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RAMSEY-TYPE THEOREMS WITH FORBIDDEN SUBGRAPHS

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Dedicated to the memory of Paul Erdős

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A graph is called H-free if it contains no induced copy of H. We discuss the following question raised by Erdős and Hajnal. Is it true that for every graph H, there exists an $\varepsilon(H)>0$ such that any H-free graph with n vertices contains either a complete or an empty subgraph of size at least $n^{\varepsilon(H)}$? We answer this question in the affirmative for a special class of graphs, and give an equivalent reformulation for tournaments. In order to prove the equivalence, we establish several Ramsey type results for tournaments.

1. Introduction

Given a graph G with vertex set $V(G)$ and edge set $E(G)$, let $\alpha(G)$ and $\omega(G)$ denote the size of the largest independent set (empty subgraph) and the size of the largest clique (complete subgraph) in G, respectively. A subset $U\subseteq V(G)$ is called *homogeneous*, if it is either an independent set or a clique. Denote by $\hom(G)$ the size of the largest homogeneous set in G, i.e., let

$$\hom(G)=\max\left(\alpha(G),\omega(G)\right).$$

If H is not an induced subgraph of G, then we say that G is an H-*free* graph.

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According to classical Ramsey theory, $\hom(G) \geq \frac{1}{2}\log_2 n$ for every graph G with n vertices [8], and there exists some G with $\hom(G) < 2\log_2 n$ (see [6]). Erdős and Hajnal [7] raised the possibility that "the following could be true."

Conjecture 1. *For every graph H, there exists a positive $\varepsilon = \varepsilon(H)$ such that every H-free graph with n vertices has a homogeneous set whose size is at least n^{ε}.*

Erdős and Hajnal confirmed their conjecture for every graph H which belongs to the class $\mathcal{H}$ defined recursively as follows:

1. K_1, the graph consisting of a single vertex, belongs to $\mathcal{H}$;
2. if H_1 and H_2 are two vertex-disjoint graphs belonging to $\mathcal{H}$, then their disjoint union as well as the graph obtained from this union by connecting every vertex of H_1 to every vertex of H_2 belongs to $\mathcal{H}$.

Gyárfás [9] noticed that it follows from a well known result of Seinsche [14] that Conjecture 1 is also true for all graphs generated by the above rules starting with P_4, a simple path with 4 vertices, and K_1.

Our first theorem extends both of these results. If Conjecture 1 is true for some graph H, then we say that H has the *Erdős–Hajnal property.*

For any graph H with vertex set $V(H) = \{v_1, \ldots, v_k\}$ and for any other graphs, $F_1, \ldots, F_k$, let $H(F_1, \ldots, F_k)$ denote the graph obtained from H by replacing each v_i with a copy of F_i, and joining a vertex of the copy of F_i to a vertex of a copy of F_j, $j \neq i$, if and only if $v_i v_j \in E(H)$. The copies of F_i, $i = 1, \ldots, k$, are supposed to be vertex disjoint.

Theorem 1.1. *If $H, F_1, \ldots, F_k$ have the Erdős–Hajnal property, then so does $H(F_1, \ldots, F_k)$.*

In other words, the Erdős–Hajnal property is preserved by replacement. This enables us to verify that Conjecture 1 is true, e.g., for the graphs depicted in Figure 1, which answers some questions of Gyárfás [9].

No non-perfect graph is known to have the Erdős–Hajnal property. Unfortunately, in this respect Theorem 1.1 cannot offer any help. Indeed, according to a result of Lovász [12], which played a key role in his proof of the Weak Perfect Graph Conjecture [11], perfectness is also preserved by replacement. It is an outstanding open problem to decide whether the smallest non-perfect graph, the cycle of length 5, has the Erdős–Hajnal property. As Lovász pointed out, there is an even simpler unsolved

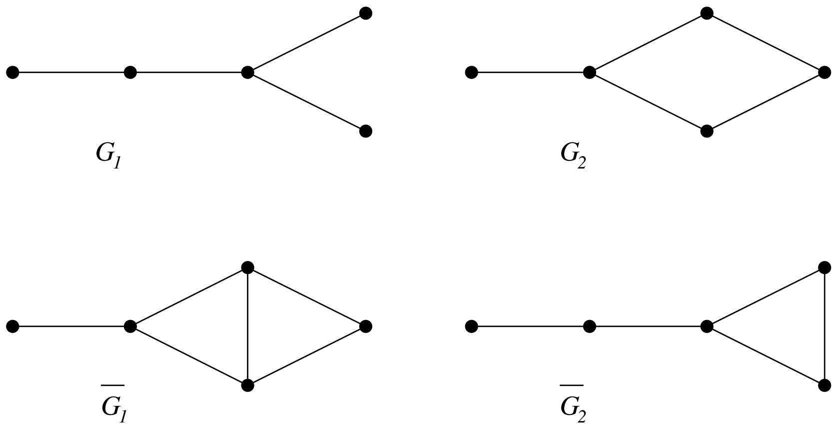

Fig. 1. G_1 and G_2 have the Erdős–Hajnal property

Problem. *Does there exist a positive constant ε so that, for every graph G on n vertices such that neither G nor its complement $\bar{G}$ contains an induced odd cycle whose length is at least 5, we have $\hom(G) \geq n^{\varepsilon}$?*

It is easy to formulate analogous questions for tournaments. A tournament with no directed cycle is called *transitive*. If a tournament has no subtournament isomorphic to T, then it is called *T-free*.

It is well known [5],[15] that every tournament of n vertices contains a transitive subtournament whose size is at least $c \log n$, and that this result is tight apart from the value of the constant.

Conjecture 2. *For every tournament T, there exists a positive $\varepsilon = \varepsilon(T)$ such that every T-free tournament with n vertices has a transitive subtournament whose size is at least n^{ε}.*

Theorem 1.2. *Conjecture 1 and Conjecture 2 are equivalent.*

In order to prove Theorem 1.2, we need a Ramsey-theoretic result for tournaments, which is interesting on its own right. A tournament T with a linear order $<$ on its vertex set is called an *ordered tournament* and is denoted by $(T,<)$. An ordered tournament $(T,<)$ is said to be a *subtournament* of another ordered tournament, $(T',<')$, if there is a function $f: V(T) \to V(T')$ satisfying the conditions

(i) $f(u) <' f(v)$ if and only if $u < v$,

(ii) $\mathbf{f(u)f(v)} \in E(T')$ if and only if $\mathbf{uv} \in E(T)$.

Theorem 1.3. *For any ordered tournament $(T,<)$, there exists a tournament T' such that, for every ordering $<'$ of T', $(T,<)$ is a subtournament of $(T',<')$. Moreover, if T has n vertices, there exists a T' with the required property with $O\left(n^3\log^2 n\right)$ vertices.*

We further show that the $O(n^3\log^2 n)$ estimate is not very far from being tight. In fact, if $(T,<)$ is any tournament on n vertices and T' satisfies the condition above for T, then T' must have at least $\Omega(n^2)$ vertices. The proof of the above theorem is very similar to the proof of the main result of [13], which deals with a similar statement for ordered induced subgraphs. This can be extended to hypergraphs as well.

By choosing a bigger tournament T', one can ensure a single tournament that contains all ordered tournaments on n vertices, in any ordering. Specifically, we prove the following.

Theorem 1.4. *Given an integer N, let n_0 be the largest integer such that*

$$\binom{N}{n_0}2^{-\binom{n_0}{2}} \geq 1,$$

and put $n=n_0-2$. Then, for all sufficiently large N, there exists an ordered tournament T' on N vertices such that in any ordering it contains every ordered tournament on n vertices.

Note that the above estimate for n is clearly tight, up to an *additive* error of 2. A similar statement holds for induced subgraphs, as shown in [3].

The rest of this paper is organized as follows. Theorem 1.1 is proved in Section 2. The proofs of Theorems 1.3 and 1.4 appear in Section 3. Section 4 contains the proof of Theorem 1.2.

2. Graphs with the Erdős–Hajnal property

In this section we prove Theorem 1.1. Obviously, it is sufficient to show the following weaker version of the theorem.

Theorem 2.1. *Let H and F be graphs having the Erdős–Hajnal property, $V(H)=\{v_1,v_2,\dots,v_k\}$. Then the graph $H(F,v_2,\dots,v_k)$, obtained by replacing v_1 with F, also has this property.*

Proof. Let H_0 denote the graph obtained from H by the deletion of v_1. For simplicity, write $H(F)$ for $H(F,v_2,\dots,v_k)$. Let G be an $H(F)$-free graph with n vertices, and assume that $\hom(G) < n^{\varepsilon(H)\delta}$. We would like to get a contradiction, provided that $\delta > 0$ is sufficiently small.

Let $m := \lceil n^{\delta} \rceil > k$. By the definition of $\varepsilon(H)$, any m-element subset of $U \subset V(G)$ must induce at least one subgraph isomorphic to H. Otherwise, we would find a homogeneous subset of $m^{\varepsilon(H)} > \hom(G)$ in the subgraph of G induced by U, which is impossible. Therefore, G has at least $\binom{n}{m}/\binom{n-k}{m-k}$ induced subgraphs isomorphic to H. For each of these subgraphs, fix an isomorphic embedding of H into G.

Since the number of embeddings of H_0 into G is smaller than $n(n-1)\cdots(n-k+2)$, there exists an embedding, which can be extended to an embedding of H in at least

$$M := \frac{\binom{n}{m}}{\binom{n-k}{m-k} n(n-1)\cdots(n-k+2)} \tag{1}$$

different ways. In other words, there are $k-1$ vertices, $v_2', \dots, v_k' \in V(G)$, and there exists an at least M-element subset $W \subset V(G)$ such that, for every $w \in W$,

$$f(v_1) = w, \quad f(v_i) = v_i' \ (i = 2, \dots, k)$$

is an isomorphic embedding of H into G.

Consider now the subgraph $G|_W$ of G induced by W. This graph must be F-free, otherwise G would not be $H(F)$-free. Since F has the Erdős–Hajnal property, we know that

$$\hom(G|_W) \geq |W|^{\varepsilon(F)} \geq M^{\varepsilon(F)}.$$

On the other hand,

$$n^{\varepsilon(H)\delta} > \hom(G) \geq \hom(G|_W).$$

Comparing the last two inequalities and plugging in the value (1) for M, we obtain that

$$\begin{aligned} n^{\delta\varepsilon(H)/\varepsilon(F)} &> \frac{\binom{n}{m}}{\binom{n-k}{m-k} n(n-1)\cdots(n-k+2)} \\ &= \frac{n-k+1}{m(m-1)\cdots(m-k+1)} > n^{1-k\delta}, \end{aligned}$$

which gives the desired contradiction, provided that

$$\delta < \frac{\varepsilon(F)}{\varepsilon(H) + k\varepsilon(F)}. \qquad \blacksquare$$

3. Ramsey-type theorems for tournaments

The proof of Theorem 1.3 uses the probabilistic method. The basic idea is a slightly simplified version of the main argument of Rödl and Winkler in [13]. We need the following lemma.

Lemma 3.1. *Let $t > n > 1$ be two positive integers, and let $S = \{a_1, a_2, \ldots, a_{tn}\}$ be a tn-element set. Let $g : S \to R = \{1, 2, \ldots, t\}$ be a function such that for every $p \in R$, we have $|\{i : g(a_i) = p\}| = n$. Further, let $f : S \to N = \{1, 2, \ldots, n\}$ be a random function obtained by choosing, for each element $a_i \in S$, randomly, independently, and with uniform distribution a value $f(a_i) \in N$. Let E be denote the event that there exist $1 \le i_1 < i_2 < \ldots < i_n \le nt$ such that $g(a_{i_j}) \ne g(a_{i_k})$ for all $1 \le j < k \le n$, and $f(a_{i_j}) = j$ for all $1 \le j \le n$.*

Then the probability that E does not hold is at most

$$\sum_{q=0}^{n-1} \binom{tn}{q} \frac{n^{q(n-1)}(n-1)^{tn-nq}}{n^{tn}} \le \left(\frac{4et}{n}\right)^n e^{-t}.$$

Proof. To estimate the number of functions f for which the event E fails, we argue as follows. Given such an f, let i_1 be the smallest integer (if it exists) such that $f(a_{i_1}) = 1$. Assuming $i_1 < i_2 < \ldots < i_{j-1}$ have already been defined, and assuming that $f(i_s) = s$ for all $s < j$ and that the elements $g(a_{i_s}), s < j$, are pairwise distinct, let i_j be the smallest integer (if it exists) satisfying $i_j > i_{j-1}$, $f(a_{i_j}) = j$ and $g(a_{i_j}) \ne g(a_{i_s})$ for all $s < j$. Note that, since the event E fails, this process must terminate after some $q \le n-1$ elements i_s have been defined. Note also that if k is an index satisfying $i_{s-1} < k < i_s$, and $g(a_k)$ differs from $g(a_{i_j})$ for all $j \le q$ (or even just for all $j < s$), then $f(a_k)$ cannot be equal to s (since otherwise we would have defined $i_s = k$). Since there is a similar restriction for the value of $f(a_k)$ for $k < i_1$ and for $k > i_q$, it follows that once the sequence $i_1 < i_2 \ldots < i_q$ has been defined, the value of $f(a_k)$ can attain at most $n-1$ values for all but at most $tn - nq$ elements a_k. Therefore, the total number of functions f for which the event E fails is at most

$$\sum_{q=0}^{n-1} \binom{tn}{q} n^{q(n-1)}(n-1)^{tn-nq}.$$

Since the total number of possible functions f is n^{tn}, the probability that E does not hold is at most

$$\sum_{q=0}^{n-1} \binom{tn}{q} \frac{n^{q(n-1)}(n-1)^{tn-nq}}{n^{tn}} \le \sum_{q=0}^{n-1} \left(\frac{etn}{q}\right)^q \left(\frac{n^n}{(n-1)^n}\right)^q \frac{1}{n^q} \left(1 - \frac{1}{n}\right)^{tn}$$

$$\leq \left(\frac{et}{n}\left(1+\frac{1}{n-1}\right)^n\right)^n e^{-t} \leq \left(\frac{4et}{n}\right)^n e^{-t}. \qquad \blacksquare$$

Proof of Theorem 1.3. Let $(T,<)$ be an ordered tournament on the set $N=\{1,2,\dots,n\}$ of n vertices, ordered naturally. We may and will assume that n is sufficiently large. Let $c>3$ be an absolute constant, and let t be the smallest integer satisfying $t>cn\log n$ such that $t-1$ is a prime. By the known estimates for the distribution of primes, $t=(1+o(1))cn\log n$. Let P be a projective plane of order $t-1$. Each line of P contains precisely t points, and the number of points in P is $(t-1)^2+t<t^2$. Replace each point $p\in P$ by a set S_p of n points, where all sets S_p are pairwise disjoint. Construct a tournament T' on the set $\cup_{p\in P}S_p$ of less than nt^2 vertices as follows. For every line l in P, let $f_l:\cup_{p\in l}S_p\to N=\{1,2,\dots,n\}$ be a random function, where each image $f_l(u)$ is chosen randomly, uniformly and independently in N, and the functions corresponding to different lines are chosen independently. For $u,v\in\cup_{p\in l}S_p$, where $u\in S_p$, $v\in S_{p'}$ and $p\neq p'$, let $\mathbf{uv}$ be a directed edge if and only if $\mathbf{f_l(u)f_l(v)}$ is a directed edge of T. The edges with two endpoints in the same set S_p are oriented arbitrarily.

To complete the proof, we show that almost surely (that is, with probability tending to 1 as n tends to infinity), T' contains an ordered copy of T in any ordering. Fix an ordering $<'$ of T', and let us estimate the probability that in this ordering $(T',<')$ contains no ordered copy of T. For each line l in the projective plane, the ordering $<'$ induces an ordering of the tn vertices $\cup_{p\in l}S_p$. Let $S=(a_1,a_2,\dots,a_{tn})$ be this induced ordering. Define $g(a_i)=p$ if $a_i\in S_p$. Then, for every $p\in l$, $|\{i:g(a_i)=p\}|=|S_p|=n$. Observe now that, by Lemma 3.1, the probability that $(T,<)$ is not a subtournament of the ordered subgraph of $(T',<')$ consisting of all edges running between distinct groups S_p $(p\in l)$, is at most $\left(\frac{4et}{n}\right)^n e^{-t}$. This follows from the fact that, if the event E in Lemma 3.1 holds for $f=f_l$, then $a_{i_1},\dots,a_{i_n}$ induce a copy of T, as required. Since the events for distinct lines are totally independent, the probability that $(T',<')$ contains no ordered copy of $(T,<)$ is at most

$$\left(\left(\frac{4et}{n}\right)^n e^{-t}\right)^{(t-1)^2+t} = e^{-(1+o(1))c^3n^3\log^3 n}.$$

The total number of orderings of T' is $(n((t-1)^2+t))!\leq e^{(1+o(1))3c^2n^3\log^3 n}$, and as $c>3$, by our choice, the probability that T' fails to contain a copy of T in some ordering is $o(1)$, completing the proof. $\blacksquare$

We next show that the $O(n^3\log^2 n)$ upper bound cannot be replaced by $o(n^2)$. We need the following well-known result.

Lemma 3.2 **([4], [2]).** *The number $|Aut(T)|$ of automorphisms of any tournament T on n vertices does not exceed $3^{(n-1)/2}$.*

Theorem 3.3. *There exists an absolute constant $b \geq \frac{1}{\sqrt{3}e^2}$ with the following property. Let $(T,<)$ be an ordered tournament on n vertices, and suppose T' is another tournament such that for every ordering $<'$ of T', $(T,<)$ is an induced subtournament of T. Then T' has at least bn^2 vertices.*

Proof. Let N be the number of vertices of T'. Then the total number of induced labelled (but not necessarily ordered) copies of T in T' is at most $\binom{N}{n}|Aut(T)|$, which, by Lemma 3.2, does not exceed $\left(\frac{eN}{n}\right)^n 3^{n/2}$. It follows that the probability that for a random ordering $<'$ of T', at least one of these copies is ordered, is at most

$$\left(\frac{eN}{n}\right)^n 3^{n/2}\frac{1}{n!} \leq \left(\frac{\sqrt{3}e^2N}{n^2}\right)^n.$$

If $N < n^2/(\sqrt{3}e^2)$, this number is less than 1, implying that there is an ordering $<'$ with no ordered copy of $(T,<)$. Thus, we have $N \geq n^2/(\sqrt{3}e^2)$, completing the proof. ∎

The discussion for tournaments can be easily adapted to induced subgraphs of graphs. A simple undirected graph H with a linear order $<$ on its vertex set is called an *ordered graph* and is denoted by $(H,<)$. An ordered graph $(H,<)$ is said to be an *induced subgraph* of another one, $(H',<')$, if there is a function $f: V(H) \to V(H')$ such that, for any $u,v \in V(H)$,

(i) $f(u) <' f(v)$ if and only if $u < v$,

(ii) $f(u)f(v) \in E(H')$ if and only if $uv \in E(H)$.

The proof of Theorem 1.3 can be easily modified to deal with ordered graphs, giving the following result of Rödl and Winkler.

Theorem 3.4 **([13]).** *For any ordered graph $(H,<)$, there exists a graph H' such that, for every ordering $<'$ of H', $(H,<)$ is an induced subgraph of $(H',<')$. Moreover, if H has n vertices, there exists an H' with the required property with $O\left(n^3 \log^2 n\right)$ vertices.*

Note that there is no nontrivial analogue of Theorem 3.3, since the number of automorphisms of an undirected graph on n vertices can be as large as $n!$. In fact, if $(H,<)$ is an ordered complete graph on n vertices, then the graph $H' = H$ has only n vertices and contains an induced ordered copy of $(H,<)$ in any ordering.

Combining the above arguments with some known results about packings, we can extend the last result to induced hypergraphs as well. Moreover, the estimate for hypergraphs with no edge of size less than 3 is slightly better than the corresponding result for graphs.

A hypergraph H with a linear order $<$ on its vertex set is called an *ordered hypergraph* and is denoted by $(H,<)$. An ordered hypergraph $(H,<)$ is said to be an *induced subhypergraph* of another one, $(H',<')$, if there is a function $f:V(H)\to V(H')$ such that, for any $u,v\in V(H)$, $f(u)<'f(v)$ if and only if $u<v$, and a set of vertices forms an edge iff its image under f forms an edge.

Theorem 3.5. *For any ordered hypergraph $(H,<)$ in which each edge contains at least 3 vertices, there exists a hypergraph H' such that, for every ordering $<'$ of H', $(H,<)$ is an induced subhypergraph of $(H',<')$. Moreover, if H has n vertices, there exists an H' with the required property with $O(n^3)$ vertices.*

Proof. Let $(H,<)$ be an ordered hypergraph on the set $N=\{1,2,\dots,n\}$ of n vertices, ordered naturally, where each edge of H is of size at least 3. Let c be an absolute constant such that $4ece^{-c}<1/2$ ($c=5$, for example, will do). Let t be the smallest prime satisfying $t>cn$ (then $t=(1+o(1))cn$.) As described in [10], there is a simple, explicit construction of a family L of t^3 subsets of a set P of size t^2 such that each member of l is of cardinality t and the intersection of no two members of L is of size more than 2. Replace each element $p\in P$ by a set S_p of n points, where all sets S_p are pairwise disjoint. Construct a hypergraph H' on the set $\cup_{p\in P}S_p$ of nt^2 vertices as follows. For every $l\in L$, let $f_l:\cup_{p\in l}S_p\to N=\{1,2,\dots,n\}$ be a random function, where each image $f_l(u)$ is chosen, randomly, uniformly and independently in N, and the functions corresponding to different members $l\in L$ are chosen independently. If $u_1,\dots,u_r$ are vertices in $\cup_{p\in l}S_p$, then $\{u_1,u_2,\dots,u_r\}$ is an edge of H' iff the vertices u_i belong to pairwise distinct sets S_p, and $\{f_l(u_1),f_l(u_2),\dots,f_l(u_r)\}$ is an edge of H. Note that, since the intersection of any two distinct members of L is of size at most 2, and H has no edges with fewer than 3 vertices, none of the edges defined above can lie in the union $\cup_{p\in l'}S_p$, for any $l'\in L, l'\neq l$.

To complete the proof, we show that almost surely H' contains an ordered induced copy of H in any ordering. Fix an ordering $<'$ of H', and let us estimate the probability that in this ordering $(H',<')$ contains no ordered induced copy of H. For each $l\in L$, the ordering $<'$ induces an ordering of the nt vertices $\cup_{p\in l}S_p$. Let $S=(a_1,a_2,\dots,a_{tn})$ be this induced ordering. Define $g(a_i)=p$ if $a_i\in S_p$. Then, for every $p\in l$, $|\{i:g(a_i)=p\}|=|S_p|=n$.

Observe now that, by Lemma 3.1, the probability that $(H,<)$ is not an induced subhypergraph of the induced ordered subhypergraph of $(H',<')$ on S, is at most $\left(\frac{4et}{n}\right)^n e^{-t} \le 2^{-n}$. This is true, because if the event E in Lemma 3.1 holds for $f = f_l$, then the vertices $a_{i_1},\dots,a_{i_n}$ induce a copy of T, as required. Since the events for distinct sets $l \in L$ are totally independent, the probability that $(H',<')$ contains no ordered copy of $(H,<)$ is at most

$$(2^{-n})^{t^3} = 2^{-(1+o(1))c^3n^4}.$$

The total number of orderings of H' is $(nt^2)! \le e^{(1+o(1))3c^2n^3\log n}$, and thus the probability that H' fails to contain a copy of H in some ordering is $o(1)$, completing the proof. ∎

It is worth noting that the argument in the proof of Theorem 3.3 also works for hypergraphs whose group of automorphisms is not too large. In particular, if the hypergraph H in the statement of the last theorem has no nontrivial automorphisms, then the number of vertices of any hypergraph H' satisfying the assertion of the theorem must be at least $\Omega(n^2)$.

Returning to tournaments, we now describe a proof of Theorem 1.4, using Talagrand's Inequality [16]. An alternative proof can be given using the methods of [3].

Proof of Theorem 1.4. Let N, n_0 and n be as in the statement of the theorem, and let T' be a random tournament on the vertices $1,2,\dots,N$, obtained by choosing, for each pair of vertices i,j of T', randomly, uniformly, and independently, either the edge **ij** or the edge **ji**. Whenever it is needed, we assume that N is sufficiently large. To complete the proof, we show that almost surely in every ordering, T' contains an ordered copy of every tournament on n vertices. To this end, fix an ordering $<'$ of T', and fix an ordered tournament T on n vertices. We use Talagrand's Inequality (see, e.g., [1], Chapter 7) to estimate the probability that in this ordering T' contains no ordered copy of T. The computation here is very similar to the one estimating the probability that the clique number of the random graph $G(n,1/2)$ is less than its expected value by at least 2.

For each set K of n vertices of T', let B_K be the event that the induced subgraph of $(T',<')$ on K is an ordered copy of $(T,<)$. Then the probability $\Pr(B_K)$ of each event B_K is precisely $2^{-\binom{n}{2}}$. Define $\mu = \binom{N}{n}2^{-\binom{n}{2}}$, and note that this is the expected number of ordered copies of $(T,<)$ in $(T',<')$. A simple computation shows that the number n_0 defined in the statement of the theorem satisfies $n_0 = (1+o(1))2\log_2 N$, implying that for the function

$f(m)=\binom{N}{m}2^{-\binom{m}{2}}$ and for every m close to n_0, we have $f(m+1)/f(m)=N^{-1+o(1)}$. Therefore, $\mu \geq N^{2-o(1)}$.

For two subsets K and K', each containing n vertices of T', let $K \sim K'$ stand for the fact that $2 \leq |K \cap K'| \leq n-1$. Define, further, $\Delta = \sum_{K \sim K'} \Pr(B_K \wedge B_{K'})$, where the sum ranges over all ordered pairs (K,K') with $K \sim K'$. Therefore, $\frac{1}{2}\Delta$ is the expected number of pairs of ordered copies of T that share an edge.

The technical part of the proof is a careful estimate of the quantity Δ/μ^2. Observe that if $|K \cap K'| = i$ (≥ 2) then

$$\Pr(B_K \wedge B_{K'}) \leq 2^{-2\binom{n}{2}+\binom{i}{2}}.$$

In fact, $\Pr(B_K \wedge B_{K'})$ is equal either to zero, or to the right-hand side of the above expression. Thus, it follows that $\Delta \leq \sum_{i=2}^{n-1} \Delta_i$, where

$$\Delta_i = \binom{N}{n}\binom{n}{i}\binom{N-n}{n-i}2^{-2\binom{n}{2}+\binom{i}{2}}.$$

Therefore,

$$\frac{\Delta_i}{\mu^2} = \frac{\binom{N}{n}\binom{n}{i}\binom{N-n}{n-i}2^{-2\binom{n}{2}+\binom{i}{2}}}{\binom{N}{n}^2 2^{-2\binom{n}{2}}}$$
$$= \frac{\binom{n}{i}\binom{N-n}{n-i}2^{\binom{i}{2}}}{\binom{N}{n}} \leq \left(\frac{n}{N}\right)^i \binom{n}{i} 2^{\binom{i}{2}} \leq \left(\frac{n^2}{N}2^{(i-1)/2}\right)^i.$$

It follows that

$$\frac{\Delta_2}{\mu^2} \leq 2\frac{n^4}{N^2}, \tag{1}$$

and that for each i satisfying, say, $3 \leq i \leq 100$, we have

$$\frac{\Delta_i}{\mu^2} = O\left(\frac{n^6}{N^3}\right). \tag{2}$$

Furthermore, for $100 < i \leq 1.9 \log_2 N$, we have

$$\frac{\Delta_i}{\mu^2} \leq \left(\frac{n^2}{N^{0.05}}\right)^i < \frac{1}{N^5}. \tag{3}$$

For every i in the range $1.9\log_2 N \le i \le n-1$, put $i=n-j$, and note that $1 \le j \le (0.1+o(1))\log_2 N$ and

$$\frac{\Delta_i}{\mu^2} \le \frac{1}{N^{2-o(1)}}\frac{\Delta_i}{\mu} = \frac{1}{N^{2-o(1)}}\frac{\binom{N}{n}\binom{n}{j}\binom{N-n}{j}2^{-\binom{n}{2}-j(n-j)-\binom{j}{2}}}{\binom{N}{n}2^{-\binom{n}{2}}}$$
$$\le \frac{1}{N^{2-o(1)}}\binom{n}{j}\binom{N-n}{j}2^{-j(n-j)} \le \frac{1}{N^{2-o(1)}}\left(nN2^{-(n-j)}\right)^j$$
$$\le \frac{1}{N^{2-o(1)}}\left(\frac{n}{N^{0.9-o(1)}}\right)^j \le \frac{1}{N^{2.9-o(1)}}.$$

Combining the last inequality with inequalities (1), (2) and (3), we conclude that $\frac{\Delta}{\mu^2} \le (2+o(1))\frac{n^4}{N^2}$.

Let $X = h(T')$ denote the maximum number of pairwise edge-disjoint ordered copies of T in $(T', <')$. We claim that the expected value of $X = h(T')$ satisfies

$$E(X) \ge \left(\frac{1}{4}+o(1)\right)\frac{N^2}{n^4}. \tag{4}$$

To see this, define $p = \frac{N^2}{2n^4\mu}$, and note that, by a simple computation, we have $p<1$. Let $\mathcal{S}$ be a random collection of ordered copies of T in T' obtained by choosing each ordered copy of T in T' to be a member of $\mathcal{S}$, randomly and independently, with probability p. The expected number of copies of T in $\mathcal{S}$ is $p\mu$, and the expected number of pairs of members of $\mathcal{S}$ that share an edge is $\frac{1}{2}\Delta p^2$. By omitting an arbitrarily chosen member of each such pair, we obtain a collection of pairwise edge-disjoint copies of T whose expected number is at least $p\mu - \frac{1}{2}\Delta p^2$. Thus

$$E(X) \ge p\mu - \frac{p^2\Delta}{2} \ge \frac{N^2}{2n^4} - \frac{N^4}{8n^8}(2+o(1))\frac{n^4}{N^2} = \left(\frac{1}{4}+o(1)\right)\frac{N^2}{n^4},$$

establishing (4).

To apply Talagrand's Inequality (in the form presented, for example, in [1], Chapter 7), note that $h(T')$ is a Lipschitz function, that is $|h(T') - h(T'')| \le 1$ if T', T'' differ in the orientation of at most one edge. Note also that h is f-certifiable for $f(s) = \binom{n}{2}s$. That is, whenever $h(T') \ge s$ there is a set of at most $\binom{n}{2}s$ oriented edges of T' such that for every ordered tournament T'' which agrees with T' on these edges, we have $h(T'') \ge s$.

By Talalgrand's Inequality we conclude that for every b and t

$$Pr[X \le b - t\sqrt{f(b)}]\ Pr[X \ge b] \le e^{-t^2/4}. \tag{5}$$

Let B denote the median of $X = h(T')$. Without trying to optimize the absolute constants, we claim that

$$B \geq \frac{N^2}{16n^4}. \tag{6}$$

Indeed, assume this is false, and apply (5) with $b = \frac{N^2}{8n^4}$ and $t = \frac{N}{4n^3}$. As $f(b) = \binom{n}{2}\frac{N^2}{8n^4} \leq \frac{N^2}{16n^2}$, we obtain that

$$Pr[X \leq \frac{N^2}{8n^4} - \frac{N^2}{16n^4}]\ Pr[X \geq \frac{N^2}{8n^4}] \leq e^{-N^2/(64n^6)}.$$

Since, by assumption, $B < \frac{N^2}{16n^4}$, the first term of the left-hand side is at least $1/2$, and we conclude that

$$Pr\left[X \geq \frac{N^2}{8n^4}\right] \leq 2e^{-N^2/(64n^6)}.$$

As $X = h(T') \leq \binom{N}{n}$ for every T', this implies that

$$E(X) \leq \frac{N^2}{8n^4} + \binom{N}{n} 2e^{-N^2/(64n^6)} = \frac{N^2}{8n^4} + o(1),$$

contradicting (4) and hence proving (6).

We can now apply (5) with $b = \frac{N^2}{16n^4}$ and $t = \frac{N}{4n^3}$ to obtain that

$$Pr[X = 0]Pr[X \geq b] \leq Pr[X \leq b - t\sqrt{f(b)}]Pr[X \geq b] \leq e^{-N^2/(64n^6)}.$$

By (6), we have $Pr[X \geq b] \geq 1/2$, and hence $Pr[X = 0] \leq 2e^{-N^2/(64n^6)}$.

Thus, we have proved that, for every fixed ordering of T' and for every fixed ordered T, the probability that T' contains no ordered copy of T is at most $2e^{-N^2/(64n^6)}$. Since the total number of orderings of T' is less than $N^N = e^{N \log N}$ and the total number of tournaments T on n vertices is $2^{\binom{n}{2}}$ we conclude that the probability that $(T', <')$ fails to contain some tournament of size n in some ordering is at most

$$e^{N \log N} 2^{\binom{n}{2}} 2e^{-N^2/(64n^6)} = o(1).$$

This completes the proof. ∎

The above proof can be modified to deal with graphs in the place of tournaments. We obtain the following, which is a very slight numerical improvement of the main result in [3].

Theorem 3.6 (see also [3]). *Given an integer N, let n_0 be the largest integer such that*

$$\binom{N}{n_0} 2^{-\binom{n_0}{2}} \geq 1,$$

and put $n = n_0 - 2$. Then, for all sufficiently large N, the following holds almost surely. The random graph $G(N,1/2)$ contains, in any ordering, an induced copy of every ordered graph on at most n vertices.

4. Tournaments and H-free graphs

In this section, we prove Theorem 1.2. We need the following wellknown

Lemma 4.1 ([8]). *For any two total orderings of the same (k^2+1)-element set V, there is a $(k+1)$-element subset $U \subseteq V$ such that either the order of any two elements of U is the same, or the order of any two elements is opposite in the two orderings.* ∎

We say that a tournament T has the *Erdős–Hajnal property* if there exists a positive $\varepsilon = \varepsilon(T)$ such that every T-free tournament with n vertices has a transitive subtournament whose size is at least n^{ε}.

To any tournament T and to any ordering $<$ of its vertex set, assign an ordered graph $(H(T),<)$ on the same vertex set, as follows. Join two vertices $u<v$ by an edge of $H(T)$ if and only if the edge connecting them in T was directed towards v. Similarly, assign to any ordered graph $(H,<)$ an ordered tournament $(T(H),<)$ with the same vertex set, by connecting $u<v$ with an edge directed towards v if $uv \in E(H)$ and with an edge directed towards u if $uv \notin E(H)$.

Now we have everything needed for the

Proof of Theorem 1.2. Assume first that Conjecture 1 is true, i.e., every graph has the Erdős–Hajnal property. Let T be a tournament. We want to show that T also has the Erdős–Hajnal property.

Choose an arbitrary ordering $<$ of the vertex set of T. Applying Theorem 3.4 to the ordered graph $(H(T),<)$ associated with T and $<$, we obtain that there exists a graph H' with the property that, for any ordering $<'$ of H', $(H(T),<)$ is an induced subgraph of $(H',<')$. By Conjecture 1, there exists an $\varepsilon(H') > 0$ such that every H'-free graph with n vertices has a homogeneous subset of size at least $n^{\varepsilon(H')}$.

Consider now a T-free tournament T' with n vertices and an ordering $<'$ of $V(T')$. Then the ordered graph $(H(T'),<')$ associated with them cannot contain an induced subgraph isomorphic to H' (because, no matter how it is ordered, this would yield a copy of T in T'). Thus, $H(T')$ must have a homogeneous set of size at least $n^{\varepsilon(H')}$. However, a homogeneous set in $(H(T'),<')$ corresponds to a transitive subtournament in T'.

The proof of the reverse statement is very similar, but the roles of graphs and tournaments have to be switched. Assume that Conjecture 2 is true, and let H be an arbitrary graph. To establish that H has the Erdős–Hajnal property, fix a linear order $<$ on $V(H)$, and denote the associated ordered tournament by $(T(H),<)$.

By Theorem 1.3, there exists a tournament T' with the property that, for any ordering $<'$ of T', $(T(H),<)$ is a subtournament of $(T',<')$. By Conjecture 2, there exists an $\varepsilon(T')>0$ such that every T'-free tournament with n vertices has a transitive subtournament of size at least $n^{\varepsilon(T')}$.

Consider now an H-free graph H' with n vertices and an ordering $<'$ of $V(H')$. Then the ordered tournament $(T(H'),<')$ associated with them cannot contain a subtournament isomorphic to T' (because, no matter how it is ordered, this would yield a copy of H in H'). Thus, $T(H')$ must have a transitive subtournament of size at least $n^{\varepsilon(H')}$. However, by Lemma 4.1, any such subtournament has at least $n^{\varepsilon(H')/2}$ vertices such that, with respect to the ordering $<'$, either all edges connecting them are directed towards their larger endpoints, or all of them are directed towards their smaller endpoints. These vertices induce a complete or an empty subgraph of H', respectively. ∎

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References

[1] N. Alon and J. H. Spencer: *The Probabilistic Method, Second Edition*, Wiley, 2000.

[2] B. Alspach: A combinatorial proof of a conjecture of Goldberg and Moon, *Canad. Math. Bull.,* **11** (1968), 655–661.

[3] G. Brightwell and Y. Kohayakawa: Ramsey properties of orientations of graphs, *Random Structures and Algorithms,* **4** (1993), 413–428.

[4] J. D. Dixon: The maximum order of the group of a tournament, *Canad. Math. Bull.,* **10** (1967), 503–505.

[5] R. C. Entringer, P. Erdős, and C. C. Harner: Some extremal properties concerning transitivity in graphs, *Periodica Mathematica Hungarica,* **3** (1973), 275–279.

[6] P. Erdős: Some remarks on the theory of graphs, *Bulletin of the American Mathematical Society*, **53** (1947), 292–294.

[7] P. Erdős and A. Hajnal: Ramsey-type theorems, *Discrete Mathematics,* **25** (1989), 37–52.

[8] P. Erdős and G. Szekeres: A combinatorial problem in geometry, *Compositio Math.,* **2** (1935), 463–470.

[9] A. Gyárfás: Reflections on a problem of Erdős and Hajnal, in: *The Mathematics of Paul Erdős* (R. L. Graham and J. Nešetřil, eds.), Algorithms and Combinatorics 14, Volume II, Springer-Verlag, Heidelberg, 1997, 93–98.

[10] N. N. Kuzjurin: On the difference between asymptotically good packings and coverings, *European J. Combin.,* **16** (1995), 35–40.

[11] L. Lovász: Normal hypergraphs and the perfect graph conjecture, *Discrete Mathematics,* **2** (1972), 253–267.

[12] L. Lovász: *Perfect Graphs,* in: Selected Topics in Graph Theory, Volume 2 (L. W. Beineke, R. J. Wilson, eds.), Academic Prcess, London–New York, 1983, 55–87.

[13] V. Rödl and P. Winkler: A Ramsey-type theorem for orderings of a graph, *SIAM Journal of Discrete Mathematics,* **2** (1989), 402–406.

[14] D. Seinsche: On a property of the class of n-colorable graphs, *Journal of Combinatorial Theory, Ser. B,* **16** (1974), 191–193.

[15] J. Spencer: Random regular tournaments, *Periodica Mathematica Hungarica,* **5** (1974), 105–120.

[16] M. Talagrand: Concentration of measure and isoperimetric inequalities in product spaces, *Inst. Hautes Études Sci. Publ. Math.,* **81** (1995), 73–205.

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