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An Application of Burnside's Theorem to Music Theory

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Introduction

Isihara and Knapp [1993] introduce fundamental ideas in an area of music theory that musicians refer to as *set theory*. At least one reader of that UMAP Module was inspired to do a more advanced treatise that enumerates mosaics [Fripertinger 1999]. Also, a very active community investigating the connections between mathematics and music has developed. In fact, there is a newly formed Society for Mathematics and Computation in Music, which publishes the *Journal of Mathematics and Music* devoted to exploring those connections; and recently, *Science* published for the first time an article about music theory [Tymoczko 2006].

We discuss an aspect of this theory that is not discussed by Isihara and Knapp but that is a topic of continuing interest to music theorists [Hook 2007], namely, *how to count the number of distinct set classes*. We emphasize using Burnside's Lemma (Neumann [1979] recounts its history) for doing the counting. This approach easily extends the results to include counting set classes of *pitch class multisets* (pitch class sets with repeated entries). Since pitch class sets that are members of the same set class sound similar [Morris 1991], composers want to know how many distinct set classes there

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are for a given size of pitch class set, since this number would describe the number of different sound combinations available (for more on composing with pitch class sets, see Morris [1987]). The reader familiar with elementary group theory and combinatorics should be well-prepared for this paper.

First, we give a brief introduction to some musical terms and operations to set the stage. Next is a brief introduction to group actions and Burnside's Theorem. Following this material is a development of the invariance properties of transformations introduced earlier. Once the invariance properties are established, we devote some attention to some examples of Burnside's Theorem in action, and an extension to orbits of multisets. The last section has some concluding thoughts.

Music Terminology

We introduce some of the terminology of atonal music theory (pitch classes and pitch class sets), define the commonly used mathematical transformations that act on pitch class sets, discuss the term "set class," and pose the question that we wish to answer.

If we start with a tone at a certain frequency, ω, and create a new tone with frequency 2ω, the two tones are one octave apart. The octave is an important interval in music theory, since the octave is usually the basis for creating musical scales.

To create a musical scale, a musician divides the octave into a finite number of tones. In atonal theory, the tones are assumed to be equally spaced and all tones separated by an octave are considered equivalent. Each of these tones creates a category that music theorists refer to as a *pitch class*.

Usually the octave is divided into 12 pitch classes; however, many composers and some non-Western cultures use more divisions. A musical system that divides the octave into more than 12 pitch classes is called *microtonal*. There are also cultures that use fewer than 12 pitch classes per octave.

For generality, we assume that the octave is divided into p pitch classes, labelled with a number starting with 0; so the set of pitch classes is $\mathbf{P} = \{0, 1, 2, \ldots, p-1\}$.

Any subset of the set of pitch classes is a *pitch class set*. A pitch class set is to atonal theory as a chord is to tonal theory. In a tonal piece of music, the chords are chosen to sound pleasing and there are aesthetic rules for chord sequencing. In atonal music, the sequence of pitch class sets is determined using mathematical operations. The aesthetics here are not necessarily based on how the music sounds, but instead are based on using pleasing mathematical principles such as symmetry.

How does a composer use mathematics to construct a piece of music? One method is to use mathematical operations to transform one pitch class set into another. We consider two kinds of operations:

- a *transposition* operator, denoted by T_n; and
- an *inversion operator*, denoted by $T_n I$.

A transposition operator acts on a pitch class set by adding a fixed constant to each element of it. Since the result must be in $\mathbf{P}$, the addition is done modulo p. Since $k + p \bmod p = k$, we have exactly p transposition operators at our disposal. For example, using $p = 12$ and $x = \{0, 4, 7\}$, we calculate $T_7(x) = \{0 + 7, 4 + 7, 7 + 7\} = \{7, 11, 14\}$, which gives the result $\{7, 11, 2\}$ since $14 \bmod 12 = 2$. Since the order of the pitch classes in a pitch class set doesn't matter, this result would usually be written $\{2, 7, 11\}$.

An inversion operator works on a pitch class set by subtracting each pitch class from p and then performing a transposition. Once again, the transposition is done modulo p. Since there are p distinct transposition operators, there are also p inversion operators. Let's apply T_3I to the pitch class set $x = \{1, 5, 8, 11\}$ with $p = 13$. First, we invert all the elements of x, $T_3I(\{1, 5, 8, 11\}) = T_3(\{12, 8, 5, 2\})$ and then perform the transposition modulo 13 to obtain $\{2, 5, 8, 11\}$.

A circle diagram is a useful visualization aid (see Johnson [2003] for more on circle diagrams). To create a circle diagram, draw a circle and put p equally spaced crossing line segments. Next, label the crossing lines with the numbers representing the pitch classes. The diagram should remind you of a clock face without hands. To represent a pitch class set, place dots on the crossing lines that correspond to the members. In **Figure 1**, we see a representation of the pitch class set $\{0, 4, 7\}$ with $p = 12$ on the left, and on the right the same pitch class set after T_7 has been applied.

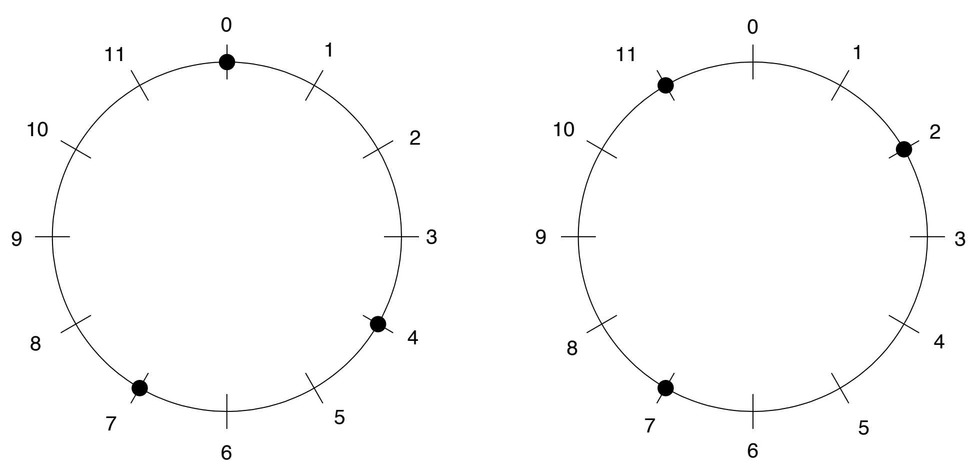

Figure 1. The pitch class set $\{0, 4, 7\}$ and $T_7(\{0, 4, 7\})$.

With a circle diagram, it is easy to see that a transposition operator corresponds to a clockwise rotation of a pitch class set. An inversion operator is a flip across the diameter of the circle through the position labelled 0, followed by a clockwise rotation. It is easy to verify that this set of operators is a group under function composition. In fact, it is the dihedral group D_p.

Suppose that you are a composer and you are working with a pitch class set x of size k. Since transpositions and inversions both preserve the intervals in a pitch class set [Morris 1991], a pitch class set and its transformation should sound similar. A composer might be interested in varying the intervals to make the music more interesting. To do so, the composer would have to find a pitch class set of size k that is not a transformation of x.

Music theorists refer to a collection of pitch class sets that are related by transposition and inversion operators as a *set class*. The more set classes there are, the more choices of sounds that a composer has. A natural question to ask is, how many distinct set classes are there for a given size pitch class set? The next section is a brief review of group actions and Burnside's Theorem, which are tools that we can use to answer this question (see Reiner [1985] and Fripertinger [1999] for a different approach, and see Hook [2007] for a tutorial on combinatorics and enumeration in music theory).

Group Actions and Burnside's Theorem

Group actions are a powerful idea from abstract algebra; we review the material about group actions that we need for our task.

Group Actions

Given a group G and a nonempty set S, a mapping $* : G \times S \rightarrow S$ is called an *action* [Nagpaul and Jain 2005] of G on S if for all $x \in S$ the following conditions are satisfied:

- $e * x = x$, where e is the identity in G; and
- $(gh) * x = g * (h * x)$ for all $g, h \in G$.

If there is an action of a group on a set S, then we say G *acts on* S and we call S a *G-set*.

Suppose that G acts on a set S. For any $x \in G$, the *orbit* of x under G is the set $\{g * x | g \in G\}$. A set class in music is an example of an orbit, as we explain shortly. From the definition of orbit, it is apparent that distinct orbits are mutually disjoint. Since every $x \in S$ must be in some orbit, we can conclude that the orbits form a partition of S. This partition is referred to as the *orbit decomposition* of S under G.

Given $g \in G$ and $x \in S$, if $g * s = s$ we say that g *fixes* s. Further, the set of all elements fixed by g is called the *fixture* of g and is written $\text{Fix}(g)$. The cardinality of $\text{Fix}(g)$ will be denoted by $F(g)$.

Burnside's Theorem

We state Burnside's Theorem without proof; for nice proofs see Nagpaul and Jain 2005] and Bogart [1991].

Theorem 1 (Burnside's Theorem) *Let G be a finite group acting on a finite set S. Then the number of distinct orbits in S under G is given by*

$$n = \frac{1}{|G|} \sum_{g \in G} F(g). \tag{1}$$

An easy way to remember Burnside's Theorem is to put **(1)** into words:

The number of distinct orbits in S under G is the average number of set elements fixed by a group element.

To apply Burnside's Theorem, we need

- a finite group,
- a finite set on which that group acts, and
- the cardinality of the fixtures of each group element.

In our problem, the group is the dihedral group D_p consisting of the p transposition operators and the p inversion operators. For pitch class sets of cardinality k, the set that our group acts on consists of all subsets of $\{0, 1, 2, \ldots, p-2, p-1\}$ of cardinality k. To determine the number of set classes (i.e., distinct orbits in S under G), we must determine the cardinality of the fixtures of the group elements. The next two sections discuss this topic.

The Fixture of Inversion Operators

We assume that the octave is divided into p equal steps and n is a number such that $0 \leq n \leq p-1$. We must also remember that we are doing arithmetic modulo p.

Recalling our discussion of the inversion operators above, each inversion operator is a flip across a line of symmetry and so is its own inverse, that is, $T_nI(T_nI(x)) = x$. Thus, the inversion operator T_nI fixes pairs of pitch classes $\{x, T_nI(x)\}$, or else a single pitch class if $T_nI(x) = x$. To compute $F(g)$ for an inversion operator g, we need to know how many singles and how many pairs are invariant. First, let's determine how many single pitch classes are fixed by an inversion operator.

Suppose that x satisfies $T_nI(x) = x$. This equation can be rewritten as

$$(p - x + n) \bmod p = x, \tag{2}$$

which is equivalent to

$$2x = n + kp. \tag{3}$$

Since $0 \leq x \leq p-1$, we need consider only $k = 0$ and $k = 1$. Since the left side of **(3)** is an even number, we have integer solutions only when the right side is also even. Since p can be even or odd and so can n, we have four cases to consider. First, suppose that p is even; then if n is odd we have no solutions, and if n is even we have two solutions. If p is odd, then there is one solution whether n is even or odd.

This information makes it possible to count the number of pitch class sets of a given size that an inversion operator fixes. We need only figure out how many pairs and singles make up a given pitch class set size.

Example: Counting the Number of Pitch Class Sets That Are Fixed. Suppose that $p = 13$ and we are interested in pitch class sets of size four. From the discussion above, we know that all the T_nI operators fix six pairs of pitch classes and one singleton pitch class. Since it takes two pairs to make four, and we have six pairs to choose from, each inversion operator fixes

$$\binom{6}{2} = 15$$

pitch class sets of size four.

If p is even, then it is slightly more complicated to do the count. Suppose that $p = 12$ and $k = 5$. All the even inversions in this case fix five pairs and 2 singles. Since five is two pairs plus one, the number of pitch class sets of size five that each even inversion fixes is

$$2\binom{5}{2} = 20.$$

The odd inversions fix six pairs, so they do not fix any set of size five.

We now know how to compute the cardinality of the fixtures of inversion operators for any number of pitch classes. To use Burnside's Theorem, we also need to know the cardinality of the fixtures of the transposition operators. This subject is the topic of the next section.

The Fixture of Transpositional Operators

Determining the fixtures of the transposition operators requires some group theory. Subgroups, cosets, and Cauchy's Theorem all make an appearance.

Cosets play a key role in determining the fixtures of the transposition operators; we recall the definition. Let G be a group, H a subgroup of G, and a any element of G. The set $Ha = \{ha|h \in H\}$ is a *right coset* of H; one can also define a left coset in an analogous manner. (Of course, in an abelian group they are the same.) We use the term "coset" to mean a right coset.

Recall that the set $\{0, 1, 2, \ldots, p-1\}$ with addition modulo p is the cyclic group Z_p, all of whose subgroups are also cyclic. So every subgroup has a generator, and by Cauchy's Theorem this generator must divide p. From these facts, we can conclude two things:

- Since every subgroup is cyclic with generator d, successive elements have the same difference, d. This means that the transposition operators $T_d, T_{2d} \ldots, T_{rd}$ all fix this subgroup. In addition, those operators also fix the cosets of that subgroup since elements of the coset are also multiples of d apart.
- The only operators T_m that can fix a pitch class set must have m not relatively prime to p or else be the identity T_0.

Cosets are not the only pitch class sets that can be fixed by a transposition operator. Invariant sets can also be unions of cosets of a given subgroup. For example, for $p = 12$ there are nontrivial subgroups of order two, three, four, and six. Invariant pitch class sets of size six can be manufactured by taking three cosets of size two, two cosets of size three, or one coset of size six. **Figure 2** shows an invariant set of T_6 that is the union of two cosets of the subgroup {0,6}.

Music theorists keep track of this information using a table that they call the T_n *cycles* [Morris 1991]. To create a T_n cycle, you create the subgroup of Z_p generated by n and then form the cosets of that subgroup. For our purposes, we need only the operators that generate nontrivial subgroups, since we know T_0 fixes everything and the rest fix nothing. One last thing to note is that if $m + n = p$, then T_n and T_m generate the same cosets, so we can list these together.

Example: Cycles and Fixed Pitch Class Sets for $p = 12$**.** The T_n cycles for $p = 12$ are:

T_2, T_{10} $\{0, 2, 4, 6, 8, 10\}$ $\{1, 3, 5, 7, 9, 11\}$
T_3, T_9 $\{0, 3, 6, 9\}$ $\{1, 4, 7, 10\}$ $\{2, 5, 8, 11\}$
T_4, T_8 $\{0, 4, 8\}$ $\{1, 5, 9\}$ $\{2, 6, 10\}$ $\{3, 7, 11\}$
T_6 $\{0, 6\}$ $\{1, 7\}$ $\{2, 8\}$ $\{3, 9\}$ $\{4, 10\}$ $\{5, 11\}$

Let's compute the number of pitch class sets of size six that are fixed by transposition operators for $p = 12$. First we note that the operators T_1, T_3, T_5, T_7, T_9, and T_{11} fix no pitch class set of size six, and T_0 fixes all 924 (that is, $\binom{12}{6} = 924$) sets of that size. Using the T_n cycles, we see that T_2 and T_{10} together fix a total of 4; T_4 and T_8 together fix 12; and

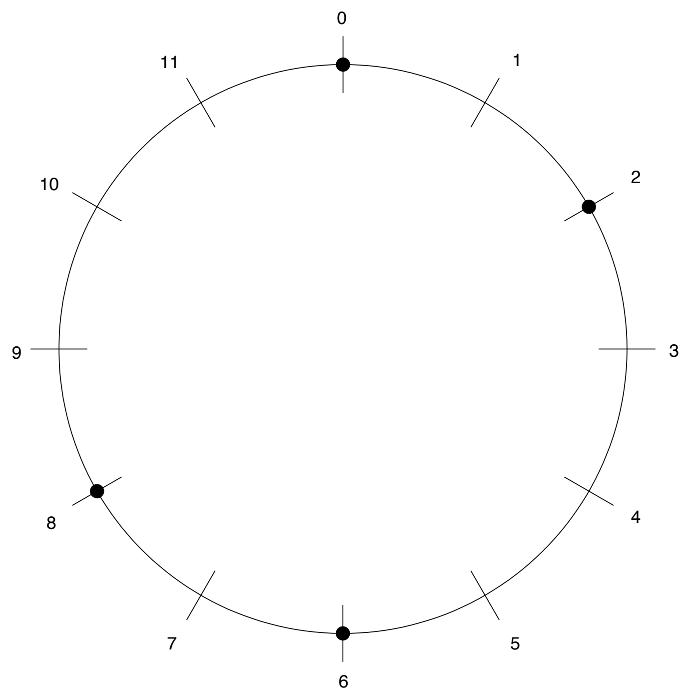

Figure 2. A union of cosets that are fixed by T_6.

T_6 fixes 20. Adding all these contributions together gives 960 pitch class sets of size six that are fixed by transposition operators.

Examples of Counting Set Classes

A Warm-Up Example

Example: Counting Set Classes. Let's take $p = 12$ and $m = 4$. From the T_n cycles above, we see that the only transposition operators that fix sets of size four are T_0, T_3, T_9, and T_6. So transposition operators fix $495 + 3 + 3 + 15 = 516$ pitch class sets of size four. The odd inversion operators fix six pairs of pitch classes each, so each of these fixes 15 sets of size four. The even inversion operators fix five pairs and two single pitch classes. Since we are counting pitch class sets of size four, we can treat the two singles as a pair; so each even inversion also fixes 15 pitch class sets. The total number of pitch class sets fixed by inversion operators is therefore 180. By Burnside's Theorem, there are $(516 + 180)/24 = 29$ set classes.

A Prime Number of Pitches

Suppose that the octave is divided into a prime number p of pitch classes, $p > 2$. We wish to know how many distinct set classes are there for pitch class sets of size k.

Since Z_p has no nontrivial subgroups when p is prime, the only transposition operator that fixes anything is T_0. The number of pitch class sets of size k that T_0 leaves invariant is

$$\binom{p}{k} = \frac{p!}{(p-k)!k!}.$$

Since p is prime greater than two, it is necessarily odd. Recalling our earlier discussion, we know that each inversion operator leaves $(p-1)/2$ pairs of pitch classes and only one singleton fixed. Bearing this fact in mind, we can calculate the number of invariant sets. Suppose that k is even, then each inversion has

$$\binom{(p-1)/2}{k/2}$$

invariant sets. If k is odd, then each inversion has

$$\binom{(p-1)/2}{(k-1)/2}$$

invariant sets.

To calculate the number N of distinct set classes of size k, we need the average number of invariant sets. For k even we get

$$N = \frac{1}{2p}\left[\binom{p}{k} + p\binom{(p-1)/2}{k/2}\right].$$

Similarly, the number of distinct set classes when k is odd is

$$N = \frac{1}{2p}\left[\binom{p}{k} + p\binom{(p-1)/2}{(k-1)/2}\right].$$

Example: Number of Distinct Set Classes. For $p = 13$ and $k = 4$, the number of distinct set classes is

$$N = \frac{1}{26}\left[\binom{13}{4} + 13\binom{6}{2}\right] = 35;$$

and for $p = 13$ and $m = 5$, the number of distinct set classes is

$$N = \frac{1}{26}\left[\binom{13}{5} + 13\binom{6}{2}\right] = 57.$$

Subgroups of Operators

Sometimes it is of interest to music theorists to count the distinct orbits for a subgroup of the $2p$ operators described earlier in this paper.

Example: Number of Orbits. Let's take $p = 12$ and let our group or operators be $G = \{T_0, T_6, T_0I, T_6I\}$. Now, let's compute the number of distinct orbits for $k = 5$. We find that T_0 fixes 792 pitch class sets of size $k = 5$, T_6 fixes 0, and the inversion operators fix 20 each. Adding these together and dividing by four gives 208 distinct orbits.

An Extension to Pitch Class Multisets

Sometimes in a piece of music with multiple voices, some voices sound the same pitch class, perhaps separated by an octave or in unison. Until now, we listed any doublings only once in a pitch class set. Some music theorists like to include the possibility of voice doublings by listing the pitch classes for each voice, for example {0,0,4,7}. In this case, we need to modify our counting to account for the possibility of repeated pitch classes in a pitch class set. Let's refer to a subset of the pitch classes that may include repeated elements as a *pitch class multiset*. Burnside's Theorem can still be used to count orbits, but we will need to recompute the size of our fixtures.

We've done most of the work to compute the fixtures of the transposition operators. We already know what types of sets can be fixed; cosets and unions of cosets. For pitch class multisets, we can add copies of cosets to our list. This addition does not change the counting much, as the following example shows.

Example: Number of Pitch Class Multiset Orbits. We consider $p = 12$ and $k = 4$. As before, we consult our T_n cycles to determine what transposition operators can fix sets of size four; we see that the possibilities are T_0, T_3, T_6, and T_9. As we count for each operator, we must allow for repeats. So T_0 fixes

$$\binom{12+4-1}{4} = 1365$$

pitch class multisets of size four, T_3 and T_9 still fix 3 each, and T_6 fixes

$$\binom{6+2-1}{2} = 21$$

multisets. Altogether, the transposition operators fix 1392 pitch class multisets of size four.

Inversion operators are more complicated in the multiset case, since accounting for the singleton fixed pitch classes is a little tricky. One approach is to figure out how many copies of the singletons you want to include in the multiset and then to fill in the rest of it with pairs. This leads to a partition of the sets that we are counting.

Example: Fixed Pitch Class Sets for Inversion Operators. Again, let $p = 12$, and $k = 4$. Recalling our discussion above, the even inversion operators fix five pairs and two single pitch classes and the odd inversion operators fix six pairs of them. The odd inversions are computed simply as fixing

$$\binom{6+2-1}{2} = 21$$

each for a total of 126. A pitch class set fixed by the even inversions could have zero, two, or four of its elements chosen from the two singletons. Suppose that there are four elements chosen from the two singletons, then there could be zero, one, two, three or four of either element in a multiset; so we see that there are five ways that four elements can be chosen from two (with repeats). Similarly, there are three ways to choose two things from two. Putting these facts together, we get that the odd inversion operators fix

$$5 + 3\binom{5}{1} + \binom{5+2-1}{2} = 35$$

pitch class sets, for a total of 210.

We can now conclude that there are (1392 + 126 + 210)/24 = 72 distinct multiset orbits for pitch class multisets of size four.

Conclusions

We have demonstrated how to use Burnside's Lemma to count set classes of pitch class sets and pitch class multisets.

For anyone reading Isihara and Knapp [1993] or taking an abstract algebra course that touches on group actions, this material could make a nice project. For example, a student could create a mathematical composition, using a $p = 5$ tone scale, that employs all the possible set classes uniformly.

Students of mathematics are likely to be surprised at the depth and type of mathematics used to answer questions in music theory. They might even be inspired to look for more connections.

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Jennifer Wilson lives in West Chester, PA. She graduated from Susquehanna University in 2005 with a B.A. in mathematics and French and a B.S. in Elementary and Early Childhood Education. While at Susquehanna, she researched the use of group theory to count pitch class sets. Jennifer is currently teaching third grade at New Holland Elementary School in New Holland, PA. She is also pursuing her master's degree in curriculum and instruction with a focus on children's literature through Penn State University. She plans to continue her study of mathematics at the graduate level in the future.