p \mod m$ if and only if p is congruent to 0 $\mod m$ see [2] and [8]).

In [9] Thomassen proved, amongst other results, that for any integer k a $2^{3^{9k}}$-connected graph either contains k vertex disjoint odd cycles or an odd cycle cover of size at most $2k-2$. The size of the odd cycle cover in his result is best possible as can be seen by considering a large complete bipartite graph in which we add a complete graph on $2k-1$ vertices in one of the partite sets. This construction shows that no condition on the connectivity can lead to smaller odd cycle covers in graphs without k vertex disjoint odd cycles.

In this paper we will improve on Thomassen's result showing that the above statement is still valid for $576k$-connected graphs. This is essentially best possible in the sense that we need to impose connectivity of $\Omega(k)$ to obtain this result. Our approach is to split each vertex of a minimum odd cycle cover into two, thereby producing bipartite auxiliary graphs and then to analyze the set of paths in the auxiliary graphs corresponding to odd cycles in the original graph.

This approach was first adopted in [7]. In that paper Reed proved, amongst other results, that for all k there is an n_k such that a graph either contains a set of $2k$ odd cycles such that every vertex is in at most two of these odd cycles or an odd cycle cover of size at most n_k. Much of our introductory discussion was taken from that paper.

We now define precisely the auxiliary graphs we study. We define for every graph G and odd cycle cover $C=\{c_1,\dots,c_l\}$ of G, a *first auxiliary (bipartite) graph* $G'=G'(G,C)$. To do so, we first choose an arbitrary partition of $G-C$ into two stable sets A' and B' (note that we have only one choice, if $G-C$ is connected). We set $V(G')=(V(G)-C)\cup\{c^A,c^B|c\in C\}$ and $E(G')=E(G-C)\cup\{c^Ay|c\in C,y\in B',cy\in E(G)\}\cup\{c^By|c\in C,y\in A',cy\in E(G)\}\cup\{c_i^Ac_j^B|c_i,c_j\in C,c_ic_j\in E(G),i<j\}$. We note that G' is bipartite with bipartition $(A=A'\cup\{c^A|c\in C\},B=B'\cup\{c^B|c\in C\})$. For any vertex c in C we define $image(c)$ as $\{c^A,c^B\}$ and $preimage(c^A)=preimage(c^B)=c$. For any vertex x in $G-C$, we set $image(x)=preimage(x)=x$. For any set S of vertices of G, the *image of* S is the union of the images of its elements. Similarily, for any set S of vertices of G', the *preimage of* S is the union of the preimages of its elements. Now, each edge e of G' corresponds to an edge of G whose endpoints are the preimages of the endpoints of e. We call this edge the *preimage* of e. The preimage function extends to subgraphs in a natural manner. For any vertex $c\in image(C)$ we define the *partner* $p(c)$ of c as the unique vertex different from c such that $preimage(c)=preimage(p(c))$ and c and $p(c)$ will be called *partners*. For any set S of vertices in $image(C)$, the set $p(S)$ of partners of S is the union of the partners of its elements.

Now, for any subset $X=\{x_1,\dots,x_k\}$ of C, if there are k vertex disjoint paths $P_1,\dots,P_k$ in $G'-image(C-X)$, such that P_i links x_i^A with x_i^B, then the preimages of these paths form k vertex disjoint odd cycles of G. In the proof of our main result we will look for such sets of paths as well as similar more complicated structures using a second auxiliary bipartite graph $G''=G''(G')$.

The graph G'' arises from G' by deleting for all $c\in C$ one of the two vertices in $image(c)$. If for any subset $X=\{x_1,\dots,x_k\}$ of C we can find k vertex disjoint paths $P_1',\dots,P_k'$ in the graph $G'[image(X)\cup V(G'')]$ such that P_i' links x_i^A and x_i^B, then these paths use at most one of the partners in $image(c)$ for any $c\in C-X$. Hence the preimages of these paths will form k vertex disjoint odd cycles of G. We will find such paths using a highly connected subgraph F of G''. If G'' has large average degree, then there are several results that imply the existence of such a highly connected subgraph. Since we are not interested in optimizing our constants, we use the following result due to Mader [6] which is one of the first and simplest of those results.

Theorem 1 **(Mader [6]).** *Every graph G of order $n(G)\geq 2k-1$ with more than $(2k-3)(n(G)-(k-1))$ edges contains a k-connected subgraph.*

The next lemma shows that we can choose for all $c \in C$ one of the partners in $image(c)$ to be deleted from G' in such a way that G'' has large average degree.

Lemma 1. *Let G be a $576k$-connected graph, $C = \{c_1, c_2, \ldots, c_l\}$ be an odd cycle cover of G and G' be the first auxiliary graph.*

There is a partition $image(C) = C_1 \cup C_2$ such that for $1 \le i \le l$ the set C_1 (and hence C_2) contains exactly one of the vertices in $image(c_i)$ and such that $G'' = G' - C_2$ has average degree at least $144k$.

Proof. For every $c \in C$ we choose uniformly at random either c^A or c^B to belong to C_1 and set $C_2 = image(C) - C_1$ and $G'' = G' - C_2$. Let $c \in C_1$ be any fixed vertex. The expected number of edges of G'' that are incident with c and a vertex in $V(G) - C$ is exactly half of the number of edges that are incident with $preimage(c)$ and a vertex in $V(G) - C$. Furthermore, the expected number of edges of G'' that are incident with c and another vertex in C_1 is exactly a quarter of the number of edges that are incident with $preimage(c)$ and another vertex in C. Let m_C and m_{C_1} denote the number of edges in G and G'' incident with vertices in C and C_1, respectively. Now, by linearity of expectation, $E[m_{C_1}] \ge \frac{1}{4} m_C$. Hence there is a choice of C_1 such that $m_{C_1} \ge \frac{1}{4} m_C$. This implies $m(G'') \ge \frac{1}{4} m(G)$. Since the minimum degree in G is at least $576k$, we have $\frac{2m(G)}{n(G)} \ge 576k$. As $n(G) = n(G'')$, this yields $\frac{2m(G'')}{n(G'')} \ge 144k$. ∎

Applying Theorem 1, we get the following.

Corollary 1. *Let G, C, G', C_1, C_2 and G'' be as in Lemma 1, then G'' contains a $36k$-connected subgraph F.*

During the proof we will have to link different pairs of vertices in F by vertex disjoint paths in F. The existence of such paths is guaranteed by the following result due to Bollobás and Thomason [1].

A graph G of order at least $2k$ is *k-linked*, if for any two disjoint sets $\{x_1, x_2, \ldots, x_k\}$ and $\{y_1, y_2, \ldots, y_k\}$ of k vertices there are k vertex disjoint paths $P_1, \ldots, P_k$ such that for $1 \le i \le k$ the path P_i has endpoints x_i and y_i. The following result ensures that a highly connected graph is also highly linked.

Theorem 2 (Bollobás and Thomason [1]). *Every $22k$-connected graph is k-linked.*

2. Result

Having constructed the two auxiliary graphs we are now in a position to prove the theorem.

Theorem 3. *Let G be a $576k$-connected graph. Then G either contains k vertex disjoint odd cycles or an odd cycle cover of at most $2k-2$ vertices.*

Remark. The constant 576 is certainly not best possible. We have made no attempt to minimize it.

Proof. Clearly $k \geq 2$. Let G be a $576k$-connected graph and let C be a minimum odd cycle cover of G. We assume that $|C| \geq 2k-1$ and prove the existence of k vertex disjoint odd cycles in G. We consider two cases.

Case 1. $|C| \leq 15k$

We prove that every $500k$-connected graph G with a minimum odd cycle C of size exactly $2k-1$ contains k vertex disjoint odd cycles. The statement of the theorem then follows by deleting all but $2k-1$ vertices from C.

Note that the graph $G-C$ is $450k$-connected. Let A', B' be a bipartition of $G-C$. Since G is $500k$-connected, every vertex has degree at least $500k$ and we can partition C in two sets C_A and C_B such that every vertex of C_A has at least $200k$ neighbours in A' and every vertex in C_B has at least $200k$ neighbours in B'. We may assume that $C_A = \{c_1, \ldots, c_{2l}\}$ and $C_B = \{c_{2l+1}, \ldots, c_{2k-1}\}$ for some $0 \leq l \leq k-1$.

If we can find a matching M_A in the graph $G[C_A \cup B']$ that covers all of C_A and a matching M_B in $G[C_B \cup A']$ that covers all of C_B, then we can find k vertex disjoint odd cycles in G as follows.

First, we choose different vertices $a_1, \ldots, a_{2l} \in A'$ and $b_{2l+1}, \ldots, b_{2k-1} \in B'$ such that for $1 \leq i \leq 2l$ the vertex a_i is a neighbour of c_i, for $2l+1 \leq i \leq 2k-1$ the vertex b_i is a neighbour of c_i and none of these vertices is incident with an edge in $M_A \cup M_B$. Such a choice is possible, since the vertices in C_A and C_B have at least $200k$ neighbours in A' and B', respectively.

For every vertex c_i in C_A that is matched to a vertex b in B, we can find an odd cycle through it by joining the vertices b and a_i by a path in $G-C$.

For every vertex c_i in C_A that is matched to a vertex c_j in C_A, we can find an odd cycle through these two vertices by joining their two neighbours a_i and a_j in A by a path in $G-C$. The cycles through vertices in C_B are constructed similarly.

We use at most $2k-1$ paths in $G-C$. Theorem 2 implies that we can choose these paths vertex disjoint. Since we need at most two vertices of C for one of these cycles, the graph G contains k vertex disjoint odd cycles.

Hence we can assume that there is no matching in $G[C_A \cup B']$, say, that covers all of C_A. Using a small extension of Tutte's matching theorem (see for instance [4]), this implies the existence of a set Z of vertices such that the graph $G[C_A \cup B'] - Z$ has at least $|Z|+1$ odd components that are contained in C_A. Let the set Z' contain exactly one vertex from each of these components. Then the graph $H = G - ((C - Z') \cup Z)$ is bipartite with bipartition $((A \cap V(H)), ((B \cup Z') \cap V(H)))$. Hence the set $(C - Z') \cup Z$ is an odd cycle cover of size $|C| - |Z'| + |Z| < |C|$ which contradicts the choice of C. This completes the proof of this case.

Case 2. $|C| > 15k$.

We consider the first auxiliary graphs G' choose a partition $image(C) = C_1 \cup C_2$ and the second auxiliary graph G'' containing a $36k$-connected subgraph F according to Lemma 1 and Corollary 1. For a set $X \subseteq C$ let $G'_X = G'[V(G'') \cup image(X)]$.

We choose a maximum size set $X \subseteq C$ such that there is no set of fewer than $2|X|$ vertices separating $image(X)$ and F in G'_X (note that in a graph H a set $S \subseteq V(H)$ is said to *separate* two sets $A, B \subseteq V(H)$, if in $H - S$ there is no path joining a vertex in A and a vertex in B).

By Menger's Theorem, there are $2|X|$ vertex disjoint paths joining $image(X)$ to F in G'_X which are internally disjoint from F. If $r := |X| \geq k$, then we choose k vertices $x_1, \ldots, x_k \in X$. For $1 \leq i \leq j$ let f_i^A and f_i^B be the endpoints of the paths joining x_i^A and x_i^B to F, respectively. Let P_i be a path from f_i^A to f_i^B in F. Since F is k-linked, we can assume that the paths $P_1, \ldots, P_k$ are vertex disjoint. For $1 \leq i \leq k$ we can now join the path from x_i^A to F, P_i and the path from x_i^B to F and get k vertex disjoint paths that are internally disjoint from C_2 and which correspond to k vertex disjoint odd cycles in G. Hence $r < k$.

We choose a set Z_0 of size $2r$ that separates $image(X)$ and F in G'_X such that the component U_0 of $G'_X - Z_0$ that contains $F - Z_0$ is as small as possible (note that $F - Z_0$ is still at least $34k$-connected.)

(2.1) There is a set Z^* of size at most $7k$ such that $Z_0 \subseteq Z^*$ and the component U^* of $G'_X - Z^*$ that contains $F - Z^*$ contains no vertex in C_1 whose partner is adjacent to a vertex in U^*.

Proof. Let $x_1, x_2, \ldots, x_j \in C_1$ be the vertices in U_0 whose partners are adjacent to a vertex in U_0. We assume $j > 0$, since otherwise we can choose $Z_0 = Z^*$.

By the choice of X, for $1 \leq i \leq j$ there is a set of at most $2r+1$ vertices that separates $image(X) \cup \{x_i, p(x_i)\}$ and F in $G'_{X \cup \{preimage(x_i)\}}$. We choose one such set Z_i such that the component U_i of $G'_{X\cup\{preimage(x_i)\}}-Z_i$ that contains $F-Z_i$ is as small as possible. Note that this choice implies $|Z_i|=2r+1$.

(2.1.1) $p(x_i)\notin Z_i$ for $1\le i\le j$.

Proof. We assume that $p(x_i)\in Z_i$ for some $1\le i\le j$ and derive a contradiction to the choice of Z_0. Let $Z'_i=Z_i-\{p(x_i)\}$ and let U'_i be the component of $G'_X-(Z_0\cup Z'_i)$ that contains $F-(Z_0\cup Z'_i)$. Let A_1 be the set of vertices in $Z_0\cup Z'_i$ that are adjacent to a vertex in U'_i and let A_2 be the set of vertices in $Z_0\cup Z'_i$ that are either in $image(X)$ or adjacent to a vertex in a component of $G'_X-(Z_0\cup Z'_i)$ containing a vertex of $image(X)$. Since $U'_i\subseteq U_0$, $U'_i\subseteq U_i$ and Z_0 and Z'_i separate $image(X)$ and F in G'_X, we obtain that $A_1\cap A_2\subseteq Z_0\cap Z'_i$. This implies $|A_1|+|A_2|\le|Z_0|+|Z'_i|=2r+(2r+1-1)=4r$. As A_2 separates $image(X)$ and F in G'_X, we have $|A_2|\ge 2r$, by the choice of X, and therefore $|A_1|\le 2r$. Now A_1 separates $image(X)$ and F in G'_X and since $U'_i\subseteq U_0$ and $x_i\in U_0-U'_i$ we obtain a contradiction to the choice of Z_0. ∎

By (2.1.1), we see that Z_i separates

$$image(X)\cup\{x_i\}\cup N(p(x_i),G'_{X\cup\{preimage(x_i)\}})\quad\text{and}\quad F \text{ in } G'_X.$$

Further, by definition, the component U_i of G'_X-Z_i is as small as possible given this condition.

(2.1.2) For $1\le i\le j$ we have $Z_i\subseteq Z_0\cup U_0$ and Z_i separates Z_0 and F in G'_X.

Proof. As in the proof of (2.1.1), let U'_i be the component of $G'_X-(Z_0\cup Z_i)$ that contains $F-(Z_0\cup Z_i)$. We choose the sets A_1 and A_2 as above and get $|A_1|+|A_2|\le|Z_0|+|Z_i|=4r+1$ and $|A_2|\ge 2r$. Hence $|A_1|\le 2r+1$. Since A_1 separates Z_i and F in G'_X, we obtain, by the choice of Z_i, that $A_1=Z_i$ and (2.1.2) is proved. ∎

(2.1.3) For any $1\le s<t\le j$ either $Z_s=Z_t$ or $Z_0\subseteq Z_s\cup Z_t$.

Proof. We assume that $Z_s\neq Z_t$. Note that U_s contains a vertex of Z_t and U_t contains a vertex of Z_s. Otherwise, we assume that $U_s\cap Z_t=\emptyset$ which implies that Z_s separates Z_t from F in G'_X and therefore $U_s\subseteq U_t$. As $Z_s\neq Z_t$, we have $U_s\neq U_t$ and we get a contradiction to the choice of Z_t. Let $U_{s,t}$ be the component of $G'_X-(Z_s\cup Z_t)$ that contains $F-(Z_s\cup Z_t)$. Let A_1 be the set of vertices in $Z_s\cup Z_t$ that are adjacent to a vertex in $U_{s,t}$ and let A_2 be the set of vertices that are in Z_0 or are adjacent to a vertex in a component of $G'_X-(Z_s\cup Z_t)$ containing a vertex of Z_0. Since Z_s and Z_t separate Z_0 from F in G'_X, we have that $A_1\cap A_2\subseteq Z_s\cap Z_t$ and therefore $|A_1|+|A_2|\le|Z_s|+|Z_t|=4r+2$. As $U_{s,t}$ is stricly contained in U_s and U_t and

A_1 separates Z_s and Z_t from F in G'_X, we see that $|A_1| \geq 2r+2$ or we get a contradiction to the choice of Z_s or Z_t. Hence $|A_2| \leq 2r$ and, as A_2 separates Z_0 and F in G'_X, we get $A_2 = Z_0$, by the choice of Z_0. This proves (2.1.3). ∎

We may assume that $Z_0 \subseteq Z_i$ for $1 \leq i \leq j'$ and that $Z_0 - Z_i \neq \emptyset$ for $j'+1 \leq i \leq j$. For every $1 \leq i \leq j'$ there is a vertex $y_i \in U_0$ such that $Z_i = Z_0 \cup \{y_i\}$. Furthermore, we may assume that all sets $Z_{j'+1}, \ldots, Z_{j''}$ are different and that for every $j''+1 \leq i \leq j$ there is some $j'+1 \leq i' \leq j''$ such that $Z_i = Z_{i'}$.

(2.1.4) *G contains k vertex disjoint odd cycles or $|\{y_1, y_2, \ldots, y_{j'}\}| \leq k-1$.*

Proof. We consider the block-structure of U_0. There is one block B_0 of U_0 that contains $F - Z_0$. By the choice of Z_i, we see that y_i is a cutvertex in B_0 such that $\{y_i\}$ separates $\{x_i\} \cup N(p(x_i), G'_{X \cup \{preimage(x_i)\}})$ and F in U_0. This implies that the vertices in $\{x_i\} \cup N(p(x_i), G'_{X \cup \{preimage(x_i)\}})$ are contained in one component of $U_0 - \{y_i\}$. We assume now that $|\{y_1, y_2, \ldots, y_{j'}\}| \geq k$. Then there is a set Y of k cutvertices in B_0 such that there are k components in $U_0 - Y$ each of which contains a path from some x_i to a neighbour of its partner. Hence there are k vertex disjoint paths in G' from some x_i's to their partners and all these paths are internally disjoint from C_2. Thus there is a set of k vertex disjoint odd cycles in G. This completes the proof of (2.1.4). ∎

We may assume now that $|\{y_1, y_2, \ldots, y_{j'}\}| \leq k-1$. For all $1 \leq i \leq j$ we have that $|Z_i - Z_0| = |Z_0 - Z_i| + 1$ and since, by (2.1.3), every vertex of Z_0 is missing from at most one Z_i for $j'+1 \leq i \leq j''$, we have $j'' - j' \leq 2r < 2k$. This implies

$$| \cup_{i=0}^{j} Z_i| \leq |Z_0| + |\{y_1, y_2, \ldots, y_{j'}\}| + \sum_{i=j'+1}^{j''} |Z_i - Z_0|$$

$$< 2k + k + 2k + \sum_{i=j'+1}^{j''} |Z_0 - Z_i| \leq 7k.$$

Hence the set $Z^* := \cup_{i=0}^{j} Z_i$ satisfies $|Z^*| \leq 7k$, $Z_0 \subseteq Z^*$ and the component U^* of $G'_X - Z^*$ that contains $F - Z^*$ contains no vertex in C_1 whose partner is adjacent to a vertex of U^*. This proves (2.1). ∎

Let $X^* = X \cup preimage(\{x_1, \ldots, x_j\})$. The set Z^* separates $image(X^*)$ and F in G'_{X^*}. Otherwise there is a path P in $G'_{X^*} - Z^*$ from $u \in image(X^*)$ to F that is internally disjoint from $image(X^*)$. Since $u \in \{x_i, p(x_i)\}$ for some $1 \leq i \leq j$ and $Z_i \subseteq Z^*$, the path P is contained in $G'_{X \cup preimage\{x_i\}} - Z_i$. This is a contradiction to the choice of Z_i.

Let $C_1^* = U^* \cap C_1$. By the construction of Z^*, the vertices in $p(C_1^*) \subseteq C_2$ are not adjacent to a vertex in U^*.

(2.2) $|C_1^*| \le 7k$.

Proof. We can partition the components of $G' - (image(C) \cup Z^* - (C_1^* \cup p(C_1^*)))$ in two sets in such a way that for every vertex in C_1^* its partner lies in a component on the other side of this partition. Hence we can find a bipartition of $G' - (image(C) \cup Z^* - (C_1^* \cup p(C_1^*)))$ such that $G - (C \cup preimage(Z^*) - preimage(C_1^*))$ arises by identifying vertices on the same side of this bipartition and is therefore bipartite. Hence $C \cup preimage(Z^*) - preimage(C_1^*)$ is an odd cycle cover of G of size at most $|C| + |Z^*| - |C_1^*|$. By the choice of C, we have $|C| \le |C| + |Z^*| - |C_1^*|$ which yields (2.2). ∎

Let $Z^{**} = Z^* \cup C_1^*$ and let U^{**} be the component of G'_{X^*} that contains $F - Z^{**}$.

(2.3) There are $16k$ vertex disjoint paths from $C_2 - image(X^*)$ to F in $G' - Z^{**}$.

Proof. If there are no such paths, then there is a set W of size less than $16k$ that separates $C_2 - image(X^*)$ and F in $G' - Z^{**}$. Since $|W \cup Z^{**}| < 16k + 14k = 30k < |image(C)|$, the set $W \cup Z^{**}$ is a cutset of G' such that the component of $G' - (W \cup Z^{**})$ that contains $F - (W \cup Z^{**}) \neq \emptyset$ contains no vertex of $image(C) - (W \cup Z^{**}) \neq \emptyset$. Hence $preimage(W \cup Z^{**})$ is a cutset of G of size at most $30k$ which is a contradiction to the connectivity of G. ∎

Hence there is a set of $16k$ vertex disjoint paths from $C_2 - image(X^*)$ to F in $G' - Z^{**}$. If we choose these paths minimal, then there is a set Y with $Y \subseteq C_2 - image(X^*)$ of $16k$ vertices which are endpoints of these paths and the paths are internally contained in U^{**} and internally disjoint from F. Let $Y = \{y_1, \ldots, y_{16k}\}$ and for $1 \le i \le 16k$ let f_i be the endpoint of the path that joins y_i to F.

(2.4) There are $|Y|$ vertex disjoint paths from Y to $p(Y)$ in $G'_{preimage(Y)}$.

Proof. If there are no such paths, then there is a set W' of size less than $|Y|$ that separates Y and $p(Y)$ in $G'_{preimage(Y)}$. As above, we can now partition the components of $G' - ((image(C) - (Y \cup p(Y))) \cup W')$ in two sets in such a way that for every vertex in Y its partner lies in a component on the other side of this partition. Hence $G - ((C - preimage(Y)) \cup preimage(W'))$ is bipartite and $(C - preimage(Y)) \cup preimage(W')$ is an odd cycle cover of G of size less than $|C| - |Y| + |Y| = |C|$ which is a contradiction. ∎

By the last claim, there is a set of $2k$ vertex disjoint paths from Y to $p(Y)$ in $G'_{preimage(Y)} - Z^{**}$. If we choose these paths $P_1, \ldots, P_{2k}$ minimal, then they are internally disjoint from $Y \cup p(Y) \cup U^{**}$. For $1 \le i \le 2k$ let the path P_i join $p(y_{s_i}) \in p(Y)$ to $y_{t_i} \in Y$. We can now define k vertex disjoint paths $P'_1, \dots, P'_k$ in G' that correspond to vertex disjoint odd cycles in G.

Let P'_1 consist of the path P_1 from $p(y_{s_1})$ to y_{t_1}, the path from y_{t_1} to f_{t_1}, a path from f_{t_1} to f_{s_1} contained in $F - Z^{**}$ and the path from f_{s_1} to y_{s_1}. Proceeding in this manner, we see that for one of the paths P'_i we have to use at most two of the $2k$ paths P_j. Since $F - Z^{**}$ is $(36-14)k = 22k$-connected and, by Theorem 2, k-linked, we can choose the k paths that are contained in $F - Z^{**}$ to be vertex disjoint. We have therefore proved the existence of k vertex disjoint odd cycles in G and the proof of the theorem is completed. ∎

A result similar to Theorem 3 cannot be obtained, if the condition on the connectivity is replaced by a condition on the minimum degree. To see this, idenfity each vertex v of an Escher wall with a vertex in a large complete bipartite graph G_v.

Furthermore, to obtain Theorem 3, we have to impose a connectivity of $\Omega(k)$ on the graph which can be seen by the following example. Take a graph G with no two vertex disjoint odd cycles and no odd cycle cover of size $2k-1$. Add $\frac{k}{2}$ vertices that are joined to each vertex of G. The constructed graph is at least $\frac{k}{2}$-connected, has at most $\frac{k}{2}+1$ vertex disjoint odd cycles but has no odd cycle cover of size $2k-1$. In view of this construction, it is concievable that the statement of Theorem 3 holds for k-connected graphs.

Acknowledgement. We want to thank the referee for his careful reading and useful remarks.

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Dieter Rautenbach

Equipe Combinatoire,
Université Pierre et Marie Curie,
75013 Paris, France,
rauten@math2.rwth-aachen.de

Bruce Reed

Equipe Combinatoire,
Université Pierre et Marie Curie,
75013 Paris, France,
reed@ecp6.jussieu.fr