Proof of Ore’s Theorem by Backwards Induction

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**Ore’s Theorem – Combining Backwards Induction with the Pigeonhole Principle**

Let \( G = (V, E) \) be a connected simple graph with \( n \geq 3 \) vertices. If \( G \) has the property that for each pair of non-adjacent vertices \( u, v \in V \), we have that \( \deg u + \deg v \geq n \) then \( G \) contains a Hamiltonian cycle.

**Proof:** by backwards induction on the number of edges in \( E \):

**Base case:** \( G_c \) is the complete graph with \( n(n-1)/2 \) edges.

Connect the vertices in \( G_c \) in any order such as \( (v_1, v_2, ..., v_n) \) to create a Hamiltonian path, and add edge \( (v_n, v_1) \) to create a Hamiltonian cycle.
Ore’s Theorem – Combining Backwards Induction with the Pigeonhole Principle

**Induction hypothesis:** the theorem is true when $G$ has $k$ edges.

- We must prove the theorem when $G$ has $k-1$ edges.
- Let $G$ be such a graph, and let $v_n$ and $v_1$ be a pair of non-adjacent vertices in $G$ such that $\deg v_n + \deg v_1 \geq n$. 
Ore’s Theorem – Combining Backwards Induction and the Pigeonhole Principle

**Induction hypothesis:** the theorem is true when $G$ has $k$ edges.

- Let $G'$ be the graph obtained by adding an edge between $v_n$ and $v_1$ in $G$. $G'$ therefore has $k$ edges.
- It follows from the induction hypothesis that $G'$ contains a Hamiltonian cycle.
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Let $H'$ be the Hamiltonian cycle in $G'$.

We must now remove the edge $(v_n, v_1)$ from $G'$ to restore $G$.

Two cases arise:

**Case 1:** $H'$ does not contain $(v_n, v_1)$. Then $H'$ is a Hamiltonian cycle in $G$, and we are done. Edge $(v_n, v_1)$ may be safely removed from $G'$.
Ore’s Theorem – Combining Backwards Induction and the Pigeonhole Principle

Case 2: $H'$ contains $(v_n, v_1)$.

- Without loss of generality let $H' = (v_1, v_2, \ldots, v_n, v_1)$.

- Delete the edge $(v_n, v_1)$ from $G'$ to recover $G$. 

$H'$
Ore’s Theorem – Combining Backwards Induction and the Pigeonhole Principle

Case 2: continued...

Since $\text{deg } v_n + \text{deg } v_1 \geq n$ it follows from the Pigeonhole Principle that here must exist vertices $v_{i-1}$ and $v_i$ such that $v_{i-1}$ is connected to $v_n$ and $v_i$ is connected to $v_1$.
Case 2: continued...

Therefore, $G$ contains the Hamiltonian cycle $H = (v_1, v_2, \ldots, v_{i-1}, v_n, v_{n-1}, v_{n-2}, \ldots, v_i, v_1)$. Q.E.D.