COMP599 Assignment 1 Solutions

Question 1: To determine if G has a K_5-e minor we first use the Lipton-Tarjan decomposition to split it, in linear time, into 3-connected pieces, each of which is a minor of G. Since K_5-e is 3-connected, as we show below (Lemma E), we can mimic the proof in the book (which concerned K_5 minors) to show that $K_5\text{-e} <_M G$ precisely if one of these pieces has K_5-e as a minor. We test for each such piece H in turn if $K_5\text{-e}<_M H$ in linear time as discussed below. We return the or of the answers.

The following results, proven below, allow us to quickly check if $K_5\text{-e}<_M H$ for 3-connected H.

Lemma A: A 3-connected non-planar graph H has no K_5-e minor precisely if it is $K_{3,3}$.

Lemma B: A 3-connected planar graph H has no K_5-e minor precisely if it is a clique of size at most 4, the complement of C_6, or a wheel (that is an induced cycle of length at least 4 and a vertex v adjacent to all this cycle).

Thus, to test if $K_5\text{-e}<_M H$ for a 3-connected H, we simply need to check if H is one of a clique of size at most 3,$K_{3,3}$, or the complement of C_6, or a wheel. The first three can be done in constant time, the last can be done in linear time as we simply check if there is a vertex of degree |V|-1, and if so, whether deleting such a vertex leaves a cycle(since H is 3-connected actually just need to check if have 1 vertex of degree n-1, and the rest of degree 3) .

Proof of Lemma A:
The graph F obtained from a $K_{3,3}$ by adding an edge between two non adjacent vertices x and y has a K_5-e minor as can be seen by contracting any edge disjoint from xy.
The graph K obtained from a $K_{3,3}$ by subdividing an edge xy and adding an edge from the new vertex to some vertex z other than x or y has F as a minor, and hence K_5-e as a minor, as can be seen by contracting the edge from the new vertex to whichever of x or y is on the same side of the $K_{3,3}$ as z.
The graph J obtained by subdividing two edges of $K_{3,3}$ and adding an edge between the two new vertices has K, and hence K_5–e as a minor, as can be seen by contracting the edge between one of the new vertices and a vertex which is in an endpoint of one but not both of the edges which were subdivided.
If H contains a copy of $K_{3,3}$ but is not $K_{3,3}$ then for any vertex v not in the $K_{3,3}$ there are three paths from v to the $K_{3,3}$ disjoint except at v. Two of these paths have endpoints on the same side of the bipartition of the $K_{3,3}$ and the union of these paths with the $K_{3,3}$ yields a subdivision of F, so G has F and hence K_5-e as a minor.
If H contains a subdivision of $K_{3,3}$ which is not a $K_{3,3}$ then we consider a path P of the subdivision with at least 2 edges between two centers x and y. Since H is 3-connected, there is a path from the interior of P to the subdivision disjoint from x and y. The union of a shortest such path with the subdivision of $K_{3,3}$ yields a subdivision of J or K in H. So by the transitivity of the minor relation $K_5\text{-e} <_M H$.
Now if H is non-planar and does not have K_5-e as a minor then by Kuratowski's theorem it contains a subdivision of $K_{3,3}$ and, by the results of the last two paragraphs, is $K_{3,3}$.

Proof of Lemma B: We first show:

Lemma C: A 3-connected plane embedded graph has no K_5-e minor precisely if it satisfies:

(*) For every pair of vertices x and y, there is a face boundary containing both x and y, or equivalently that there is no cycle C such that x and y lie on different sides of C.
We then show

Lemma D:The only 3-connected planar graphs satisfying (*) are wheels and the complement of C_6.

Proof of Lemma C:
If a plane embedding of a graph F satisfies (*) then adding any edge to F results in a planar graph. This implies that F cannot contain a K_5-e model as adding an edge e between the trees of the model corresponding to the nonadjacent vertices of $K_{3,3}$ would yield a K_5 model in the graph obtained from adding e to F.

On the other hand if F does not satisfy (*), then we let x and y be two vertices which do not lie together on the boundary of a common face in the unique embedding of F. Deleting x and the edges incident to it from this embedding yields an embedding of F-x. Since F-x is 2-connected, the face of this embedding containing x is bounded by a cycle C. By the 3-connectivity of G, there are 3 internally vertex disjoint paths from x to y. To complete the proof we need only show that $K_5 <_M F$ for any F embedded in the plane which contains a cycle C and 3 internally disjoint paths between two vertices x and y on different sides of C.

Consider a minor minimal counterexample F to this statement. F contains no vertex or edge not in one of the cycles or the 3-paths as deleting it would contradict the minimality of F. None of the three paths shares an edge with C as contracting such an edge contradicts minimality. If there is an edge e of one of the paths joining two vertices of C, then one of the two cycles consisting of this edge and a path of C also separates x from y and could be used in place of C. So again, contracting e contradicts minimality. If there is an edge of one of the paths which has one endpoint which is neither on C nor one of x or y, then contracting this edge also contradicts the minimality of F. So each of our paths is xzy for some z on the cycle, and these paths along with the cycle are a subdivision of K_5-e. This contradiction completes the proof of Lemma C. QED.

We note that with lemma C in hand we could test if a 3-connected planar G has a K_5-e as a minor in quadratic time by a brute force listing of all pairs of vertices lying together on a face boundary.

Proof of Lemma D: It is easy to see that a wheel, a clique of size at most 4 and the complement of C_6 satisfy (*), so we need only prove the other direction of (*). We give two different profs.

(1) Consider a 3-connected planar graph F satisfying (*). If F is a clique, it is a clique of size at most 4, since K_5 is not planar, and we are done. Otherwise, F contains two nonadjacent vertices x and y. By the three-connectivity of F, there are three internally vertex disjoint paths from x to y. We choose x,y and the three such paths P_1,P_2,P_3 so as to minimize the total length of these three paths. This implies that each of the paths is induced.
The graph formed by the union of these paths has a unique embedding in which each face is bounded by a cycle formed by the union of two of the paths. We note that the three paths span F, as any point of the plane not on one of these paths is cut off by one of these cycles from the interior vertices of the third path. We note further that if we have edges x_1y_2, x_2y_3, and x_3y_1 such that x_i and y_i are on the

interior of P_i then taking the subpath of P_i from x_i to y_i for each i along with these three edges, yields a cycle separating x and y, which is a contradiction. So WLOG, we can assume that every edge not in one of the paths has an endpoint on P_1. Now, if P_1 has only one interior vertex w , then every edge of F between the paths has one endpoint w. Further, since F is 3-connected, each of its vertices has degree 3 and hence must see w. In this case F is a wheel. Thus, we can assume that P_1 has at least two vertices. We note that since F has minimum degree three, every vertex on the interior of P_3 or P_2 has an edge to the interior of P_1. We consider the shortest subpath Q of the interior of P1 such that one endpoint of Q has a neighbour on the interior of P_2 and the other has an endpoint on the interior of P_3.
Since every interior vertex of Q has degree 3 and hence has a neighbour in the interior of P_2 or P_3, the minimality of Q implies that it has no interior vertices. I.e. it is an edge or a vertex.
Now, we let a be an interior vertex of P_2 seeing one endpoint of Q and b be an interior vertex of P_3 seeing an endpoint of Q. Then a and b are non-adjacent and the cycle formed by P_3 cup P_2 decomposes into two induced paths between them. Q is the interior of a third path between them. So, by our choice of x, y, and P_1,P_2,P_3 we have that Q is all of the interior of P_1. Thus Q is an edge, and the interior of P_1 consists of one vertex w_2 seeing all of P_2 and another w_3 seeing all of P3. By the symmetry between x and y we can assume xw_3 and yw_2 are edges of P_1.
Now, if P_2 and P_3 both have one vertex then H is the complement of C_6, also known as Simba. Otherwise WLOG we can assume that the interior of P_2 has at least 2 vertices. Now, the cycle formed by P_3, the edge yw_2, the edge from w_2 to the neighbour c of x on P_2 and the edge cx separetes those vertices of P_2 which are not on it from w_3. QED.

(2) Consider a 3-connected planar graph F satisfying (*). F clearly contains a cycle we let C be a shortest cycle in F, which must be induced. The 3-connectivity of F implies that if there is no vertex v off of C then C is a triangle and we are done. So we can assume that such a v exists. Now, 3-connectivity implies that there are 3 paths from v to V(C) in F disjoint except at v, which we can clearly chose to be internally disjoint from V(C). These paths and C yields a subdivision of a K_4 in F. This subdivision, and indeed every K_4 subdivision in F must span V as in the unique embedding of a subdivision of a K_4 every point off the embedding is in a face bounded by a cycle containing only 3 centers of the subdivision and on the opposite side of this cycle from the fourth center. It follows that every path of the subdivision corresponding to an edge of the K_4 is induced, as otherwise we could find a shorter non-spanning subdivision.
If the only edges of F are in this subdivision then the 3-connectivity of F implies that F is a K_4 and we are done. Otherwise, adding an edge not on the subdivision to it yields either a subdivision of simba spanning V or a subdivision of a wheel with 5 vertices spanning V.
The graph obtained from Simba by subdividing an edge in a triangle does not satisfy (*) and hence neither does any subdivision of it. So, if F contains a spanning subdivision of Simba, this subdivision consists of two triangles joined by three disjoint induced paths. If the graph has any other edge xy , then this edge creates a cycle which separates two of the six endpoints of these three paths (one endpoint is on the path containing x, the other is on the path containing y) and hence does not satisfy (*). So, by 3-connectivity any graph satisfying (*) which contains a subdivision of Simba is Simba.
Thus, we can assume that F contains a subdivision of a wheel with 5 vertices. We let x be the vertex of degree 4 in the subdivision. The 4 paths of the subdivision corresponding to edges of the wheel which have one endpoint x must be edges or G contains a non-spanning subdivision of K_4, which is impossible. If every edge not in the subdivision has one endpoint x then the 3-connectivity of G implies it is a wheel, and we are done. So there is an edge off the subdivision which does not have x as an endpoint. But adding any such edge creates a non-spanning K_4 subdivision, which is impossible, QED.

Lemma E: If X is a minimal cutset of size at most 2 in a graph G, then G contains a K_5-e minor precisely if for some component U of G-X, the graph G_U obtained from G[U union X] by adding an edge between the two vertices of X if |X| is 2 and this edge does not exist contains a K_5-e minor.

Proof of Lemma E: Each G_U is a minor of G so if one of them has K_5-e as a minor so does G. Conversely if G has a K_5-e model, then there is a component U of G-X completely containing a tree of the model. Since K_5-e is 3-connected, there is a path of G within the model disjoint from X joining any two trees of the model disjoint from X. So U is unique. It is easy to verify that for every tree of the model, the restriction of the vertex set of the tree to U cup X yields a connected subgraph of G_U and that there is an edge of G_U between any two of these subgraphs. The result follows.

Question 2: Choose a subdivision with centers in C and a family F of 2k vertex disjoint paths from S union T to C so as to minimize the union of the edge sets of the subdivision and the paths in F. Clearly this implies that no path of F has an interior vertex in C, so the set U of unused centers which are in no element of F has size k.
For any two centers a and b we let Q(a,b) be the path of the subdivision between these centers which contains no other center. We enumerate U as $u_1,\ldots,u_k$ and for each x in S union T let x' be the vertex of C joined to x by a path in F. We denote this path P(x). For each i between 1 and k, we construct a path R_i from s_i to u_i which consists of the subpath of $P(s_i)$ from s_i to the vertex s_i^* of $Q(u_i,s_{i'})$ which is closest to s_i along $P(s_i)$ and the subpath of $Q(u_i,s_i')$ from s_i* to u_i. We construct a path R_i' from t_i symmetrically.
We claim, and prove below, that for every i between 1 and k and x in S cup T, the only path of F to intersect $Q(u_i,x')$ is P(x). It follows that the concatenation of R_i and the reverse of R_i' is a path P_i from s_i to t_i and that the P_i are disjoint. So, to complete our proof it remains to prove our claim.
To this end, assune we have a u_i and x for which the claim is false. Let v be the vertex of $Q(u_i,x')$ closes to u_i along this path which is also on P(y) for some y. Replacing P(y) by the concatenation of the subpath of P(y) from y to v and the subpath of $Q(u_i,x')$ from v to u_i would contradict our choice of the subdivision and F unless the subpath of P(y) from v to y' is completely contained in the subdivision. But this implies that y' is x', and that the claimed result holds.

Question 3: No solution will be written up. I'm tired. It was easy.