Computational Aspects of Musical Rhythms:

COMP 251 Course Notes
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1. Necklaces and Bracelets,
2. Homometric Rhythms,
3. The Hexachordal Theorem,
4. Patterson’s Theorems, and
5. Flat Rhythms and Deep Rhythms
Consider a rhythm with \( k \) onsets in a time-span (clock) of \( n \) (even) units.

The rhythm clock determines \( n/2 \) different possible durations between pairs of pulse points.

A rhythm is called all-interval flat if it contains all the \( n/2 \) duration intervals, and each of the intervals is used precisely once.

Example of all-interval flat rhythm for \( n=6, k=3 \)
An \((k = 4, n = 12)\) All-Interval flat Rhythm Bracelet

Another \((k = 4, n = 12)\) All-Interval flat Rhythm Bracelet
The Two-Bracelets Theorem

For $n = 12$ there exist only two all-interval flat bracelet yielding 16 all-interval rhythms.

The equation $k(k-1)/2 = 6$ has only one solution: $k = 4$. Therefore only rhythms with 4 onsets are candidates.

There are two cases: the longest interval is either: (1) a diagonal or (2) an edge of the resulting quadrilateral.
case 1: *diameter determined by diagonal*
The Two-Bracelets Theorem - cont.

**case 1:** diameter determined by diagonal
case 2: diameter determined by edge
case 2: diameter determined by edge
Points with specified distance multiplicities

Paul Erdös - 1986

Can one find $n$ points in the plane (no 3 on a line and no 4 on a circle) so that for every $i$, $i = 1, 2, ..., n-1$ there is a distance determined by these points that occurs exactly $i$ times?

Solutions have been found for $n = 2, 3, ..., 8$. Ilona Palásti for $n = 7$ and 8.
Patterson’s example of a **homometric pair**

A. Lindo Patterson,  

A simple **homometric pair**.
Complementary homometric rhythms

V. G. Rau, L. G. Parkhomov, V. V. Ilyukhin and N. V. Belov, 1980

Every $n$-point subset of a regular $2n$-gon is homometric to its complement.
Patterson’s **first theorem**

A. Lindo Patterson,


If two subsets of a regular $n$-gon are **homometric**, then their **complements** are.
**Patterson’s second theorem**

A. Lindo Patterson,

Every $n$-point subset of a regular $2n$-gon is homometric to its complement.
Erdös infinite family of homometric pairs

Paul Erdös,
in personal communication to A. Lindo Patterson,
Physical Review, March, 1944.

\[ a < \frac{1}{4} \]
The Hexachordal Theorem

**Theorem:** Two *complementary* hexachords have the same *interval content*.

*First observed empirically:* Arnold Schoenberg, ~1908.
The Hexachordal Theorem: Music-Theory Proofs

Theorem: Two complementary hexachords have the same interval content. First observed empirically: Arnold Schoenberg, 1908.

Proofs:

1. Milton Babbitt and David Lewin - 1959, topology
2. David Lewin - 1960, group theory
3. Eric Regener - 1974, elementary algebra
4. Emmanuel Amiot - 2006, discrete fourier transform
The **Hexachordal Theorem: Crystallography Proofs**

**First observed experimentally:** Linus Pauling and M. D. Shappell, 1930.

**Proofs:**

1. Lindo Patterson - 1944, *claimed proof not published*

2. Martin Buerger - 1976, *image algebra*


4. Steven Blau - 1999, *elementary induction*
The interval-content theorem of Iglesias

Juan E. Iglesias,

**Theorem:** Let \( p \) of the \( N \) vertices of a regular polygon inscribed on a circle be black dots, and the remaining \( q = N - p \) vertices be white dots. Let \( n_{ww} \), \( n_{bb} \), and \( n_{bw} \) denote the multiplicity of the distances of a specified length between white-white, black-black, and black-white, vertices, respectively.

Then the following relations hold:

\[
p = n_{bb} + \left(\frac{1}{2}\right)n_{bw}
\]
\[
q = n_{ww} + \left(\frac{1}{2}\right)n_{bw}
\]
Lemma: Any given duration value $d$ occurs with multiplicity $N$.

(1) If $d = 1$ or $d = N-1$ the multiplicity equals the number of sides of an $N$-vertex regular polygon.

(2) If $1 < d < N-1$, and $d$ and $N$ are relatively prime, the multiplicity equals the number of sides of an $n$-vertex regular star-polygon.

(3) If $d$ and $N$ are not relatively prime then the multiplicity equals the total number of sides of a group of convex polygons. There are $g.c.d. (d, N)$ polygons with $N/g.c.d(d, N)$ sides each.
Proof of Iglesias’ theorem:
For each duration value $d$

\[
p = n_{bb} + (1/2)n_{bw}
\]

\[
q = n_{ww} + (1/2)n_{bw}
\]

case 1

case 2

case 3
Iglesias’ Proof of Patterson’s Theorems

**Theorem 1:** If two different black sets form a homometric pair, then their corresponding complementary white sets also form a homometric pair.

**Proof:** If the black sets are homometric they must have the same number of points. Then, for each duration value $d$

$$p = n_{bb} + (1/2)n_{bw} = n_{bb}^* + (1/2)n_{bw}^*$$

$$q = n_{ww} + (1/2)n_{bw} = n_{ww}^* + (1/2)n_{bw}^*$$

and thus

$$p - q = n_{bb} - n_{ww} = n_{bb}^* - n_{ww}^*$$

Since the black sets are homometric $n_{bb} = n_{bb}^*$ and thus $n_{ww} = n_{ww}^*$

**Theorem 2:** If $p = q$ the two sets are homometric.

**Proof:** If $p = q$ then

$$n_{bb} + (1/2)n_{bw} = n_{ww} + (1/2)n_{bw}$$

and thus

$$n_{bb} = n_{ww}$$
Popular (2/4)-time folk-dance rhythms of northern Transylvania

Ubiquitous rhythms in *African*, *rockabilly*, and *world music*. The *Habanera* rhythms.

Which *necklaces* have the property that they are *deep* and have *deep complementary necklaces*?
Complementary deep rhythms

Dave Brubeck, *Unsquare Dance*,
Columbia Records, CS 8490 (stereo).
Deep scales have been studied in music theory at least since 1966 by Terry Winograd. Carlton Gamer, *Journal of Music Theory*, 1967.