A Mathematical Analysis of African, Brazilian and Cuban *Clave* Rhythms

Godfried Toussaint*

School of Computer Science

McGill University

Montréal, Québec, Canada

May 13, 2002

This is an extended version of the paper in: *Proceedings of BRIDGES: Mathematical Connections in Art, Music and Science*, Towson University, Towson, MD, July 27-29, 2002.

Abstract

Several geometric, graph theoretical and combinatorial techniques useful for the teaching, analysis, generation and automated recognition of rhythms are proposed and investigated. The techniques are illustrated on the six fundamental 4/4 time *clave* and *bell* rhythm timelines most frequently used in African, Brazilian and Cuban music. It is shown that it is possible with three simple geometric features to automatically classify the rhythms without knowledge of their starting note. Three measures of rhythm complexity are compared. Pressing's measure agrees well with the difficulty of performing these clave rhythms whereas the Lempel-Ziv measure appears to be useless. An analysis of the rhythms using several similarity measures and visualization tools reveals that the clave *Son* is most like all the other clave rhythms and perhaps provides an explanation for its worldwide popularity. Finally, a combinatorial technique based on permutations of multisets suggests a fruitful approach to automated generation of new rhythms.

1 Introduction

Imagine a clock which has 16 hours marked on its face instead of the usual 12. Let us assume that the hour and minute hands have been broken off so that only the second-hand remains. Furthermore assume that this clock is running fast so that the second-hand makes a full turn in about 2 seconds. Such a clock is illustrated in Figure 1. Now set the clock ticking starting at "noon" (16 O'clock) and let it keep running for ever. Finally let us strike a bell at 16 O'clock and at the 3, 6, 10 and 12 positions for a total of five strikes per clock cycle. These times are marked with a bell in Figure 1.

The resulting pattern rings out a seductive rhythm which in a short span of fifty years during the last half of the 20th century has managed to conquer the planet. It is known

*This research was supported by NSERC and FCAR. e-mail: godfried@cs.mcgill.ca

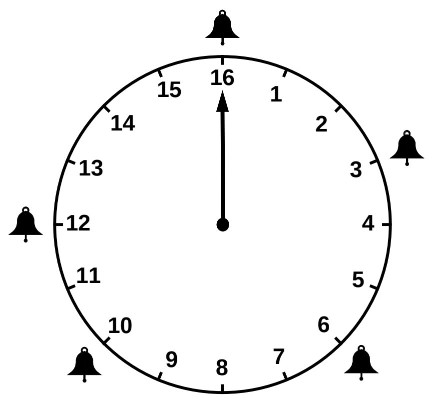

Figure 1: A clock divided into sixteen intervals of time.

around the world mostly as the *Clave Son* from Cuba. However, it is common in many African rhythms and probably travelled from Africa to Cuba with the slaves [48], [49]. In Cuba it is played with two sticks made of hard wood also called claves [39]. In Africa it is traditionally played with an iron bell. In a section below a mathematical argument is offered to explain the world-wide popularity of this rhythm.

The *Clave Son* rhythm is usually notated for musicians using standard music notation which affords many ways of expressing a rhythm. Four examples are given in the top four lines of Figure 2. The fourth line shows it with music notation using the smallest convenient notes and rests. The bottom line shows a popular way of representing rhythms for percussionists that do not read music. It is called the *Box Notation Method* developed by Philip Harland at the University of California in Los Angeles in 1962 and is also known as TUBS (Time Unit Box System). If we connect the tail to the head of this last diagram and draw it in the form of a circle in clockwise direction we obtain the clock representation in Figure 1, where the squares in Figure 2 filled with black dots correspond to the positions of the bells in Figure 1. The box notation method is convenient for simple-to-notate rhythms like bell and clave patterns as well as for experiments in the psychology of rhythm perception, where a common variant of this method is simply to use one symbol for the strike and another for the pause [15]. Thus for the clave son a common way to write it is simply as [x . . x . . x . . . x . x . . .]. Finally, in physiology, where the study of cardiac rhythms [3] is important, as well as in computer science the clave son would be written as the 16-bit binary sequence: [1 0 0 1 0 0 1 0 0 0 1 0 1 0 0 0].

Playing these five notes is more difficult than it appears at first hearing. When a Cuban performer passionately invites the audience to participate in clapping this *clave Son*, the non-musicians in the crowd invariably execute their second clap at 4 o'clock instead of 3 o'clock.

1.1 The clave patterns as time keeping metronomes?

In a classical music concert the conductor marks out the beat visually and silently with a baton. All the members of the orchestra can see the conductor and the baton. This situation

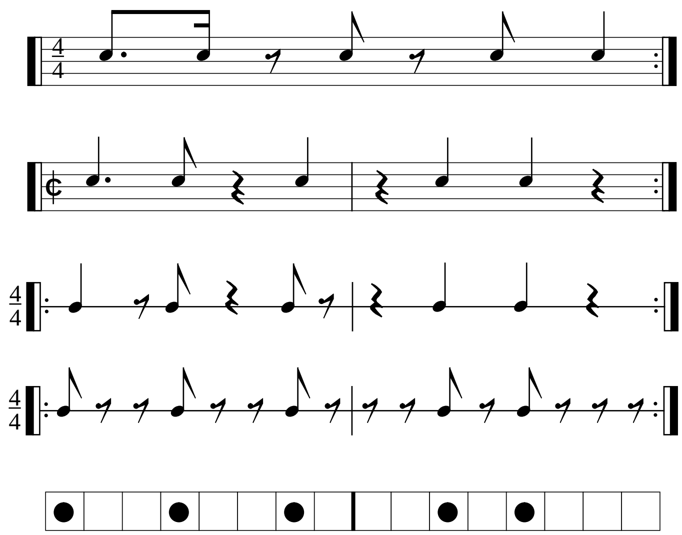

Figure 2: Five ways of representing the *clave Son* rhythm.

helps to keep all the musicians together on the beat of the music. The clave and bell patterns used in African derived music also have a similar function of keeping all the drummers together on the beat [7]. However, unlike the conductor with the baton, the drummers need not see the bell player since the bell can be heard. The African metal bells and the hard wooden claves used in Brazil and Cuba have a highly penetrating sound that can "cut" through a score of loud drums so that all the drummers can hear it. One may ask then why the bell patterns are not played like the simple rhythmic movements of the conductors' baton. One reason is that it would be too monotonous. The silence of the baton bothers no one. This is one reason why the bell patterns are much more intricate than the simple 1-2-1-2-1-2 or 1-2-3-1-2-3 pulse of the baton. However, the main reason for their variability and complexity is that the bell patterns are in fact the core of the rhythm. Not only do the individual pattern notes supply anchors for the various drum parts being played, but they provide the basic underlying feel of the piece. In the words of M. E. Nzewi [37], the role of the bell is that of "a phrasing beacon for the other ensemble instruments which have the freedom to develop their themes." For this reason the clave and bell timelines are also called the *heartbeat* of the rhythm [22]. Furthermore, in the choreographed dance ensembles of the *Igbo* the bell is used as a master instrument directing the choreography. Thus we see that these bell patterns and the bell itself are much more than mere time-keeping metronomes.

There exist literally hundreds of such timeline patterns for bells, claves and woodblocks used in music throughout Africa, Brazil and the Caribbean. This is not surprising when one considers the number of combinations one can create out of five notes played in the sixteen available positions of two bars. Add to that the patterns made with six, seven and up to eleven notes; add to that the patterns that use four bars and in addition the 6/8 time rhythms and we quickly obtain a combinatorial explosion. In this preliminary study however, we are concerned only with the six fundamental five-note 4/4 time clave and bell timelines most frequently used in African, Brazilian and Cuban music. These rhythms are known under

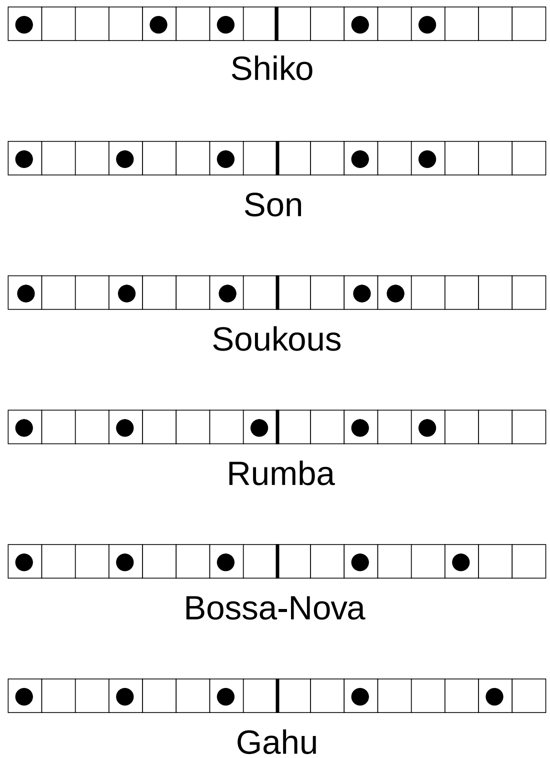

Figure 3: The six fundamental 4/4 time clave and bell patterns in box notation.

many names in different countries but for the purpose of this study I will call them: *Shiko*, *Son*, *Soukous*, *Rumba*, *Bossa-Nova* and *Gahu*. Figure 3 shows all six of them in box notation. In the following sections a lot more will be said about these rhythms. For the moment the most important feature to notice is that all six consist of five notes.

A glance at the box-notation representation of these bell patterns in Figure 3 reveals several relationships between them. All six patterns have the first and fourth notes in the same positions. All but Shiko and Rumba have the same first bar consisting of the first three notes. Son can be converted to Shiko by advancing the second note by one time unit, Son to Rumba by advancing the third note by one time unit, and Soukous to Son, Bossa-Nova and Gahu by advancing the fifth note by one, two and three time units, respectively.

1.2 A geometric representation of rhythms

Consider again the standard musical notation for the clave Son illustrated in the top row in Figure 2. Can the rhythm be played backwards starting at a suitable note so that it sounds exactly the same? Answers to questions such as these are not immediately evident with such a notation. The box notation at the bottom of Figure 2 allows this question to be answered more easily. The answer is yes if we start on the third note. In other words the clave Son is a shifted (or weak) *palindrome*.

An even better representation for such cyclic rhythms is obtained by starting with the clock idea of Figure 1 and connecting consecutive note locations with edges to form a convex polygon. Such a representation not only enhances visualization but lends itself more readily to mathematical analysis. It has been used by Becker [2] to analyse Javanese Gamelan music, by McLachlan [34] to analyze rhythmic structures from Indonesia and Africa using group theory and Gestalt psychology, and by London [33] to study meter representation in general. The six clave bell patterns are represented as convex polygons in Figure 4 and analysed in more detail in the following section. Note that in Figure 4 the dashed lines indicate either the base of an isoceles triangle or an axis of mirror symmetry.

2 The Six Fundamental 4/4 Time Bell Patterns

2.1 Shiko

The first bell pattern shown in Figure 3 is a common pattern found in Africa and the Caribbean. For the purpose of this study we shall refer to it as *Shiko* as it is known in Nigeria. Notating it in one bar instead of two it is [x . x x . x x .]. Note that therefore in this notation the pattern uses eight units of time instead of sixteen. The Shiko is also played in the *Moribayasa* rhythm among the Malinke people of Guinea and in the *Banda* rhythm used in Voudoo ceremonies in Haiti. In Cuba it is played on a wooden block in the *Makuta* rhythm. These patterns are also played by starting on the second and the fourth notes. For example, the *Timini* rhythm in Senegal is [x x . x x . x .] which is equivalent to starting the Shiko on the second note. This bell pattern is also played for the *Adzogbo* dance of the *Fon* people of Benin [7] On the other hand, the *Kromanti* rhythm of Surinam is [x x . x . x x .] which is equivalent to starting the Shiko on the fourth note. In this study we consider cyclic shifts of a rhythm as essentially the same rhythm and focus only on the most representative of such cyclic rhythms. The Shiko rhythm is also found as the first bar of several two-bar rhythms. For example, a well known Arabic rhythm, the *Wahda Kebira* given by [X . x x . x x . | X . X . x . x x] contains the Shiko as its first bar. Here the symbol | separates the two bars and since this is played on a drum, rather than bell, the capital X's denote low sounds and the small x's high-pitched slaps. The Shiko variants also appear in other two-bar rhythms. For example, The *Kassa* from Guinea given by [x x . x . x x . | x . x . x . x .] has the Kromanti as its first bar. Finally we remark that starting the Shiko on the second note is also a popular pattern found in Arabic rhythms played on a drum. Here again some notes are low sounds whereas others are high pitched. For example the *Wahda wa Noss* is given by [X x . x X . x .]. The *Baladi* is given by [X X . x X . x .]. The *Masmudi* is a slow Baladi and the *Saidi* is given by [X X . X X . x .]. The time-line pattern in these three Arabic rhythms is the same and it is only the pitch of the notes that changes from rhythm to rhythm.

2.2 Son

The second bell pattern shown in Figure 3 is of course most well known around the world as the *clave Son*. It was transported by the Son music from Cuba and later by its New York off-spring, Salsa [44]. In both of these musics it is played on the claves. It is also played on claves in the more traditional Cuban rhythms such as the *Guaguancó*, one of the three

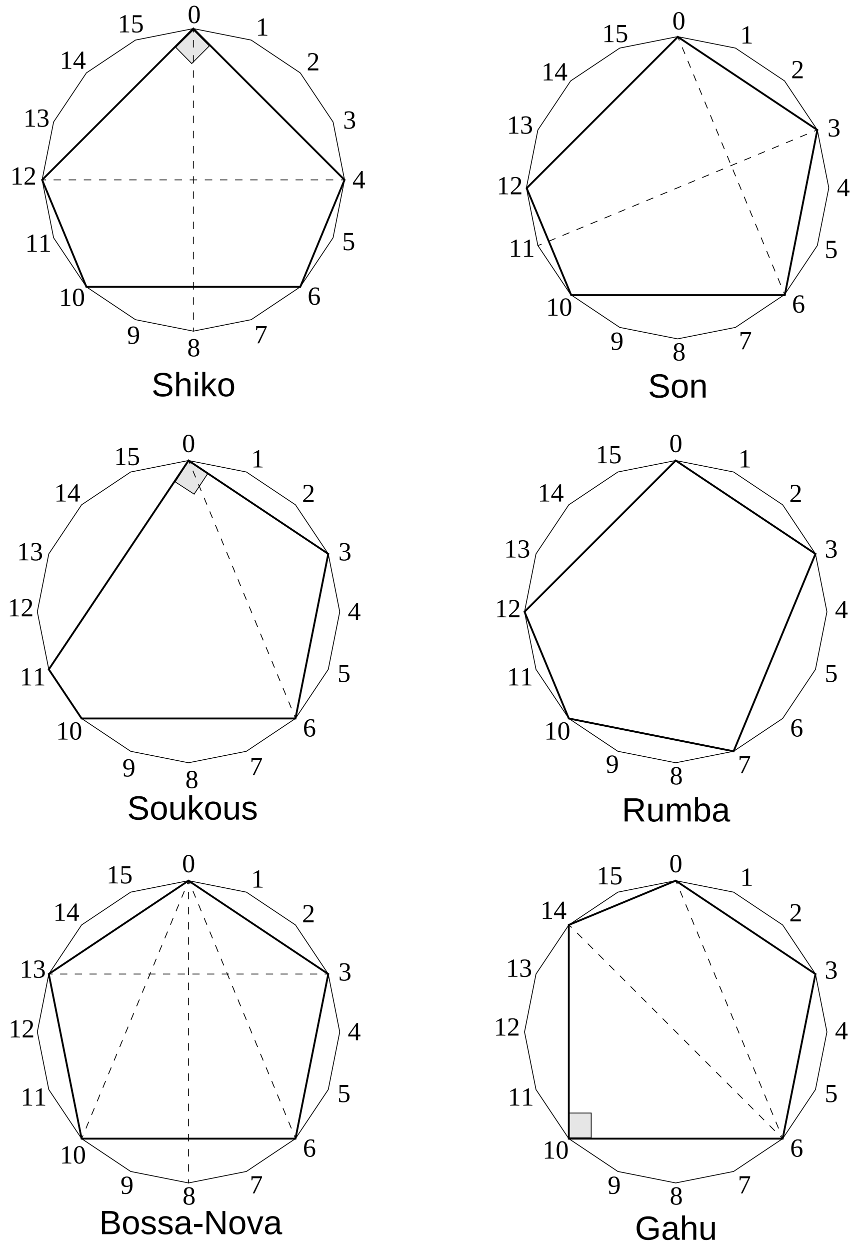

Figure 4: The six fundamental 4/4 time clave and bell patterns represented as convex polygons inscribed in an imaginary circle.

styles of *Rumba*. In Africa this pattern is played most popularly on an iron bell called the *Ngongo* in the *Kpanlogo* rhythm (pronounced panlogo), a recreational music of the *Ga* people of Ghana [22]. The same pattern is played on a secondary bell in the *Kassa* rhythm from Guinea. One of West Africa's most popular modern dances, the Highlife, is also based on this pattern [7].

In some Cuban music the two bars of the clave Son are transposed. Note that the first bar [x . . x . . x .] contains three notes whereas the second bar [. . x . x . . .] contains two notes. For this reason the pattern [x . . x . . x . | . . x . x . . .] is also called the 3-2 clave Son. Some Cuban music uses instead the 2-3 clave Son given by [. . x . x . . . | x . . x . . x .]. However, since the 2-3 clave is a cyclic shift of the 3-2 clave and the latter is the most representative, we restrict this study to the 3-2 clave.

2.3 Rumba

The fourth pattern in Figure 3 given by [x . . x . . . x | . . x . x . . .] has also travelled through much of the world and goes mainly by its Cuban name: clave Rumba. The Rumba is one of the most well known Afro-Cuban folkloric song-and-dance styles popular at large feasts. There are three styles of Rumba: the 6/8 time *Columbia* and the two 4/4 time rhythms, the fast *Guaguancó* and the slower *Yambú*. It is these two last rhythms that use the clave Rumba time-line. This pattern is also used in several other Cuban rhythms such as the *Conga de Comparsa* and the *Mozambique*, both used mainly for Carnavals. The same time-line pattern is played on a bell in a processional music of the *Ibo* and *Yoruba* peoples of Nigeria [22]. Like the clave Son, the Rumba clave has its 3-2 version shown above and also a 2-3 version, namely [. . x . x . . . | x . . x . . . x]. Again, the 2-3 version is a cyclic shift of the 3-2 version and in this study we restrict ourselves to the 3-2 clave Rumba in 4/4 time.

2.4 Soukous

Rhythm did not travel with the slaves along a one-way street from Africa to America. After the Second World War Cuban music became popular in central Africa. Rumba in particular hit a sympathetic chord with dancers and was speeded up to create what was first called *Congolese* and later became *Soukous*. Soukous literally means "to shake" and what may account for the strong heartbeat pulse is the time-line pattern [x . . x . . x . | . . x x . . . .]. with its last two notes closer together. This pattern is usually played on either a wood-block or even a snare drum. It is not surprising that Africans would resonate with a new music that had African roots. Neither is it surprising that they would speed it up to suit their more frenetic dances. What may seem surprising at first glance is that they would change the rhythm itself to such an extent that the resulting time-line ended up closer to Son than to Rumba. (To transform the Son to Soukous one merely plays the last note of the Son one time-unit earlier. On the other hand, to transform the Rumba to Soukous one must play both the third and fifth notes of the Rumba one time-unit earlier.) However, further analysis reveals that even when the tempo of the Rumba is increased, the relatively late third note slows it down subjectively. Advancing this third note by one time-unit gives it the rolling drive of the clave Son.

2.5 Bossa-Nova

The fifth time-line pattern in Figure 3 given by [x . . x . . x . | . . x . . x . .] is known as the *Bossa-Nova* clave and is played in Bossa-Nova music, in *Samba* music and in Afro-Brazilian folk music from Bahia [22]. The Bossa-Nova, an offspring of Samba, is a style of music that was developed in the late 1950's in Rio de Janeiro by musicians such as Joao Gilberto and Stan Getz [26]. The Bossa-Nova clave is usually played either on claves, wood block or the rim of a snare drum.

2.6 Gahu

Gahu is a polyrhythmic drumming music of the *Ewe* people of Ghana. The word Gahu means either "money dance" or "airplane." It appears to have been created by Yoruba speakers of Benim and Nigeria as a "form of satirical commentary on the modernization in Africa" [32] and first taken to Ghana in the early 1950's. The Gahu time-line, the last pattern in Figure 3 given by [x . . x . . x . | . . x . . . x .] is played on a double bell called the *Gankogui*. The first note is traditionally played on the low-pitched bell and the remaining four on the high-pitched bell. Although objectively it is similar to the Bossa-Nova clave (only the last note of the Bossa-Nova is played one time-unit later) subjectively it has a significantly different feel.

3 Geometric Analysis of Rhythms

3.1 Geometric measurements of cyclic rhythms

The representation of the six clave and bell patterns described above as convex polygons illustrated in Figure 4 readily suggests a variety of discriminating geometric properties useful for comparison, analysis and automatic classification of rhythms. Discovering such properties by analyzing classical music notation, or even box-notation, is not self evident. For example, it is imediately obvious from examination of Figure 4 that three timelines, namely Shiko, Soukous and Gahu contain a right interior angle as one of the vertices in their polygons, whereas the other three (Son, Rumba and Bossa-Nova) do not. It is equally irresistible to notice that the former three rhythms evolved in Africa whereas the latter three in America; Bossa-Nova in Brazil and Son and Rumba in Cuba. Does a right angle translate to a stronger beat? Note that a right angle (90 degrees) at one vertex of the polygon implies that the remaining four vertices all lie in one and the same semi-circle. Is the presence of the right angle as a discriminating feature between African and American popularity of rhythms more than mere coincidence?

We also see immediately that Shiko and Bossa-Nova are palindromes. They sound the same played forwards or backwards. This can be seen from the mirror symmetry of the polygons about the line (axis) through positions (0,8). On the other hand, Son is a *weak palindrome* in the sense that there exists a position other than (0) from which the rhythm sounds the same when played forwards or backwards. In this case the position is (3) since the polygon has mirror symmetry about the line (3,11).

The number of isoceles triangles determined by adjacent edges provides another geometric feature. Shiko, Son and Soukous have one isoceles triangle each. Note that an isoceles triangle

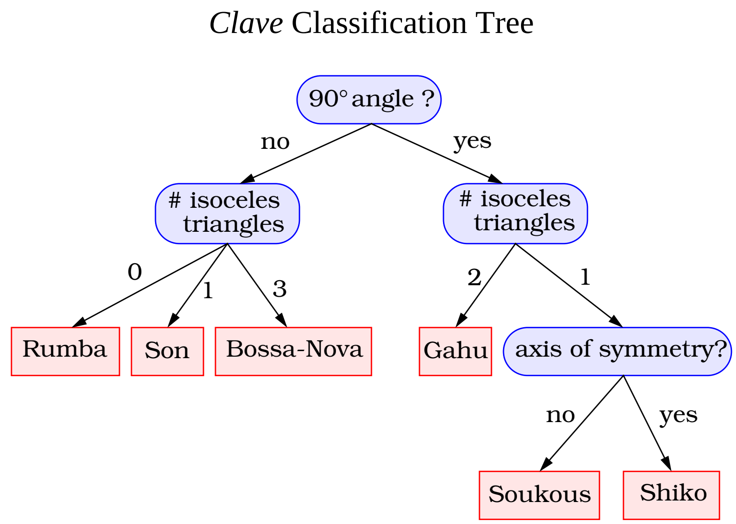

Figure 5: One possible decision tree to automatically classify the six clave patterns.

indicates two equal consecutive time intervals between notes. Gahu has two isoceles triangles and Bossa-Nova has three. The Bossa-Nova is a *maximally-even* set [8]. A maximally-even set is one in which a subset of the elements has its elements as evenly spaced as possible. The Bossa-Nova has four inter-note intervals of length three and one of length four. In contrast Rumba is the only rhythm with no isoceles triangles, no axis of mirror symmetry and no right angles. Rumba is, at least geometrically speaking, quite an extraordinary rhythm indeed.

3.2 Automatic classification of rhythms

Geometric features such as those described above on the convex polygons representing the rhythms can be used to help automatic classification and recognition of rhythms. There are several stages in the rhythm recognition process starting from the acoustic signal [11]. A fundamental and difficult step is the detection of all the note onsets. Once these are established a matching is sought between the rhythm to be classified and stored templates. This matching problem is made easier if the underlying *beat* is also known. By *beat* is meant one of a series of perceived pulses marking equal units of time [29]. However, this problem in itself, known as *beat induction*, is also a difficult problem [11]. One approach to solve this problem is to look for a match over all cyclic shifts between the unknown pattern and the stored templates. Geometric features such as those proposed here may also help. The idea is illustrated below with a decision tree to automatically classify the six clave patterns without knowing where the “start” of the beat is.

The decision tree is illustrated in Figure 5. It is assumed that we have already obtained the note onsets in cyclic order (and computed the polygon) although we do not know which note starts the rhythm. First determine if the polygon has a 90 degree angle. If the answer is NO then we know the rhythm is either Rumba, Son or Bossa Nova. These three can be differentiated by computing the number of isoceles triangles contained at vertices of the polygon: zero for Rumba, one for Son and three for Bossa Nova. If the polygon has a 90 degree angle then again we compute the number of isoceles triangles. If there are two isoceles triangles we have found the Gahu rhythm. Otherwise, if there is only one isoceles triangle determine if there exists an axis of symmetry. If the answer is YES we have the Shiko and otherwise we have the Soukous. Note that none of the three measurements used depend on knowing which is the starting note of the rhythm and all three can be very easily computed. The first two are trivial to compute and simple algorithms for finding the axis of symmetry are described by Peter Eades [14].

4 Measuring the Complexity of Rhythms

One natural feature useful for a variety of applications including automated recognition of rhythms is *rhythm complexity*. A great deal of attention has been devoted to measuring the objective complexity of sequences in the field of information theory [6], [35]. However, when dealing with rhythm one cannot restrict investigation to consider only objective phenomena. As with visual stimuli, aural stimuli exhibit a variety of perceptual illusions. One of the earliest observations of this kind is concerned with the perception of *beat*. Already in 1894 T. L. Bolton discovered that an isochronous train of identical pulses, such as clock "tics," elicits in the human subject an experience of alternating strong and weak beats, a phenomenon known as *subjective rhythm* [4]. To paraphrase Ian Cross [10], the *physicalist* view of rhythm is largely wrong.

4.1 Objective, cognitive and performance complexities

Everyone can understand the principle behind juggling three balls as well as instructions on how to juggle them. On the other hand, picking up three balls and juggling them is another matter. In other words we all know that perceptual or cognitive complexity is not the same as performance complexity. It is easier to recognize something than to reproduce it. In the same way there is no logical a priori reason why an objective mathematical measure of complexity should agree with either cognitive or performance complexities. In this section we discuss and compare three measures of the complexity of rhythm with respect to the six clave rhythms considered in this paper. One of the measures is new.

4.2 The Lempel-Ziv complexity

In 1976 Lempel and Ziv [30] proposed an information-theoretic measure of the complexity of a finite sequence in the context of data compression. Their novel approach evaluates the complexity of a finite sequence by scanning the given sequence from left to right looking for the shortest subsequences (words) that have not yet been seen during the scan. Every time such a word is found it forms part of a growing vocabulary. When the scan is finished the size of this vocabulary is the measure of complexity of the sequence. For cyclic sequences such as the rhythms considered here it is sufficient to examine a concatenation of only two instances of the rhythm pattern since no new subsequences will be found by examining longer

Clave Son

(a) 1001001000101000

(b) 10010010001010001001001000101000

(c) 1◆0◆01◆001000◆101◆000100◆1001000101000
1 2 3 4 5 6

Figure 6: Illustrating the computation of the Lempel-Ziv complexity of the clave Son.

sequences of the rhythm. As an example consider the clave Son illustrated in Figure 6. The rhythm in binary notation is shown in Figure 6 (a). Repeating it a second time yields the 32 bit-pattern in Figure 6 (b). Figure 6 (c) shows each new subsequence found by the scan separated by a diamond marker and labelled with an index number underneath. For the clave Son six new subsequences are generated by this process and therefore the Lempel-Ziv complexity is equal to six.

This measure is relatively simple to compute [18] and it is completely objective in the sense that it is defined purely mathematically. Therefore it has been explored as a measure of rhythm complexity by musicologists interested in the automated recognition of rhythms. However, preliminary experimental evaluation of the Lempel-Ziv measure compared to the complexity perceived by human subjects yielded negative results [45]. The comparison of this measure with the other measures of rhythm complexity discussed below, with respect to the six clave patterns described above, confirms these results. Furthermore, looking at the scores obtained for the six clave rhythms in Figure 7 (without even comparing to the other measures) shows that this measure is quite defficient. There is almost no variance in the scores: all are either 5 or 6. Also the scores do not make sense to anyone experienced in teaching or playing these rhythms. For example Shiko is the simplest of the six rhythms, and Gahu one of the most complex, both to recognize and to play, yet the Lempel-Ziv complexities are 5 for both of these rhythms. It appears that information theoretic measures are not able to capture well the human perceptual, cognitive and performance complexities of rhythms.

4.3 The cognitive complexity of rhythms

Jeff Pressing proposed a measure of the cognitive complexity of a rhythm based on psychological principles and the syncopation present in the rhythm at different levels of pulse [40]. The reader is referred to Pressing [40] for the theory and details on how to measure and calculate the cognitive complexity of rhythmic patterns. The cognitive complexities of the ten 4-unit patterns made up of one-note patterns and two-note patterns computed with Pressing's measure are shown in Figure 8. If we take the 16-bit patterns of the clave rhythms, divide them into four units of four bits each, and add the Pressing-complexities of the four corresponding units we obtain values for the Pressing cognitive complexities of the six clave rhythms. For example, the Shiko pattern consists of the concatenation of the patterns [1 0 0 0], [1 0 1 0], [0

Rhythm Complexity Measures

	Lempel-Ziv	Pressing	Metric
Shiko	5	6	2
Son	6	14.5	4
Soukous	6	15	6
Rumba	6	17	5
Bossa-Nova	5	22	6
Gahu	5	19.5	5

Figure 7: A comparison of three measures of rhythm complexity.

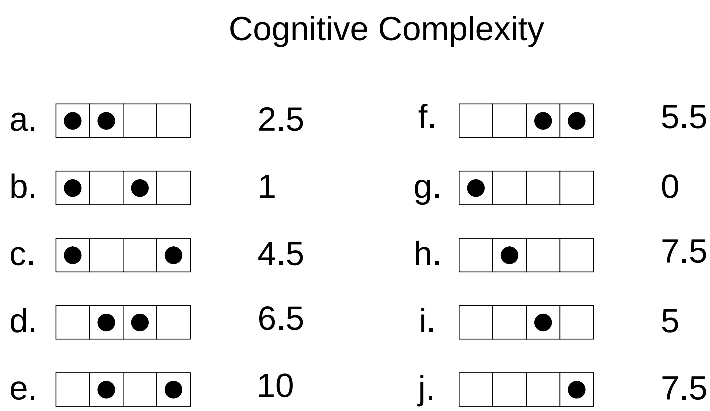

Figure 8: The cognitive complexities of ten basic rhythm patterns according to Pressing [40].

0 1 0] and [1 0 0 0]. Referring to Figure 8 we obtain the complexities 0, 1, 5 and 0 for a total of 6. On the other hand the Rumba yields a Pressing cognitive complexity of 4.5 + 7.5 + 5 + 0 = 17. Examining the Pressing cognitive complexities of all six clave rhythms in Figure 7 reveals much more information than the Lempel-Ziv complexities. All the scores are different and the variance is quite large ranging from 6 for the Shiko to 22 for the Bossa-Nova. The scores are also in good agreement with teaching and performing experience. Shiko is easy, Rumba is harder than Son, and Bossa-Nova and Gahu are the hardest of these six rhythms.

4.4 Metricity and metric complexity

The concept of *meter* traditionally refers to the number of pulses between more or less regularly ocurring accents. Meter is implicit rather than being notated in classical music notation [33]. It is a hierarchical arrangement of beats of different strengths. Based on work

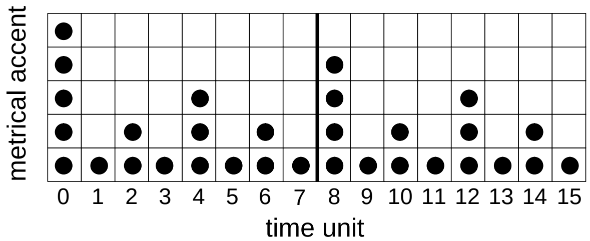

Figure 9: The *metrical structure* of Lerdahl and Jackendoff [31] for a time line pattern of 16 time units.

by Komar [27], Lerdahl and Jackendoff [31] proposed a construct called a *metrical structure* for describing the temporal psychological organization of rhythmic patterns at all metrical levels. Their metrical structure for the 16-time unit interval relevant to the clave and bell patterns considered here is illustrated in Figure 9. The structure defines a function that maps the index of the time unit to the relative strength of its metrical accent. The easiest way to describe the function is by summing levels of beats. At the first level of metrical accent a beat is added at every time-unit starting at unit 0. At the second level a beat is added at every *second* time unit starting at unit 0. At the third level a beat is added at every *fourth* time unit starting at unit 0. We continue in this fashion doubling the time interval between time units at every level. In this way time unit 8 receives 4 beats and finally time unit 0 receives 5 beats. Thus time units 2, 6, 10, and 14 are weak beats, time units 4 and 12 are medium strength beats, unit 8 is stronger and unit 0 is the strongest beat.

A new measure of the complexity of a rhythm can be defined based on the above concept of meter. First define a measure of the total metric strength of a rhythm and call it *metricity*. It is simply the sum of all the metrical accents of the beats present in a rhythm. For example, the clave Son has notes at time units 0, 3, 6, 10 and 12. The metrical accents in Figure 9 corresponding to these time units are 5, 1, 2, 2 and 3, respectively. Therefore the metricity of the clave Son according to the hierarchical arrangement of Lerdahl and Jackendoff [31] is equal to 13. Since this metrical structure has a simple and highly structured hierarchy exhibiting a great deal of symmetry, the metricity function defined here can be viewed as a measure of metric *simplicity*. As such it is inversely proportional to what will be called *metric complexity*. Note that the maximum value of the metricity for five notes in a sixteen unit time scale cannot exceed 17, and therefore the metric complexity will be defined as 17 minus the metricity. Therefore the metric complexity of the clave Son, for example, is $17 - 13 = 4$.

Although the metrical structure defined by Lerdahl and Jackendoff is based on Western or European music and may thus be limited in scope [28] it is nevertheless an interesting lens through which to view African and Afro-American rhythms. However, even if this measure has little psychological significance or even universal musical merit, at worst it is as valid an objective mathematical measure as is the Lempel-Ziv complexity and may find useful application as a feature extraction method for the automated classification of rhythms. Examination of the metric complexity scores of the six clave rhythms in Figure 7 reveals that this is nevertheless a much better measure than the Lempel-Ziv complexity. The variance is reasonable, Shiko has the lowest score of 2, Rumba (5) has a higher score than Son (4) and

Hamming Distance Matrix

	Shiko	Son	Soukous	Rumba	Bossa	Gahu
Shiko	0	2	4	4	4	4
Son		0	2	2	2	2
Soukous			0	4	2	2
Rumba				0	4	4
Bossa-Nova					0	2
Gahu						0
Σ	18	10	14	18	14	14

Figure 10: The Hamming distance matrix of the six rhythms. The bottom row indicates for each rhythm the sum of the distances it is from the other five.

Bossa-Nova is amongs the most complex with a score of 6. However, the metric complexity measure is not as good as Pressing's cognitive complexity. Bossa-Nova and Gahu are more difficult to play than Soukous, for example, and the metric complexity measure ranks Soukous as harder than Gahu.

5 Measuring the Similarity of Rhythms

At the heart of any algorithm for comparing, recognizing or classifying a rhythm lies a measure of the similarity between two rhythms. There exists a wide variety of methods for measuring the similarity of two rhythms represented by a string of symbols. Indeed the resulting approximate pattern matching problem is a classical problem in pattern recognition and computer science in general [12]. Traditionally similarity between two patterns is measured by a simple template matching operation. More recently similarity has been measured with more complex functions such as the effort required to morph one pattern into another [5].

5.1 The Hamming distance

When the two strings are binary sequences a natural measure of distance or non-similarity between them is the Hamming distance [21] widely used in coding theory. The Hamming distance is simply the number of places in the strings where elements do not match. For example the Gahu and Soukous rhythms differ in the position of their last note. Therefore there are two locations in the 16-bit binary string where a missmatch occurs and the Hamming distance between Gahu and Soukous is equal to 2. Hence it is one of the simplest kinds of template matching [12]. Note that in this approach each rhythm is represented by a vector $X = (x_1, x_2, ..., x_{16})$ where the x_i represent binary valued features of the rhythm. If a note is played at time unit i then $x_i = 1$ and otherwise $x_i = 0$. Thus rhythms are represented as points in a 16-dimensional binary-valued vector space (usually called a hypercube). The Hamming distance between two points $X = (x_1, x_2, ..., x_{16})$ and $Y = (y_1, y_2, ..., y_{16})$ in this space is given by:

$$d_H(X, Y) = \sum_{i=1}^{16} |x_i - y_i| \tag{1}$$

where $|x|$ denotes the absolute value of x. The Hamming distance matrix is shown in Figure 10. The bottom row in Figure 10 shows for each rhythm the sum of its Hamming distances to all the other five rhythms. This is a measure of how dissimilar (according to this measure) a rhythm is from the rest of the group. Note that according to this measure Son is the most similar rhythm to all the others.

The Hamming distance is not very appropriate for our problem of rhythm similarity because although it measures a missmatch, it does not measure how far the missmatch occurs. Furthermore, if a note is moved a large distance the resulting rhythm will sound more different than if it is moved a small distance. For example, the Hamming distance between Gahu and Soukous is the same as that between Gahu and Bossa-Nova but clearly the latter two are much closer to each other.

There exists a large variety of generalizations of the Hamming distance [12]. One such generalization is the *edit* distance which allows for insertions and deletions of notes. A discussion of the application of the *edit-distance* to the measurement of similarity in music is given by Orpen and Huron [38]. However, in our study here all the rhythms have the same number of notes and so this generalization is not relevant.

Some rhythm detection algorithms [36] and systems for machine recognition of music patterns [9] use inter-onset intervals as a basis for measuring similarity. These are the intervals of time between consecutive note onsets in a rhythm. Coyle and Shmulevich [9] represent a music pattern by what they call a *difference-of-rhythm vector*. If $T = (t_1, t_2, ..., t_n)$ is a vector of inter-onset time intervals for the notes of a rhythm then they define the difference-of-rhythm vector as $X = (x_1, x_2, ..., x_{n-1})$, where $x_i = t_{i+1}/t_i$. This approach is more appropriate than the Hamming distance for measuring the similarity of rhythms. In the next subsection the six clave rhythms are compared with respect to a feature vector defined by successive inter-onset intervals in a slightly different way.

5.2 The interval vector distance

Consider the representation of the six clave rhythms as convex polygons in Figure 4. These polygons immediately suggest a variety of possible shape feature vectors for characterizing the rhythms. Concentrating on the vertices of the polygon $V = (v_1, v_2, v_3, v_4, v_5)$ one could take v_i to be the internal angle of the ith vertex. On the other hand viewing the polygon as a sequence of edges $E = (e_1, e_2, e_3, e_4, e_5)$ one could take e_i to be the length of the ith edge of the polygon. In addition, one could use a variety of global shape features of the polygon such as the ratio of perimeter to area, or various moments of inertia [46]. Here each rhythm will be represented by a vector of five numbers that characterize these five intervals. More

Interval Vector Distance Matrix

	Shiko	Son	Soukous	Rumba	Bossa	Gahu
Shiko	0	1.41	2	2.45	2	3.16
Son		0	1.41	1.41	1.41	2.83
Soukous			0	2	2.83	4.24
Rumba				0	2	3.16
Bossa-Nova					0	1.41
Gahu						0
Σ	11.02	8.47	12.48	11.02	9.65	14.80

Figure 11: The distance matrix of the interval vectors with the Euclidean metric.

specifically a rhythm will be represented by $X = (x_1, x_2, x_3, x_4, x_5)$, where x_i is the number of vertices skipped by the ith polygon edge starting at vertex labelled 0. This is essentially the same as the sequence of inter-onset time intervals since the time interval is the number of vertices skipped plus one. The dissimilarity between two rhythms $X = (x_1, x_2, ..., x_5)$ and $Y = (y_1, y_2, ..., y_5)$ is measured by the Euclidean distance between the two vectors X and Y in this 5-dimensional interval vector space which is given by:

$$d_E(X, Y) = \sqrt{\sum_{i=1}^{5} (x_i - y_i)^2} \qquad (2)$$

The distance matrix based on these vectors is shown in Figure 11. As before, the bottom row in Figure 11 shows for each rhythm the sum of its distances to all the other five rhythms. Most noteworthy are the highest (14.80) and lowest (8.47) values for Gahu and Son, respectively. David Locke [32] has argued forcefully, from the music theory point of view, the uniqueness of the Gahu bell pattern. The results obtained here provide mathematical confirmation of Locke's musical analysis. They also provide mathematical evidence that the clave Son is most like all the others. This may explain the world-wide popularity of the Son.

5.3 The minimum spanning tree

The minimum spanning tree is a powerful and useful visualization tool for understanding the structure of data in higher dimensional spaces such as that encountered here [50]. For this reason it has been used with great success in cluster analysis [12]. The distance matrix of Figure 11 defines a complete weighted graph G. The six rhythms correspond to the six nodes of the graph. The graph is complete because every pair of nodes is connected by an edge. The weight on each edge connecting two nodes is the distance between the corresponding

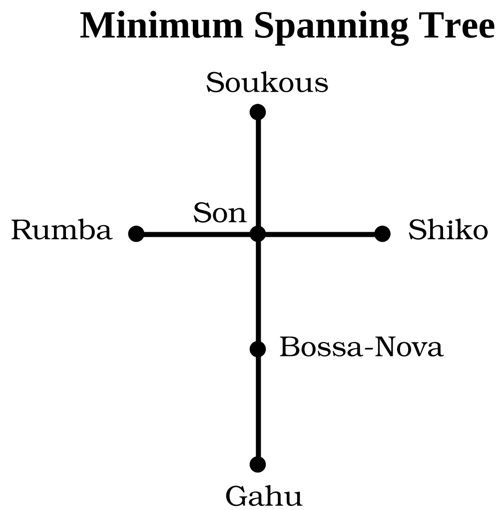

Figure 12: The minimum spanning tree determined by the distance matrix of interval vectors.

two rhythms. The minimum spanning tree (MST) of G is the subgraph of G that is a tree (has no cycles), connects (spans) all six nodes, and has minimum weight among all such trees. Here the weight of the tree is just the sum of the weights of all its edges. Note that in general the minimum spanning tree may not be unique. There exist many algorithms for computing the minimum spanning tree [17]. The simplest from a conceptual point of view is Kruskal's algorithm. This algorithm first sorts all pairs of nodes (edges) by increasing weight (distance) and then scans through this sorted list picking edges to add to a growing tree as long as no cycles are created. Proceeding in this way with the distance matrix in Figure 11 quickly results in the minimum spanning tree shown in Figure 12. The topology of the minimum spanning tree in Figure 12 provides at a glance a good idea of how the smallest distances (1.41) in the distance matrix relate to one another qualitatively. Here one can visualize instantly that Gahu is most unlike the remaining rhythms whereas Son is most similar to the other five rhythms. In fact Son is the center of gravity of the tree in the sense that it is the node that minimizes the sum of distances (within the tree) to all other nodes.

5.4 Phylogenetic trees

The drawing of the minimum spanning tree in Figure 12 does not attempt to quantitatively visualize *all* the distances in the distance matrix of Figure 11. For this purpose there exist more powerful (and more difficult to compute) tools such as the *phylogenetic trees* of Gaston Gonnet [16] and others. For details on how to construct such trees see [16]. The Computational Biochemistry Research Group of the Swiss Federal Institute of Technology (ETH) offers a web service for computing phylogenetic trees from distance matrices submitted to their server. Although designed for applications to gene sequence analysis in molecular biology, phylogenetic trees can be constructed for any distance matrix such as the ones obtained from sequences of notes in rhythms. Such trees serve to describe the clustering relationships between objects much like the more traditional hierarchical cluster analysis techniques used

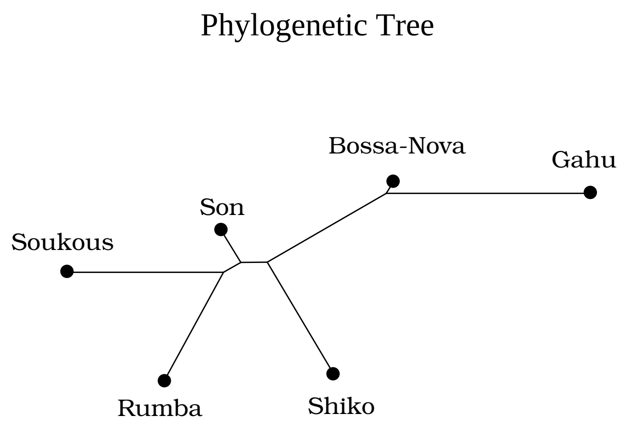

Figure 13: The standard phylogenetic tree constructed from the distance matrix in Figure 11.

in data analysis [12].

Two types of phylogenetic trees that provide insighful visualization were computed from the interval-vector distance matrix of Figure 11. The first is the standard tree shown in Figure 13. Here the tree is drawn such that (as much as possible) the shortest Euclidean distance travelled *in the tree* between any two nodes corresponds to the distance between the pair of rhythms in the distance matrix. Such a tree constructed from the distance matrix of Figure 11 is shown in Figure 13. This tree again clearly shows that Son is the most similar rhythm to the rest whereas Gahu is the most different. Gahu and Soukous are evidently the furthest pair. The tree displays visually at a glance an approximation to all the numerical information contained in the distance matrix.

The second type of phylogenetic tree is the "vertically" oriented *rooted* tree. Such a tree constructed from the distance matrix of Figure 11 is shown in Figure 14. In this tree the minimum *vertical* distance travelled in traversing the tree between two nodes (up and down) is proportional to the distance between the pair of rhythms in the distance matrix. For example, the distance between Gahu and Soukous is 4.24. In the rooted phylogenetic tree this distance is proportional to the distance travelled upwards (vertical component only) from the Gahu node to the root plus the distance travelled downwards (vertical component only) from the root to the Soukous node. The rooted phylogenetic tree shows clearly the hierarchical clustering structure that exists between all the rhythms. The first partition is in the two clusters: Bossa-Nova and Gahu in the right cluster and the other four into the left cluster. The left cluster then breaks up into Shiko and the other three: Son, Soukous and Rumba. Finally, Son breaks away from the pair Soukous and Rumba.

5.5 Proximity graphs

By definition the minimum spanning tree and the phylogenetic trees considered above necessarily describe the distance structure in the distance matrix as a tree. However, one may be interested in knowing how appropriate such a tree-like structure is. In the bioinformatics

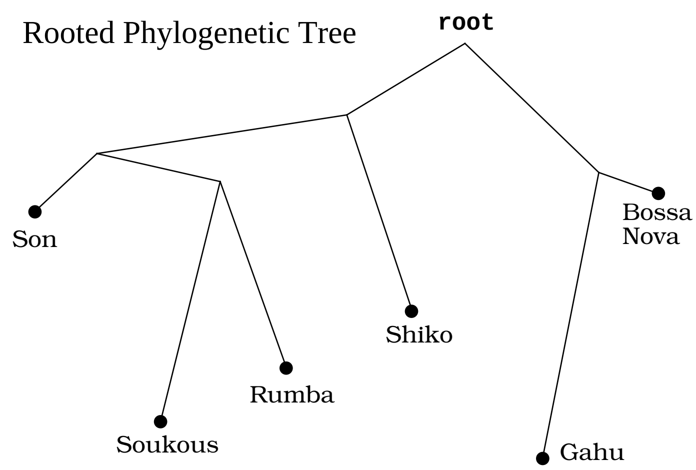

Figure 14: The rooted phylogenetic tree constructed for the distance matrix in Figure 11.

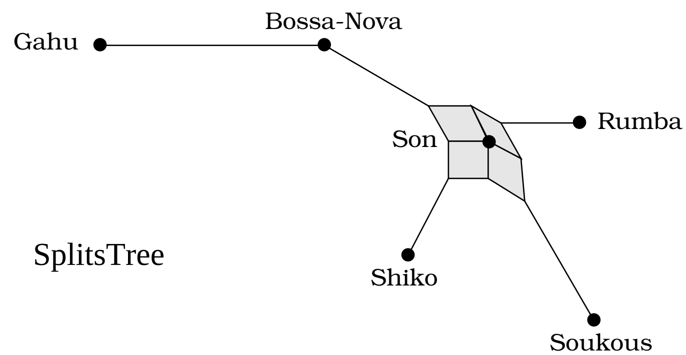

Figure 15: The SplitsTree constructed for the distance matrix in Figure 11.

literature there exist techniques which provide this information in a graph that is a generalization of a tree. One such example is the *SplitsTree* [24]. The University of Bielefeld in Germany also offers a web service for computing SplitsTrees. The SplitsTree constructed from the distance matrix of Figure 11 is shown in Figure 15. If the tree structure does not match the data perfectly then edges are split to form parallelograms whose size is proportional to the missmatch. In Figure 15 the four edges incident to the Son have been split. However, the parallelograms are relatively small, indicating that the underlying structure is strongly tree-like.

A different approach to determining if the underlying graph of a distance matrix is tree-like is obtained by using proximity graphs [25]. Consider for a moment a set of points in the plane. For example Figure 16 shows two sets of points (ignoring the edges connecting them for the moment). The set on the left is clearly tree-like whereas the one on the right is not. A set of points in the plane determines a distance matrix. The distance between two nodes in the complete graph corresponding to x and y is simply the Euclidean distance between the locations of x and y in the plane. We would like to define a graph that we can easily

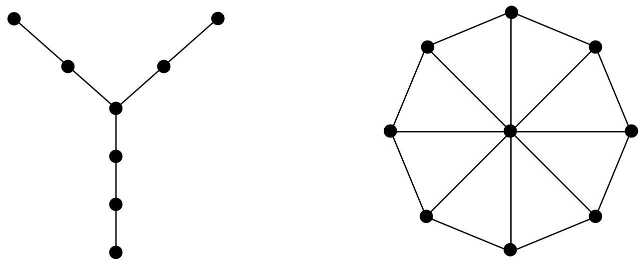

Figure 16: The relative neighborhood graphs (RNG) of two sets of points.

compute from the distance matrix such that the graph is a tree if, and only if, the structure of the points forms a tree. One such graph that can be used to solve this problem is the *relative neighborhood* graph [47]. To construct this graph we add an edge between a pair of vertices (nodes, points) if the points defining the pair are closer to each other than to any other point in the set. More precisely, two points x and y are joined by an edge if for every other point z the maximum of $d(z, x)$ and $d(z, y)$ is greater than or equal to $d(x, y)$. If we compute this graph for the two sets of points in Figure 16 then we obtain the edges shown. The edges on the left form a tree whereas the edges on the right form a wheel. Thus the relative neighborhood graph captures the underlying graph structure of the distance matrix without imposing some a priori structure such as a tree.

Turning now to the distance matrix of Figure 11 for the six clave rhythms, the reader may verify that the relative neighborhood graph computed from it yields precisely the same topological graph as the minimum spanning tree shown in Figure 12.

6 Combinatorial Analysis of Rhythms

The application of combinatorics to the analysis of music is not new. However, almost all such analyses have been applied to the "vertical" tone scale rather than the "horizontal" time scale [13], [43], [41], [8]. Exceptions to this trend are the two papers by Joel Haak [19], [20] and the one by Justin London [33].

Steve Reich [42] composed an interesting piece called *Clapping Music* for two people clapping hands. Both performers clap exactly the same 12-time-unit rhythm shown in Figure 17. One performer repeats the sequence continually throughout the piece but the second performer, after having repeated the sequence 12 times, shifts the sequence by one beat. The second player continues to shift the pattern by one time unit in the same direction every time the pattern has been played 12 times. The piece finishes when the second performer returns in phase with the first. Thus the last 12 combined patterns sound the same as the first 12 when both performers are perfectly in phase.

There are 8 notes (claps) in Steve reich's *Clapping Music*. The number of ways we can select 8 out of 12 time units in which to clap is $(12!)/(8!)(4!) = 495$. Joel Haak [19] raises

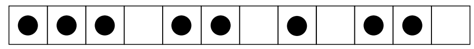

Figure 17: The sequence used by Steve Reich in *Clapping Music* [42].

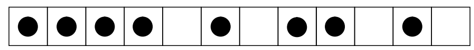

Figure 18: The other sequence found by Joel Haak [19].

the interesting question of how Reich might have come to select the pattern in Figure 17 out of all the possible 495. Did he listen to all 495 and select the pattern he liked best? Haak then suggests a mathematical response to the question. If we take into consideration that a piece should begin with a clap and not a silent pause, and if cyclic permutations of a pattern are considered equivalent, and if during the execution of the entire piece the combined pattern made by both performers clapping does not repeat itself, and finally, if we do not allow consecutive repetitions of the number of claps between two consecutive pauses, then only 2 of the 495 patterns satisfy these constraints. One is the [3,2,1,2] note pattern Reich chose in Figure 17. The other is the [4,1,2,1] note pattern of Figure 18. Haak does not offer a criterion for choosing between the last 2 remaining candidates. But of course if we want to minimize the maximum range of the lengths of consecutive claps, then we obtain Reich's pattern. Alternately, we may invoke the criterion of maximally-even sets [8] to arrive at Reich's pattern. I am sure Reich will not mind if we continue to guess how he might have arrived at his pattern. I offer an alternate non-mathematical hypothesis. Reich studied African drumming in 1970 at the Institute of African studies at the University of Ghana. Now one of the fundamental West-African bell patterns in 6/8 time is the *Yoruba* pattern shown in Figure 19. If we insert a note between the first two notes in the Yoruba pattern we obtain Reich's pattern.

6.1 Permutations and multisets

In the clave rhythm patterns considered in this paper five notes are played in sixteen available time units. The number of ways to select 5 out of 16 is $(16!)/(5!)(11!) = 4368$. This is a large number of patterns. Furthermore most of these may be useless as good time-line patterns for powerful percussive dance music. How can we reduce this large number to an interesting small subset? Note that in the five clave patterns other than Soukous, the minimum and maximum inter-onset intervals are 2 and 4, respectively. Therefore we can modify the combinatorial question to take such constraints into account. Consider the clave Son pattern in binary sequence representation [1 0 0 1 0 0 1 0 0 0 1 0 1 0 0 0] and its interval-vector (2 2 3 1 3) corresponding to the time units skipped (zero's) in between the notes played. One may ask how many permutations exist of the pattern (2 2 3 1 3)? Note that these are *multisets* now since repetitions of the elements are permitted [23]. We have 5

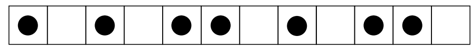

Figure 19: One of the classic 6/8 time bell patterns in Yoruba music [1].

objects of three different classes: 1 of class one, 2 of class two and 2 of class three. Therefore the total number of different permutations of (2 2 3 1 3) is $(5!)/(1!)(2!)(2!) = 30$.

The reader is invited to play these. It turns out all 30 of these permutations sound great. Among these 30 are also found the Rumba, the Gahu as well as the backward versions of the Son, Rumba and Gahu. This also becomes evident from examining Figure 4. If one rhythm may be obtained from another by a permutation of its interval vector the two rhythms will be said to belong to the same *interval combinatorial class*. Thus Son (2 2 3 1 3), Rumba (2 3 2 1 3) and Gahu (2 2 3 3 1) belong to the same interval combinatorial class, whereas Shiko (3 1 3 1 3), Soukous (2 2 3 0 4) and Bossa-Nova (2 2 3 2 2) each belong to their own distinctive classes.

Returning to the 30 rhythms of the Son-Rumba-Gahu interval combinatorial class, and excepting the Son played backwards because it is a weak palindrome, the remaining 26 rhythms have an eerie resemblance to the Son, Rumba and Gahu but sound more modern, more jazzy somehow. Any one of them could be successfully incorporated in new music. This interval combinatorial technique suggests a fruitful approach to automated generation of new rhythms.

7 Concluding Remarks

Several general and specific conclusions may be drawn from this study. First consider the general conclusions. Three measures of the complexity of rhythms were compared with respect to the six fundamental clave rhythms in African, Brazilian and Cuban music. Pressing's congnitive complexity measure gives results which match well with experience. The Lempel-Ziv measure appears to be useless and this finding confirms previous preliminary studies on this measure in another context. The interval-vector distance was found to be good for comparing the similarity of rhythms. Phylogenetic trees appear to be useful for the visualization and cluster analysis of rhythms. Viewing the interval vectors as multisets and generating permutations of these multisets provides an easy manner of generating new and interesting rhythms from a given rhythm.

Turning to the specific conclusions about the six clave rhythms we may sumarize the main results as follows. It has been shown that it is possible with three simple geometric features to automatically classify the six rhythms without knowledge of their starting note The Son clave pattern (the only weak palindrome in the set) is most like all the others. It is in some sense the center of all the claves. It has elements of all the other claves and perhaps because of this it is accessible to a wider audience. It is also one of the simplest rich rhythms. Perhaps these are the reasons for its worldwide popularity. The Son, Rumba and Gahu belong to the same interval-combinatorial family. Only Bossa-Nova and Shiko are strong palindromes, which indicates they possess a strong symmetry structure. Interestingly, these two rhythms have diametrically opposite cognitive complexities according to Pressing's measure [40]. Shiko has the lowest value of 8 and Bossa-Nova the highest value of 22. The fundamental clustering evident from the phylogenetic tree construction algorithm is (1) Gahu and Bossa-Nova, (2) the remaining four rhythms. Rumba is the most unique of the six claves from the geometric point of view in that its polygon contains no isoceles triangles, no right angles, and no axis of symmetry. Gahu is one of the most complicated rhythms according to the three measures of complexity considered here. From the mathematical point of view it is the most unique in the sense that it is the most different from the rest. These results add mathematical support to the musical analysis of the Gahu rhythm provided by Locke [32].

8 Acknowledgements

I would like to thank Jeff Pressing for correspondence concerning his measure of the cognitive complexity of rhythms [40], and Mike Hallett for discussions on phylogenetic trees.

References

[1] Willie Anku. Circles and time: A theory of structural organization of rhythm in African music. *Music Theory Online*, 6(1), January 2000.

[2] J. Becker. Hindu-Buddhist time in Javanese gamelan music. In *Study of Time IV*. 1981. Ed.,J. T. Fraser.

[3] H. Bettermann, D. Amponsah, D. Cysarz, and P. Van Leeuwen. Musical rhythms in heart period dynamics: a cross-cultural and interdisciplinary approach to cardiac rhythms. *Proceedings of the American Physiological Society*, pages H1762–H1770, 1999.

[4] T. L. Bolton. Rhythm. *American Journal of Psychology*, 6:145–238, 1894.

[5] Sung-Hyuk Cha and Sargur N. Srihari. On measuring the distance between histograms. *Pattern Recognition*, 35:1355–1370, 2002.

[6] Gregory J. Chaitin. Information theoretic computational complexity. *IEEE Transactions on Information Theory*, IT-20:10–15, 1974.

[7] John Miller Chernoff. *African Rhythm and African Sensibility*. The University of Chicago Press, Chicago, 1979.

[8] J. Clough and J. Douthett. Maximally even sets. *Journal of Music Theory*, 35:93–173, 1991.

[9] E. J. Coyle and I. Shmulevich. A system for machine recognition of music patterns. In *Proceedings of the IEEE International Conference on Acoustics, Speech, and Signal Processing*, Seattle, Washington, 1998.

[10] Ian Cross. Music and science: three views. *Revue Belge de Musicologie*, LII:207–214, 1998.

[11] Simon Dixon. Beat induction and rhythm recognition. In *Proceedings of the Australian Joint Conference on Artificial Intelligence*, pages 311–320, Perth, Australia, 1997.

[12] Richard O. Duda and Peter E. Hart. *Pattern Classification and Scene Analysis.* John Wiley, New York, 1973.

[13] Andrew Duncan. Combinatorial music theory. *Journal of the Audio Engineering Society*, 39:427–448, June 1991.

[14] Peter Eades. Symmetry finding algorithms. In Godfried T. Toussaint, editor, *Computational Morphology*, pages 41–51. North-Holland, 1988.

[15] Douglas Eck. A positive-evidence model for classifying rhythmical patterns. Technical Report IDSIA-09-00, Instituto Dalle Molle di studi sull'intelligenza artificiale, Manno, Switzerland, 2000.

[16] Gaston H. Gonnet. New algorithms for the computation of evolutionary phylogenetic trees. In S. Suhai, editor, *Computational Methods in Genome Research.* Plenum Press, 1994.

[17] Michael T. Goodrich and Roberto Tamasia. *Algorithm Design.* John Wiley and Sons, Inc., New York, 2002.

[18] D. Gusfield. *Algorithms on Strings, Trees, and Sequences: Computer Science and Computational Biology.* Cambridge University Press, Cambridge, 1997.

[19] Joel K. Haak. Clapping Music - a combinatorial problem. *The College Mathematics Journal*, 22:224–227, May 1991.

[20] Joel K. Haak. The mathematics of Steve Reich's *Clapping Music.* In *Proceedings of BRIDGES: Mathematical Connections in Art, Music and Science*, pages 87–92, Winfield, Kansas, 1998.

[21] R. W. Hamming. *Coding and Information Theory.* Prentice-Hall, Englewood Cliffs, New Jersey, 1986.

[22] Royal Hartigan, Abraham Adzenyah, and Freeman Donkor. *West African Rhythms for Drum Set.* Manhattan Music, Inc., 1995.

[23] T. C. Hu and B. N. Tien. Generating permutations with non-distinct items. *American Mathematical Monthly*, 83(8):193–196, October 1976.

[24] Daniel H. Huson. SplitsTree: Analyzing and visualizing evolutionary data. *Bioinformatics*, 14:68–73, 1998.

[25] J. W. Jaromczyk and Godfried T. Toussaint. Relative neighborhood graphs and their relatives. *Proceedings of the IEEE*, 80(9):1502–1517, 1992.

[26] Tőm Klőwer. *The Joy of Drumming: Drums and Percussion Instruments from Around the World.* Binkey Kok Publications, Diever, Holland, 1997.

[27] A. J. Komar. *Theory of Suspensions.* Princeton University Press, Princeton, New Jersey, 1971.

[28] Edward W. Large. *Dynamic Representation of Musical Structure.* PhD Thesis, The Ohio State University, Columbus, 1994.

[29] Edward W. Large and John F. Kolen. Resonance and the perception of musical meter. *Connection Science*, 6:177–208, 1994.

[30] A. Lempel and J. Ziv. On the complexity of finite sequences. *IEEE Transactions on Information Theory*, IT-22(1):75–81, 1976.

[31] F. Lerdahl and R. Jackendoff. *A Generative Theory of Tonal Music.* MIT Press, Cambridge, Massachussetts, 1983.

[32] David Locke. *Drum Gahu: An Introduction to African Rhythm.* White Cliffs Media, Gilsum, New Hampshire, 1998.

[33] Justin London. Hierarchical representations of complex meters. In *6th International Conference on Music, Perception and Cognition*, Keele University, United Kingdom, August 5-10 2000.

[34] Neil McLachlan. A spatial theory of rhythmic resolution. *Leonardo*, 10:61–67, 2000.

[35] Abraham Moles. *Information Theory and Esthetic Perception.* The University of Illinois Press, Urbana and London, 1966.

[36] Bernard Mont-Reynaud and Malcolm Goldstein. On finding rhythmic patterns in musical lines. In *Proceedings of the International Computer Music Conference*, pages 391–397, San Francisco, California, 1985.

[37] O'dyke Nzewi. The technology and music of the Nigerian Igbo *Ogene Anuka* bell orchestra. *Leonardo Music Journal*, 10:25–31, 2000.

[38] Keith S. Orpen and David Huron. Measurement of similarity in music: A quantitative approach for non-parametric representations. In *Computers in Music Research*, volume 4, pages 1–44. 1992.

[39] Fernando Ortiz. *La Clave.* Editorial Letras Cubanas, La Habana, Cuba, 1995.

[40] Jeff Pressing. Cognitive complexity and the structure of musical patterns. In *Proceedings of the 4th Conference of the Australasian Cognitive Science Society*, Newcastle, Australia, 1997. [CD-ROM].

[41] Ronald C. Read. Combinatorial problems in the theory of music. *Discrete Mathematics*, 167-168, April 1997.

[42] Steve Reich. *Writings about Music.* The Press of the Nova Scotia College of Art and Design, Halifax, Canada, 1974.

[43] David L. Reiner. Enumeration in music theory. *American Mathematical Monthly*, 92:51–54, January 1985.

[44] Carlos Saavedra Reyes. *Compendio de Percusión Afrolatina.* Editorial Fundación UNA, Costa Rica, 1999.

[45] I. Shmulevich and D. Povel. Rhythm complexity measures for music pattern recognition. In *Proceedings of the IEEE Workshop on Multimedia Signal Processing*, Redondo Beach, California, December 7-9 1998.

[46] Godfried T. Toussaint. Generalizations of π: some applications. *The Mathematical Gazette*, pages 291–293, December 1974.

[47] Godfried T. Toussaint. The relative neighbourhood graph of a finite planar set. *Pattern Recognition*, 12:261–268, 1980.

[48] Chris Washburne. Clave: The African roots of salsa. In *Kalinda!: Newsletter for the Center for Black Music Research.* Columbia University, New York, 1995. Fall-Issue.

[49] Chris Washburne. Clave: The African roots of salsa. *CLAVE*, 1(1), April 1998.

[50] Charles T. Zahn. Graph-theoretical methods for detecting and describing gestalt clusters. *IEEE Transactions on Computers*, C-20:68–86, 1971.