The f-factor Theorem of Tutte

Given a non-negative integer $f(v)$ for each vertex v in a graph G, an f-factor is a subgraph H of G with $d_H(v) = f(v)$ for every vertex v in G. In this setting, we set $f(Y) = \sum_{v \in Y} f(v)$ for every subset Y of V.

We state and proof a theorem of Tutte characterizing when an f-factor exists below. To do so, we need to apply the characterization of which graphs have perfect matching to an auxiliary graph G_f defined as follows:

$$V(G_f) = \{v_1, ..., v_{f(v)} | v \in V(G)\} \cup \{e_u, e_v | e = uv \in E(G)\}$$

$$E(G_f) = \{v_1 e_v, ..., v_{f(v)} e_v | v \in V(G), e = uv \in E(G)\} \cup \{e_u e_v | e = uv \in E(G)\}.$$

We note that each vertex of G_f corresponds either to a vertex of G or to an edge of G.

Lemma: *G_f has a perfect matching precisely if G has an f-factor*

Proof: Given an f-factor H of G, we can enumerate the edges of H incident to v as $e(v,1), ..., e(v, f(v))$. Our perfect matching uses the edges $v_i e(v,i)_v$ for each vertex v and i between 1 and $f(v)$, and also the edge $e_u e_v$ for each edge uv of G which is not in H.

Given a perfect matching M in G_f, we set H to be the subgraph of G consisting of $\{e | e = uv, e_u e_v \notin M\}$. For any such edge $e_u u_i$ and $e_v v_j$ are edges of M for some i and j. Conversely, if $v_i e_v$ is an edge for somne $e = uv$ then $e_u e_v$ is not in M. Thus, since M is perfect, each vertex v of G is incident to precisely $f(v)$ edges of H. QED.

This lemma is the key to the proof of the following:

Theorem: *G has an f-factor precisely if there do not exist disjoint X and Y contained in V satisfying:*

$$(*) f(Y) > f(X) - |O(X,Y) + \sum_{v \in Y} (d(v) - |N(v) \cap X|),$$

where $O(X,Y)$ is the set of components U of $G - X - Y$ for which the parity of $f(U)$ and $E(U,Y)$ differ.

Proof: $f(X) + \sum_{v \in Y} d(v) - |N(v) \cap X|$ is a bound on the sum of the degrees of the vertices in Y in any f-factor, where we bound the number of edges to X and to $V - X$ separately. Furthermore, for any component U in $O(X,Y)$, parity conditions ensure that in any f-factor either there is an edge from U to X in H or there is an edge from U to Y in $E(G) - E(H)$. Thus, if some Y satisfies (*) then G cannot have an f-factor. It remains to prove the converse.

To this end, suppose G has no f-factor, then G_f has no perfect matching so we can find a set Z in G_f such that the number of odd components of $G_f - Z$

exceeds $|Z|$. We choose such a set Z so as to minimize the number of vertices in Z.

We note that if some v_i is in a component U of $G_f - Z$ then for any v_j in Z, the components of $G_f - (Z - v_j)$ are exactly the components of $G_f - Z$ except that U is replaced by $U + v_j$. So, our choice of Z ensures that if any v_i is in Z, all of $\{v_1, ..., v_{f(v)}\}$ is in Z. We let X be the set of vertices for which this is the case. We note that if there is some edge $e = uv$ with $v \in X$ and $e_v \in Z$ then e_v sees vertices in at most one component of $G - Z$ (that containing e_u), and hence $Z - e_v$ contradicts the minimality of Z. So there is no such e_v in Z.

Suppose, for a contradiction, that for some vertex $v \in V - X$, there are two edges e and f such that f_v is in Z and e_v is not in Z. Then $v_1, .., v_{f(v)}$ are all in the same component U of $G - Z$ as e_v, so the components of $G - (Z - f_v)$ containing f_v conists of the union of U, f_v and possibly the component of $G - Z$ containing f_w for the other endpoint w of f. In any case, the number of odd components of $G - (Z - f_v)$ differs by at most one from the number of odd components of $G_f - Z$ (for if we lose two, then we gain one). Thus, by the minimality of Z, there is no such pair f_v, e_v, and we can find a subset Y of $V - X$. such that $Z = \{e_v | v \in Y\} \cup \{v_i | v \in X\}$.

Now, for any edge uv of G disjoint from $X \cup Y$, e_u, e_v and all the copies of u and v are in $G_f - Z$. Furthermore, there are copies of both u and v as X contains all vertices w with $f(w) = 0$. Thus, for any component U of $G - X - Y$ there is a component W of $G_f - Z$ consisting of all of the copies of the vertices of Z and e_v for every vertex v of U and edge e incident to v. Note that the edges between vertices of U contribute two vertices to W, so W is odd precisely if U is in $O(X, Y)$. The remaining components of W consists of $e_u e_v$ for every edge uv within X, every vertex copy of a vertex in Y and e_u for every edge uv with u in X and v in Y. Thus, We have that

$$|O(X, Y)| + f(Y) + E(X, Y) > |Z| = f(X) + \sum_{y \in Y} d(Y).$$

QED.