Algebraic methods

$1, \omega, \omega^2, \ldots, \omega^{p-1}$ form a linear basis for $\omega$ be a primitive $p$th root of unity. Let $Z[\omega]$ be the ring that the quotient ring $Z[\omega]/((1 - \omega) \cdot Z[\omega])$ is a factor of $(1 - \omega)$.

$\omega$ be a primitive $p$th root of unity, let $\eta_1, \ldots, \eta_k$ be distinct integers in $\mathbb{Z}$ and has at most $k$ non-zero coefficients. And Lemma 9.26, show that $P$ is a multiple of an infinite descent argument, obtain another least in the case $q = p$, which is all one needs. 9.52 from Lemma 9.51. (Hint: Lemma 9.51) of the Fourier matrix $(e^{2\pi i (k/l)})_{\mathbb{Z} \times \mathbb{Z}}$ are now that Theorem 9.52 implies the $q = p$ case.

$\{ z \in \mathbb{C} : z^p = 1 \}$ be the $p$th roots of unity, and a polynomial with $\deg(P) < p$. Show that that $n$ in $G$ cannot exceed the number of non-zero elements group $G$ and any real number $k$. let $\theta(Z; k)$

\[ \tilde{f}(f) = \sup_{G \subseteq Z} \frac{1}{G} \left( \frac{\theta(G; k)}{\theta(Z/G; 1)} \right) \]

Corollary 9.54, conclude via an inductive zero function $f$ in $L^2(Z)$, the lattice point on or above the convex hull of the points over all subgroups of $Z$.

10

Szemerédi’s theorem for $k = 3$

A surprisingly fruitful and deep problem in additive combinatorics is that of determining whether a given set $A$ contains non-trivial (i.e., proper) arithmetic progressions of a given length. We have already seen some special cases of this problem; in Section 4.7, we saw that sum sets such as $A + A + A + A$, or $2A - 2A$, contained very long arithmetic progressions (and generalized arithmetic progressions), while in Section 6.3 we saw that if we colored a large finite group (or a large interval of integers) into a small number of color classes, then one of the color classes must necessarily contain a long arithmetic progression. In this chapter and the next we shall discuss perhaps one of the deepest theorems known to additive combinatorics, namely Szemerédi’s theorem:

**Theorem 10.1 (Szemerédi’s theorem) [345]** Let $A$ be a subset of the positive integers with positive upper density $\bar{\sigma}(A) > 0$. Then $A$ contains arbitrarily long arithmetic progressions.

This theorem was originally proved by Szemerédi in 1975 by a sophisticated combinatorial argument, introducing for the first time the powerful Szemerédi regularity lemma, which we discuss in Section 10.6. There are several other deep and important proofs of this theorem, including the ergodic-theoretic proof of Furstenberg [125], the additive combinatorial proof of Gowers [138], and the hypergraph regularity proofs of Gowers [140] and Nagle, Rödl, Schacht, and Skokan [254], [282], [283], [284]. These proofs will be discussed in the next chapter.

One can formulate Szemerédi’s theorem in a more quantitative manner, using the following definition.

**Definition 10.2 (Erdős–Turán constant) [99]** Let $A$ be an additive set, and let $k \geq 1$. We let $r_k(A)$ denote the size of the largest subset of $A$ which does not contain any proper arithmetic progressions of length $k$.

\[ \sigma_k(A) = \lim_{n \to \infty} \frac{\left| A \cap [1, n] \right|}{n} \]
Examples 10.3 We have $r_k(A) = 0$ and $r_2(A) = 1$ for any additive set $A$. Clearly $r_1(A)$ is non-decreasing in $A$, and we have the trivial bound $r_k(A) \leq |A|$ for any $A$ is a linear in a group; for example, $r_k(\mathbb{F}_2) = |A|$ for all $k > 1$.

Theorem 10.1 is then easily shown to be equivalent to the following, which was first conjectured by Erdős and Turán [99].

**Theorem 10.4 (Szemerédi’s theorem, second formulation)** Let $k \geq 1$ and $N \geq 1$. Then $r_k([1, N]) = o_N(N^{k/2})$ and $r_k(\mathbb{Z}_N) = o_N(N)$.

One in fact has the following generalization:

**Theorem 10.5 (Szemerédi’s theorem, in an arbitrary group)** Let $k \geq 1$ and let $Z$ be a finite additive group. Then $r_k(Z) = o_Z(N)$.

This generalization either follows from the density Hales-Jewett theorem [124] or from the hypergraph proofs of Szemerédi’s theorem [140], [254], [282], [283], and will be discussed in Section 11.6.

A further famous conjecture of Erdős and Turán remains open:

**Conjecture 10.6 (Erdős–Turán conjecture)** Let $A \subset \mathbb{Z}^+$ be such that $\sum_{n=1}^{N} \frac{1}{n} = \infty$. Then $A$ contains arbitrarily long proper arithmetic progressions.

Up to very small factors, such as $\log^{O(1)} N$, this conjecture is essentially equivalent to asking for $r_k([1, N]) = O_k(N/\log N)$ for all $k$ and $N$ (Exercise 10.0.6). This conjecture remains unsolved even for progressions of length 3 (though see Theorem 10.30 below). However, a special case of this conjecture, restricted to the prime numbers $P = \{2, 3, 5, \ldots\}$, has recently been proven by Green and Tao.

**Theorem 10.7 (Green-Tao theorem)** Let $k \geq 1$ and $N > 1$. Then $r_k(\mathbb{Z} \cap [1, N]) = o_N(N)$ for any $k \geq 1$. In particular, the primes contain arbitrarily long arithmetic progressions.

Note that the sum $\sum_{n=1}^{N} \frac{1}{n}$ is divergent.

For general $k$, Szemerédi’s theorem and the Green–Tao theorem are rather involved and will be treated in Chapter 11. However, the $k = 3$ case is amenable to Fourier-analytic methods, and we have the following theorem of Roth:

**Theorem 10.8 (Roth’s theorem)** We have $r_3([1, N]) = o_N(N)$ for all $N > 1$. More generally, for any finite additive group $Z$ of odd order we have $r_3(Z) = o_Z(|Z|)$.

The generalization to arbitrary additive groups $Z$ of odd order is due to Meeks [28]. Note that the restriction that $Z$ be odd is necessary, since for 2-torsion groups, there are no proper progressions of length three and hence $r_3(Z) = |Z|$.

Exercise 10.0.1 Establish the inequalities $r_k([1, N]/k) = o_N(N)$ for any $N > k > 1$. This shows $r_k([1, N]) = \alpha_N(N)$.

10.0.2 Show that Theorem 10.4 is equivalent to Theorem 10.1 from Theorem 10.1 and, by contradiction, any non-trivial arithmetic progression is contained in a subset of $(\mathbb{Z}/n\mathbb{Z})$ of density $\alpha_n(n)$.

10.0.3 Show that Theorem 10.4 is equivalent to Theorem 10.1 and, by contradiction, any non-trivial arithmetic progression is contained in a subset of $(\mathbb{Z}/n\mathbb{Z})$ of density $\alpha_n(n)$.

10.0.4 Show that Szemerédi’s theorem (Exercise 6.3.7).

10.0.5 Give an example to show that the color class contains an infinite progression of length $k$ in that case.
Roth’s theorem and Szemerédi’s theorem have a surprising diversity of different proofs, using such techniques as harmonic analysis, ergodic theory, graph theory, hypergraph theory, inverse sum set theory, and Ramsey theory. However, they all revolve around a fundamental dichotomy, namely the dichotomy between arithmetically structured sets (e.g. arithmetic progressions, Bohr sets, sets of small doubling, sets of large additive energy, almost periodic sets) and arithmetically unstructured sets (e.g. random sets, pseudo-random sets, “mixing” sets). The point is that one needs very different arguments to deal with either of the two cases, and so any proof of the above theorems must first decompose a general set somehow into a structured component and an unstructured one. To make such a decomposition rigorous, one needs some powerful tools, for instance from harmonic analysis, ergodic theory, or graph theory.

The purpose of this chapter is to give several proofs of Roth’s theorem, both for general $Z$ and in special cases, and to also discuss some variants of this theorem. These proofs serve as models for the more difficult Szemerédi and Green–Tao theorems, to be discussed in the next chapter. It turns out that linear Fourier analysis (as developed in Chapter 4) is a particularly well-adapted tool to detect progressions of length 3, as we shall see however in the next chapter, progressions of longer length will require a quadratic or higher-order Fourier analysis.

Exercises

10.0.1 Establish the inequalities

$$r_3([1, N/k]) \leq r_3(Z_N) \leq r_3([1, N])$$

for any $N > k > 1$. This shows that the two forms $r_3(Z_N) = o_{N \to \infty}(N)$ and $r_3([1, N]) = o_{N \to \infty}(N)$ of Theorem 10.4 are equivalent.

10.0.2 Show that Theorem 10.4 is equivalent to Theorem 10.1. (Hint: to deduce Theorem 10.1 from Theorem 10.4 is rather easy. For the converse direction, argue by contradiction, obtaining dense subsets of $[1, N]$ without any proper arithmetic progressions, and paste those subsets together in some suitable way to contradict Theorem 10.1.)

10.0.3 Show that Theorem 10.1 is equivalent to the statement that every subset of the integers of positive upper density contains infinitely many progressions of length $k$, for each $k \geq 1$.

10.0.4 Show that Szemerédi’s theorem implies van der Waerden’s theorem (Exercise 6.3.7).

10.0.5 Give an example to show that if the positive integers $Z^*$ are partitioned into two color classes, then it is not necessarily the case that one of the color class contains an infinitely long proper arithmetic progression.
10.5.5 Let $f : Z \to \mathbb{C}$ be normalized so that $\|f\|_{L^1(Z)} = \|\hat{f}\|_{L^\infty(Z)} = 1$. Suppose that one can cover the set $\{T^h f : h \in Z\} \subset L^2(Z)$ by $M$ balls of radius $r$ in the $L^2(Z)$ metric. Show that $f$ is $O_{M, r}(1)$-almost periodic. (Hint: use the pigeonhole principle and the Fourier transform to establish a lower bound for

$$P_{\mathbb{Z}^d} \left( \sum_{h \in Z} |f(\xi + h)|^2 \cdot |\hat{f}(\xi)|^2 \right).$$

Remove the $K$ largest Fourier coefficients from $f$, for some $K = O_M, r(1)$ to be chosen later, and apply the previous exercise to conclude an upper bound on the $L^2$ norm of the remaining Fourier coefficients. This result, combined with Exercise 10.5.2, gives a way to define almost periodicity purely in terms of the precompactness of the orbit $\{T^h f : h \in Z\}$, without explicit mention of the Fourier transform.

10.6 The Szemerédi regularity lemma

In the original proof of Szemerédi’s theorem (Theorem 10.1), Szemerédi introduced an important result in graph theory, the Szemerédi regularity lemma. This lemma has since become one of the main tools in discrete mathematics. It asserts, roughly speaking, that any dense large graph can be decomposed into a relatively small number of disjoint subgraphs, most of which behave pseudo-randomly. A more “ergodic” way of viewing the lemma is as an assertion that the indicator function of a graph can be decomposed into a “low-complexity” component and a “pseudo-random” component.

To state the lemma, we need some notation.

Definition 10.41 ($\epsilon$-regularity) Let $G(V, E)$ be a graph. If $X, Y$ are disjoint nonempty subsets of $V$, we define the edge density $d(X, Y)$ between $X$ and $Y$ to be the quantity

$$d(X, Y) := \frac{\sum_{x \in X, y \in Y} \mathbb{1}(xy \in E)}{|X| \cdot |Y|}.$$

If $\epsilon > 0$, we say that the pair $(X, Y)$ is $\epsilon$-regular if we have

$$|d(X', Y') - d(X, Y)| \leq \epsilon$$

whenever $X' \subseteq X$, $Y' \subseteq Y$ are such that $|X'| \geq \epsilon |X|$ and $|Y'| \geq \epsilon |Y|$.

A partition $V = V_1 \cup V_2 \cup \cdots \cup V_k$ is near-uniform if $-1 \leq |V_i| - |V_j| \leq 1$. Szemerédi’s Regularity Lemma asserts that given a positive constant $\epsilon$ and a graph $G$, one can find a near-uniform partition of $V$ in not too many parts so that most of the pairs $(V_i, V_j)$ are $\epsilon$-regular.

Lemma 10.42 (Regularity Lemma) Let and $G = (V, E)$ a graph. If $|V| \to \infty$ then there exists a near-uniform partition $O_{\epsilon, n}(1)$ such that all but at most $\epsilon^k$ of

Remark 10.43 The Regularity Lemma is regular, only that $1 - \epsilon$ of the pairs are one cannot expect regularity of all the $p_i$.

Remark 10.44 The theorem requires $|V|$ to put it another way, one needs $\epsilon = \alpha$. Lemma allows us to have $\epsilon = O_{\alpha, n}(1)$ exponential $n \to \epsilon \mapsto n$, defined recursively. Quite amazingly, Gowers showed, for any sufficiently large $|V|$, the $\epsilon$-regular partition of $\epsilon$ larger than $\epsilon$.

The proof of the regularity lemma can be found in Section 11.6. We shall give a number similar to that of the previous set-theoretic perspective on the lemma, and $\epsilon$.

The survey paper [208] contains a wider lemma. In this section, we restrict to combinatorics, and in particular to Roth’s.

To prove Roth’s theorem via the regularity lemma, some graph-theoretic results. Let $G = G(V, E)$ forms a matching if $e_1, \ldots, e_k$ are mutual, the subgraph spanned by its endpoints does already in the matching.

Proposition 10.45 [104] Let $G = G(V, E)$ of $|V|$ induced matchings. Then $|E| = \alpha |V|$.

Proof The strategy will be to apply the regu- lative fact that a dense $\epsilon$-regular graph cannot

Assume that the proposition failed. Then

arbitrarily large graphs $G(V, E)$ with $|E| \leq |V|$ was the union of $|V|$ induced matchings

Fix one of these large graphs. Applying the we obtain a partition $V = V_1 \cup \cdots \cup V_k$ with

$O(1)$ for all $i, j$, and such that all but at most regular.
regularity lemma

The Szemerédi regularity lemma (Theorem 10.1), Szemerédi's introduction, the Szemerédi regularity lemma. This main tools in discrete mathematics. It asserts graph can be decomposed into a relatively most of which behave pseudo-randomly. A lemma is as an assertion that the indicator into a "low-complexity" component and a

otation.

Lemma 10.42 (Regularity Lemma) Let $\epsilon$ be a positive constant, $m \geq 1$ an integer, and $G = G(V, E)$ a graph. If $|V|$ is sufficiently large depending on $\epsilon$ and $m$, there exists a near-uniform partition $V = V_1 \cup \cdots \cup V_k$ for some $m \leq k \leq O_{m, \epsilon}(1)$, such that all but at most $\epsilon k^2$ of the pairs $(V_i, V_j)$ are $\epsilon$-regular.

Remark 10.43 The Regularity Lemma does not assert that all pairs $(V_i, V_j)$ are regular, only that $(1 - \epsilon)$ of the pairs are. In fact, there are examples showing that one cannot expect regularity of all the pairs (Exercise 10.6.5).

Remark 10.44 The theorem requires $|V|$ to be large depending on $\epsilon$ and $m$, or to put it another way, one needs $\epsilon = o_{\log(1/\epsilon)}(1)$. The proof of the Regularity Lemma allows us to have $\epsilon = O_{m, \epsilon}(\frac{1}{\log \log |V|^2})$, where $\log i$ is the inverse to the tower exponential $n \mapsto e \uparrow \uparrow n$, defined recursively by $e \uparrow 1 = e$ and $e \uparrow (n+1) := e^{e \uparrow n}$. Quite surprisingly, Gowers shown that this bound is essentially tight, namely, for any sufficiently large $|V|$, there are graphs where one cannot find an $\epsilon$-regular partition with $\epsilon$ larger than $\epsilon = \frac{1}{\log \log |V|^2}$.

The proof of the regularity lemma can be found in various textbooks on graph theory; in Section 10.6 we shall give a proof of this lemma using "ergodic" techniques similar to that of the previous section. See also [339] for an information-theoretic perspective on the lemma, and [239] for an analytic perspective.

The survey paper [208] contains a wide range of applications of the regularity lemma. In this section, we restrict our self to a few applications in additive combinatorics, and in particular to Roth's theorem.

To prove Roth's theorem via the regularity lemma, it is convenient to first prove some graph-theoretic results. Let $G = G(V, E)$ be a graph. A set $\{e_1, \ldots, e_k\}$ in $E$ forms a matching if $e_1, \ldots, e_k$ are mutually disjoint. A matching is induced if the subgraph spanned by its endpoint does not contain any edge other than those already in the matching.

Proposition 10.45 [339] Let $G = G(V, E)$ be a graph whose edge set is the union of $|V|$ induced matchings. Then $|E| = o_{\sqrt{\log |V|^2}}$.

Proof The strategy will be to apply the regularity lemma, combined with the intuitive fact that a dense $\epsilon$-regular graph cannot support any large induced matchings.

Assume that the proposition failed. Then one could find an integer $m \geq 1$ and arbitrarily large graphs $G(V, E)$ with $|E| \geq \frac{2}{m^2}|V|^2$ (say) such that each of the graphs $G$ was the union of $|V|$ induced matchings.

Fix one of these large graphs. Applying the regularity lemma (with $\epsilon := 1/m$) we obtain a partition $V = V_1 \cup \cdots \cup V_k$ with $m \leq k \leq O_{m, \epsilon}(1)$ with $|V_i| = \frac{1}{2}|V|$ or $O(1)$ for all $i$, such that all but at most $\frac{1}{2}k^2$ of the pairs $(V_i, V_j)$ are $\epsilon$-regular.
Call an edge $e$ of $G$ bad if one of the following three events occurs:

- $e$ is contained in one of the $V_i$;
- $e$ connects $V_i$ to $V_j$, where $d(V_i, V_j) \leq \frac{1}{m}$;
- $e$ connects $V_i$ to $V_j$, where $(V_i, V_j)$ is not $\frac{1}{m}$-regular.

One can easily verify that the total number of bad edges is at most

$$(1 + o_{V \to \infty}(1)) \left( \frac{|V|}{k} \left( \frac{|E|}{k} + O(1) \right) + \frac{1}{m} \left( \frac{1}{k^2} + \frac{1}{m} \frac{|V|^2}{k^2} \right) \right) \leq \frac{3}{m} |V|^3,$$

if $V$ is large enough depending on $m$. Thus, if we let $E' \subseteq E$ be the edges of $E$ that are not bad, we still have $|E'| \geq \frac{1}{m} |V|^3$. By the pigeonhole principle, we can thus find an induced matching $F$ of $G$ which contains at least $\frac{1}{m} |V|^3$ edges from $E'$.

Call a set $V_i$ poor if it contains at most $\frac{1}{m} |V|$ vertices from $F$. If we delete all the poor sets $V_i$ (and their associated edges) from $F$, we will have deleted at most $\frac{1}{m} |V|$ edges in all. Thus the remaining matching $F'$ will still contain an edge from $E'$. By definition, this edge connects two distinct sets $V_i, V_j$ which are not poor, which have edge density at least $\frac{1}{m}$, and is $\frac{1}{m}$-regular. If we let $V_{i,F}$ and $V_{j,F}$ be the vertices from $F$ in $V_i, V_j$ respectively, we thus have

$$d(V_{i,F}, V_{j,F}) \geq d(V_i, V_j) - \frac{1}{m} \geq \frac{1}{m}.$$

On the one hand, since $F$ is an induced matching, the number of edges in $V_{i,F}$ and $V_{j,F}$ cannot exceed $|V_{i,F}|$, so the edge density cannot exceed $1/|V_{i,F}|$. We conclude that

$$|V_{i,F}| \leq m.$$

On the other hand, we have $|V_{i,F}| \geq \frac{1}{m} |V_i|$ (since $V_i$ is not poor) and $|V_j| = \frac{1}{m} |V| + O(1)$. We conclude that $|V| = O_m(1) = O_m(1)$, contradicting the hypothesis that $V$ could be arbitrarily large depending on $m$. The claim follows.

There are several equivalent formulations of the above theorem; see the exercises. A slightly stronger version of the theorem is as follows.

**Lemma 10.46 (Triangle removal lemma)** [304] Let $G = G(V, E)$ be a graph which contains at most $\delta |V|^3$ triangles. Then it is possible to remove $o_{\delta \to \infty}(|V|^2)$ edges from $G$ to obtain a graph which is triangle-free (it contains no triangles whatsoever).

Lemma 10.46 can be proven by the same method used to prove Proposition 10.45 and is left as an exercise. In fact one can easily use Lemma 10.46 to deduce Proposition 10.45.

Now we use Proposition 10.45 to give rem. 10.8.

**Proof** Fix a finite additive group $Z$ of contains no arithmetic progressions. It suffices to define a bipartite graph $G$ as follows.

Define a bipartite graph $G$ as follows. We draw an edge between $(a + r, 1)$ with $(a + 2x)$, for each $a \in Z$. All the edges are a contradiction of the form $1 + 2x$, which would give $2x = r \in Z$. If $G$ contains an arithmetic progression of length three, we have a contradiction, and hence we have at most $o_{Z \to \infty}(1)$ in $G$ is clearly $|A||Z|$, the claim follows.

In fact, the above methods yield the following.

**Proposition 10.47** [3] Let $Z$ be a finite abelian group such that $A$ contains no right-angled triangles $a, b, r \in Z$ and $r \neq 0$. Then $|A| = o_{Z \to \infty}(1)$.

We leave the proof of Proposition 10.47 to the exercises.

It is of interest to obtain more quantifiable results. By using an explicit quantity, we can sharpen the $o_{|V| \to \infty}(|V|)$ bound to $O(|V|^2/\log_2 |V|)$, and similarly for Lemma 10.47. Thus the quantitative bounds achieved by the Fourier method, (which are slightly easier to extend to the case of general analytic results) are significant.

**Question 10.48** [139] Prove Proposition 10.45, the Regularity Lemma, find a better constant.

In the case of Proposition 10.47, there has been significant progress [314]. In particular, the best known due to Shkredov [314].

**Exercises**

10.6.1 [304] Show that Proposition 10.45 is equivalent: Let $G(V, E)$ be a graph such that $E$ contains exactly one triangle. Then $|E| = o_{V \to \infty}(1)$.
10.6 The Szemerédi regularity lemma

Now we use Proposition 10.45 to give another proof of Roth's theorem. Theorem 10.8.

Proof Fix a finite additive group $Z$ of odd order, and a subset $A$ of $Z$ which contains no arithmetic progressions. It suffices to show that $|A| = o_{\log |Z|}(1)$.

We define a bipartite graph $G$ as follows. The color classes are the sets $Z \times \{1\}$ and $Z \times \{2\}$. We draw an edge between $(a + r, 1)$ and $(a + 2r, 2)$ for every $a \in Z$ and $r \in A$. For each $a \in Z$, the edges between $(a + r, 1)$, $(a + 2r, 2)$ for $r \in A$ form a matching. We claim that this matching is induced. For, if there was another edge connecting $(a + r, 1)$ with $(a + 2s, 2)$ for some distinct $r, s \in A$, then by construction we would have $2s - r \in A$. But then $r, s, 2s - r$ would be a proper progression of length three in $A$, a contradiction. Thus $G$ is the union of $|Z|$ induced matchings, and hence has at most $o_{\log |Z|}(|Z|^2)$ edges. Since the number of edges in $G$ is clearly $|A||Z|$, the claim follows.

In fact, the above methods yield the following stronger form of Roth's theorem.

Proposition 10.47 [3] Let $Z$ be a finite additive group, and let $A \subseteq Z \times Z$ be such that $A$ contains no "right-angled triangles" $(a, b), (a, b + r), (a + r, b)$ with $a, b, r \in Z$ and $r \neq 0$. Then $|A| = o_{\log |Z|}(|Z|^2)$.

We leave the proof of Proposition 10.47 (and its connection to Roth's theorem) to the exercises.

It is of interest to obtain more quantitative bounds for the $o(|Z|)$ term in the above results. By using an explicitly quantitative formulation of the regularity lemma, one can sharpen the $o_{\log |Z|}(|Z|^2)$ expression in Proposition 10.45 to $O(|Z|^2/\log^4 |Z|^2)$, and similarly for Lemma 10.46, Roth's theorem and Proposition 10.47. The quantitative bounds achieved by this method compare poorly to that achieved by the Fourier method. However, the graph-theoretical method is slightly easier to extend to the case of general $k$; see the next chapter. Given that the bounds of Roth's theorem are significantly better than what is achieved by the regularity lemma, one is then naturally led to ask the following question:

Question 10.48 [139] Prove Proposition 10.45 (or Lemma 10.46) without using the Regularity Lemma. Find a better quantitative bound.

In the case of Proposition 10.47, there has been some recent progress on this question [381], [314]. In particular, the best known bound here is $A = O_{k, \log k, \log |Z|}$ due to Shkredov [314].

Exercises

10.6.1 [314] Show that Proposition 10.45 is equivalent to the following statement: Let $G(V, E)$ be a graph such that each edge is contained in at most one triangle. Then $|E| = o_{\log |V|^2}(1)$. 
10.6.2 (6, 3)-theorem [304] Show that Proposition 10.45 is equivalent to the following statement: let $G = G(V, E)$ be a 3-uniform hypergraph (thus each "edge" in $E$ is a collection $\{x, y, z\}$ of three vertices in $V$) such that there is no set of six vertices in $V$ which contain three or more edges in $E$. Then $|E| = o_{\epsilon \to 0}(|V|^3)$. 

10.6.3 [304] Show that Lemma 10.46 implies Proposition 10.45. (Hint: first reduce to the case of a bipartite graph which is the union of induced matchings. Add $|E|$ additional vertices to the graph, one for each induced matching, and connect each new vertex to all the vertices in an induced matching. This creates a tripartite graph with rather few triangles, but which requires many edges to be removed in order to make it triangle-free.)

10.6.4 [304] Use the regularity lemma to prove Lemma 10.46.

10.6.5 [8] Let $V_1 = \{v_1, \ldots, v_n\}$, $V_2 = \{w_1, \ldots, w_n\}$ be disjoint collections of vertices, let $V := V_1 \cup V_2$, and let $G = G(V_1, V_2, E)$ be the bipartite graph formed from all those edges $\{v_i, w_j\}$ for which $i \leq j$. Use this to show that even for very simple graphs one must require an exceptional set of pairs $(V_i, V_j)$ which is not regular.

10.6.6 By modifying the proof of Roth's theorem, use Proposition 10.45 to prove Proposition 10.47.

10.6.7 [323] Use Lemma 10.46 to prove Proposition 10.47, without going through Proposition 10.45. (Hint: consider a graph whose vertices are the vertical lines $\{a, b\} : a = \text{const}$, horizontal lines $\{a, b\} : b = \text{const}$ and diagonal lines $\{a, b\} : a + b = \text{const}$ in $Z^2$, and with two vertices connected by an edge if their associated lines have distinct orientations and intersect in a point in $A$.)

10.6.8 Show that Proposition 10.47 implies Roth's theorem. (Hint: if $A \subseteq Z$, consider sets of the form $\{a, b\} \in Z \times Z : a + 2b \in A\}$.)

10.6.9 [136] Let $V_1$, $V_2$ be disjoint finite sets, and let $f_1 : V_1 \to \{-1, +1\}$ and $f_2 : V_2 \to \{-1, +1\}$ be functions. Let $G = G(V_1, V_2, E)$ be the bipartite graph formed by creating an edge between $x_1 \in V_1$ and $x_2 \in V_2$ if and only if $f_1(x_1) = f_2(x_2)$. Let $X_1 \subseteq V_1$ and $X_2 \subseteq V_2$ be non-empty, and let $0 < \epsilon < 1$. Show that if $(X_1, X_2)$ is $\epsilon$-regular, then

$$|E_{x_1 \in X_1, f_1(x_1)}, E_{x_2 \in x_2, f_2(x_2)}| \geq 1 - O(\epsilon).$$

This shows that any partition of $V_1$ and $V_2$ into regular pairs will have to essentially be a refinement of the sets $\{x_1 \in V_1 : f_1(x_1) = \pm 1\}$ and $\{x_2 \in V_2 : f_2(x_2) = \pm 2\}$. 

10.6.10 [136] Let $V$ be a large finite set. Show that there exist $n$ functions $f_1, \ldots, f_n : V \to \{-1, +1\}$ for some $n = \Omega(\log |V|)$ with the property that for any distinct $x, y \in V$ and $i, j \in [n]$ with $i \neq j$, the values $f_i(x)$ and $f_j(y)$ are independent.

10.7 Szemerédi's theorem for $k = 3$

In this section we give another proof of Roth's theorem, e.g. [143]. This argument gives slightly bet regularity lemma, but still worse than the argument. However, it has the advantage of rather short. A more complex version of this establish Szemerédi's theorem for progression
show that Proposition 10.45 is equivalent to the
$G = G(V, E)$ be a 3-uniform hypergraph (thus
$\{x, y, z\}$ of three vertices in $V$) such that
vertices in $V$ which contain three or more edges in
$V \setminus e$).

10.46 implies Proposition 10.45. (Hint: first
bipartite graph which is the union of induced
bipartite verticals to the graph, one for each
connect each new vertex to all the vertices in an
creates a tripartite graph with rather few trian-
many edges to be removed in order to make it
lemma to prove Lemma 10.46.

1. $V_2 = \{v_1, \ldots, v_{2k}\}$ be disjoint collections
of $\mathcal{V}$, and let $G = G(V_1, V_2, E)$ be the bipartite
pair union of edges $\{v_i, v_j\}$ for which $i \le j$. Use this to
graphs one must require an exceptional
$h$ is not regular.

Roth’s theorem, use Proposition 10.45 to prove
5 to prove Proposition 10.47, without going
(Hint: consider a graph whose vertices are
$\{a, b\} \in Z \times Z : \{a + 2b \in A\}$.)
finite sets, and let $f_i : V_i \to \{-1, +1\}$ be
ctions. Let $G = G(V_1, V_2, E)$ be the bipartite
edge between $x_1 \in V_1$ and $x_2 \in V_2$ if and
with $X_1 \subseteq V_1$ and $X_2 \subseteq V_2$ be non-empty, and let
$(X_1, X_2)$ is $\varepsilon$-regular, then
\[ \left| \sum_{x \in V} \hat{\lambda}(x) f_i(x) \right|^2 \le 1 - \Omega(\varepsilon). \]

Conclude in particular that $\left| \sum_{x \in V} \hat{\lambda}(x) f_i(x) \right| \le 1 - \Omega(\varepsilon)$ for at least
$\Omega(\varepsilon n)$ values of $i$.

10.6.11 [136] Let $V$ be a large finite set, and let $f_1, \ldots, f_n : V \to \{-1, +1\}$
be as in the preceding exercise. Let $W$ be another large finite set, let
$G$ be the graph with vertex set $[1, n] \times V \times W$, with any two distinct
vertices $(i, x, w), (j, y, z)$ being connected by an edge if and only if
$f_i(y) = f_j(x)$. Let $\varepsilon > 0$, and suppose that $[1, n] \times V$ is partitioned into
$[1, n] \times V \times W = V_1 \cup \cdots \cup V_k$ as in the regularity lemma. Suppose
further that for all but $O(\varepsilon k)$ of the sets $V_i$, there exists an $i_0 \in [1, n]$ such that $\lambda_{i_0}((i_0) \times V \times W) \ge (1 - O(\varepsilon))(|V|)$; thus up to errors
of $O(\varepsilon)$, most of the cells $V_i$ of the partition are essentially contained
in one of the $[i] \times V$. Conclude that for all but $O(\varepsilon k)$ of the sets $V_i$, there exists $i_0 \in [1, n]$ and $x \in V$ such that $\lambda_{i_0}((i_0) \times \{x\} \times W) \ge (1 - O(\varepsilon))(|V|)$; thus any regular partition which essentially refines the
partition $[i] \times V \times W$, must automatically essentially refine the finer
partition $[i] \times \{x\} \times W$. (This is a more complicated version of Exercise
10.6.9, and requires use of the previous exercise, with $\hat{\lambda}(x)$ being
equal to the relative density of $V_i \cap (i_0) \times \{x\} \times W$ in $V_i \cap (i_0) \times V \times W$.) An iteration of this fact can be used to establish a lower bound
of lower type for the Szemerédi regularity lemma; see [136].

10.7 Szemerédi’s argument

In this section we give another proof of Roth’s theorem due to Szemerédi (see
[143]). This argument gives slightly better bounds than that obtained from
the regularity lemma, but still worse than that given from the Fourier-analytic
argument. However, it has the advantage of being completely elementary and
rather short. A more complex version of this argument was also used in [343] to
establish Szemerédi’s theorem for progressions of length 4, but the general $k$ case.