Assignment 1 Solutions

COMP 760 — MATH 762

2006-01-11

Question 1

The following are two of many examples of a family such as that required in Question 1.

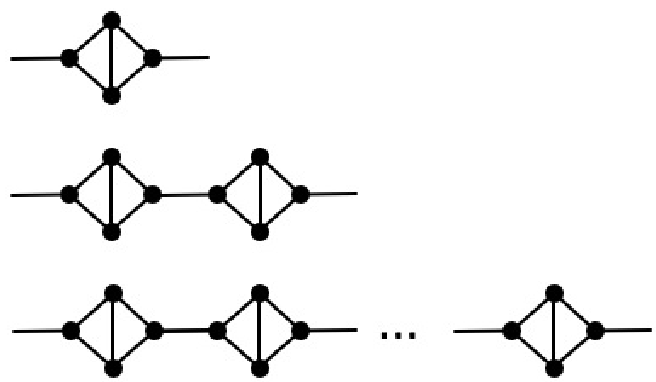

Figure 1: Graph G_k has $4k + 2$ vertices, $6k + 1$ edges, a unique perfect matching and at least 2^k perfect matchings.

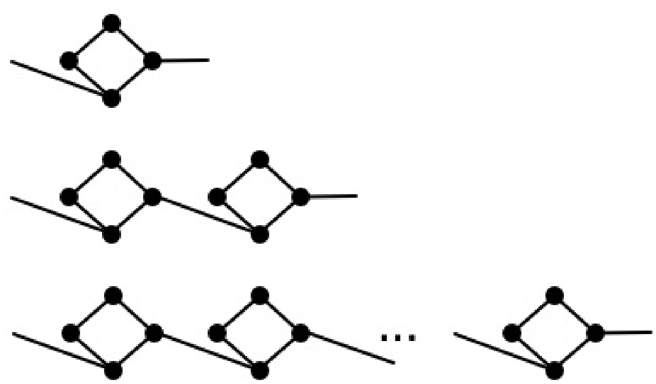

Figure 2: Graph G_k has $4k + 2$ vertices, $5k + 2$ edges, a unique perfect matching and at least 2^k perfect matchings.

Question 2

Given an orientation of G in which every vertex satisfies $d^-(v) \leq k$ (a *good* orientation), it follows immediately that $|E(G)| = \sum_{v \in V} d^-(v) \leq k|V(G)|$.

Conversely, let G be any graph and orient G to form a digraph D. Let S be the set of vertices with $d^-(v) > k$ and let T be the set of vertices with $d^-(v) < k$. By our assumption on the average degree, the badness $b = \sum_{v \in S}(d^-(v) - k)$ is at most $\sum_{v \in T}(k - d^-(v))$. We choose D to minimize b; if $b = 0$ the orientation is good so assume that $b > 0$.

In this case, let H be the component of G consisting of S together with all vertices at the head of an oriented path to S. If there were a vertex $t \in V(H) \cap S$, then reversing all the edges on a path from t to S would reduce b, contradicting its minimality. Therefore $V(H)$ is disjoint from T. By the definition of H, $d^-_H(v) = d^-(v)$ for every $v \in V(H)$; by the definition of T it follows that $d^-_H(v) \geq k$ for every vertex in H. As S is nonempty, there is some $s \in V(H)$ for which $d^-_H(s) > k$. Therefore $|E(H)| = \sum_{v \in H} d^-_H(v) > k|V(H)|$; this completes the proof.

Question 3

We start with any orientation D of G and define S, T as in Question 2. We repeatedly search for an oriented $S - T$ path and reverse all its edges until no such path exists. If S is empty then we have exhibited the desired orientation; if it is not, then by Question 2 no such orientation exists.

This procedure runs in polynomial time as each step consists an application of depth-first search in an oriented graph. Each step reduces the badness b by at least 1. The initial badness is surely at most n^2; thus the whole procedure takes time at most n^3.

Question 4

Let G be a graph satisfying the conditions of Question 4. G contains no isolated vertices and any component with at least 2 vertices contains strictly greater than $2n/3$ vertices - thus G is connected.

Let P be a path of maximum length in G. Denote by v, w the endpoints of P.

Suppose $G[P]$ is Hamiltonian and $y \notin P$ has a neighbor $u \in P$. Letting Q be a Hamiltonian path in $G[P]$ starting at y, yQ is a path of length strictly longer than P, contradicting the fact that P is maximum. Thus if $G[P]$ is maximum then it is a component, and thus $G[P] = G$ as G is connected.

We suppose that $G[P]$ is not Hamiltonian and derive a contradiction. Say that a is *before* b on P (or b is *after* a) if a is between v and b along P. Say that a is *just before* b (or b is *just after* a) if a is before b and ab is an edge of P. Recall from class that Posa's flipping technique essentially consists of the observation that if $G[P]$ is not Hamiltonian the following property must hold

(*) If a is just before b and $wa \in E$ then $vb \notin E$.

Given a set of vertices S and vertices $a, b \in P$, let S^{ab} be the intersection of S with the segment of P between a and b. For any vertex a, let $M(a)$ be the neighbours of a on P, let $B(a)$ (resp. $F(a)$) be the vertices that are just before (resp. just after) elements of $M(a)$. We then must have:

(1) $B(v)$ and $M(w)$ are disjoint,

for otherwise $(*)$ will fail to hold.

Note that by the maximality of P, the neighbourhoods of v and w are both contained in P, so in particular both v and w have at least two neighbours in P. Let x be the first neighbour of v that is not just after v and let z be just before x. There is a path Q from z to w obtained by following P from z to v, then traversing edge vx and following P from x to w. A condition (**) akin to (*) can then be stated for the path Q; the precise statement is ommited. We then have

(2) $B^{xw}(z)$ and $N(w)$ are disjoint, and

(3) $F^{vz}(z)$ and $N(w)$ are disjoint,

or else (**) fails. (The details are left to the reader.) Note that $F^{vz}(z) - x$ is disjoint from $B(v) - z$, by our choice of x. Therefore $S = (F^{vz}(z) - x) \cup B^{xw}(z) \cup (B(v) - z)$ has cardinality at least that of $B(z) \cup B(v) - z - x$, which is at least $2n/3 - 4$ by assumption. Furthermore, $w \notin S$ by definition. As S is disjoint from $N(w)$ and $|N(w)| > \frac{n}{3} + 4$ this implies that $|S \cup N(w)| \geq n$, so $|P| \geq |S \cup N(w) \cup w| > n$, a contradiction as $|P| < |V(G)| = n$.