COMP599 Assignment 2 Solutions

1(a) Let \( h(k) \) be \( \max(4, g(k, k) + 1) \) for the function \( g \) of Lemma 4.2. Suppose that the desired result is false and consider a counterexample \((G, H, X, v)\) chosen to minimize \(|V(H)| + |E(H)|\). Let \((Y_1, Y_2, \ldots, Y_p)\) be a partition of \( X \) which is realizable in \( G \) but not \( G-v \). Let \( T_1, \ldots, T_p \) be a realization of this partition in \( G \) chosen so as to minimize the number of vertices in these trees. Clearly \( v \) is in some \( T_i \) wlog \( T_1 \).

If \( w \) is a vertex of \( G-X-v \) in no \( C_i \) and no \( T_i \) then deleting it contradicts the minimality of \( G \). If \( e \) is an edge of \( G \) which is in no \( C_i \) and no \( T_i \) then deleting it contradicts the minimality of \( G \). If \( e \) is an edge which is in some \( C_i \) and does not join two distinct trees of our realization then contracting it contradicts the minimality of \( G \). It follows that the edges of the realization are precisely those edges which are not in some \( C_i \) and the realization is unique. It also follows that the set of cycles is unique. It also follows that every vertex is in some \( T_i \). Finally, if \( e \) is an edge of \( G \) which is in some \( T_i \) and does not either join \( v \) to \( C_i \) or join \( C_i \) to \( C_j \) for distinct \( i \) and \( j \), or join two vertices of \( C_{h(k)} \) then contracting it contradicts the minimality of \( G \). It follows from this that the vertex set of \( G \) consists of the union of the vertices of the cycles and \( v \). It also follows that \( v \) has degree 2.

If \( l \) is a leaf of some \( T_i \) which is not in \( X \) then replacing \( T_i \) by \( T_i-l \) contradicts our choice of a realization. It follows that every vertex \( w \) not in \( X \) is joined to \( X \) by two paths of some \( T_i \) disjoint except at \( w \). In particular, there are two paths \( P_1 \) and \( P_2 \) from \( v \) to \( X \) which intersect only at \( v \).

Note that no edge of \( G \) which is not in \( H \) has an endpoint in \( V(H)-C_{h(k)} \). Thus, each \( C_i \) separates \( v \) from \( X \). It follows that both \( P_1 \) and \( P_2 \) must pass through every \( C_i \) (before leaving the closed disc bounded by \( C_i \)). Thus \( G \) is 2-connected.

Let \( H' \) be the planar graph obtained from \( H \) by contracting the component of \( H-C_{h(k)-1} \) containing \( C_{h(k)} \) into a single vertex \( b \) and deleting \( v \) and adding an edge between its two neighbours on \( C_1 \). We also modify the tree of our realization containing \( v \) by deleting \( v \) and adding an edge between \( v \)'s neighbours. We note that \( b \) is adjacent only to vertices of \( C_{h(k)-1} \) in \( H' \).

Assume that \( H' \) is not 3-connected. Let \( Z \) be an arbitrary minimal cutset of size at most 2 in \( H' \). Thus every vertex of \( Z \) has a neighbour in each component of \( H'-Z \).

If \( b \) is in \( Z \) then \( Z \) intersects \( C_{h(k)-1} \) in at most one vertex so \( C_{h(k)-1}-Z \) is connected and lies in one component of \( H'-Z \). Thus, \( b \) contains a neighbour in the other component of \( H'-Z \) which is disjoint from \( C_{h(k)-1} \), a contradiction.

So \( b \) is not in \( Z \), and \( Z \) is a cutset in \( H \) such that some component \( U \) of \( H-Z \) completely contains \( C_{h(k)} \). Thus, by the definition of \( H \), every other component of \( H-Z \) is a component of \( G-Z \). Let \( K \) be any other component of \( G-Z \) and let \( w \) be a vertex of \( K \). Now \( K \) is disjoint from \( X \), so \( K \) contains no leaf of any \( T_i \). Since \( w \) is not in \( X \), there are two paths of \( T_i \) from \( w \) to \( X \), disjoint except at \( w \), which must pass through \( Z \). Thus there is a path \( Q \) of \( T_i \) whose interior is in \( K \), joining the two vertices of \( Z \). It follows that there is at most one choice for \( K \), and hence \( G-Z \) has two components. Since there are no leaves of any \( T_i \) in \( K \), we see that \( K \) intersects our realization precisely in the interior of \( Q \). This implies that every vertex of \( K \) is on the interior of \( Q \). It also implies that for any other path \( Q' \) joining the vertices of \( Z \) whose interior is in \( Q \), replacing \( T_i \) by \( T_i-Q+Q' \) yields a realization of the partition \((Y_1, \ldots, Y_p)\). Thus, by the
uniqueness of the partition, Q is induced. It follows that if a $C_i$ intersects $K$ then Q is a subpath of $C_i$. But now contracting any edge of Q which is not incident to v contradicts the minimality of G (it is in some $T_i$ does not join disjoint cycles and does not touch v). It follows that $Z$ consists of the neighbours of v, and $V(K)$ is v. But now, $Z$ is not a cutset of $H'$ because v is not in $H'$.

It follows that $H'$ is a a 3-connected graph which has a unique embedding. Thus so does the graph $H^*$ obtained by adding back v in the middle of the edge of $H'$ between its neighbours via a subdivision. We can take obtain this embedding by either contracting from our planar embedding of G or by contracting from the planar embedding of G which shows that v is $h(k)$-protected. Thus, just as in the latter embedding, in the planar embedding of G, $C_1$ to $C_{h(k)-1}$ are nested cycles separating v from $C_{h(k)}$.

Thus, since there are no vertices of G apart from v which do not lie in one of the nested cycle, the boundary of every face incident to a vertex of X intersects $C_{h(k)}$. For each vertex w in $X$, we choose a face whose boundary contains w. Draw a small disc in that face which intersects G only at w. Make these discs small enough that they do not intersect. The desired result now follows from Lemma 4.2.

(b) We consider a minimal counterexample and minimal realization as in 1(a). We again derive that: every edge either is in a $C_i$ and joins distinct trees of the realization or is in a $T_i$ and joins distinct cycles, or joins v to $C_i$. This implies that the realization is unique, the set of protecting cycles is unique and that $V(G)$ is $V(H)$. We also again obtain that every vertex w not in X is joined to X by two paths of some $T_i$ disjoint except at w. Applying this to v, we see that G is 2-connected.

We will show that in fact G is planar. To do so, we show that the Lipton-Tarjan decomposition splits B into triangles and one 3-connected graph which has no $(3,3)$-cut which, by Wagner’s Theorem, completes the proof.

We begin with a detour. We ask if there are three vertex disjoint paths of G from $C_1$ to X. If not there is a cutset Z of size 2 separating $C_1$ from X. Since $C_1$ separates v from X, Z also separates v from X. We let A be the union of those components of G-Z which contain v or a vertex in $C_1$. Since X is disjoint from A, the restriction of our realization to the subgraph of G induced by A cup Z is a path P between the vertices of Z. Since the realization is unique there is only one such path with interior in A. But there are two disjoint paths from $C_1$ to Z which are subpaths of P1 and P2 and hence two paths with endpoints in Z and interior in A (we can go either way around $C_1$), a contradiction. So the desired three paths exist. We note that they actually exist in G-v since $C_1$ separates v from X.

We next ask if there are four vertex disjoint paths of G from $C_1$ to X. If not there is a cutset Z of size 3 separating $C_1$ from X. Since $C_1$ separates v from X, Z also separates v from X. Since both of v's neighbours are on $C_1$ and not in X we have that v is not in Z (as otherwise Z-v would separate $C_1$ from X, contradicting the last paragraph). We let A be the union of those components of G-Z which contain v or a vertex in $C_1$. Since X is disjoint from A, the restriction of our realization to the subgraph of G induced by A cup Z is a forest F whose leaves are in Z. But the fact that there are three paths from X to $C_1$ in G-v imply that there are 3 paths from Z to $C_1$ with their interior in A-v. These along with the cycle $C_1$ show that there is a forest $F'$ in A cup Z-v with the same leaves as $F$ such that replacing $T_i$ with $T_i$-v yields a realization of $(Y_1, ..., Y_p)$ in G-v, a contradiction. Hence the desired four paths do exist.

Now consider some cutset $Z$ of size at most 3 in G. There is a component $K_2$ of G-Z completely containing some $C_i$. $K_2$ intersects all four of the disjoint paths of the last paragraph and completely
contains one of them (which is disjoint from $Z$). Thus it also completely contains $C_i - Z$ for every cycle not intersecting $Z$ in 2 vertices, and intersects every $C_i$. It will be useful to know this in what follows.

Suppose now that $G$ has a cutset $Z$ of size 2. We let $K$ be some component of $G - Z$ other than $K_2$. If $K$ is disjoint from every $C_i$ then the minimality of $G$ implies that it is $v$. If $K$ intersects some $C_i$ then, since it does not contain all of any $C_i$ it must contain a path $Q$ of $C_i$ between the two vertices of $Z$, and no other vertices of any $C_i$. Now every edge of $C_i$ joins distinct trees of our realization, applying this to edges of $C_i$ from $Z$ into $K$, we see that there is a $T_j$ intersecting $K$ which contains at most one vertex of $Z$. Since every vertex $w$ of $T_j - X$ is joined to $X$ by two paths which are disjoint except at $w$, we see that $X$ intersects $K$ and thus $i$ is $h(k)$. We know that $K_2$ contains (i) all of every $C_i$ with $j$ less than $h(k)$, (ii) a vertex of $C_h(k)$ and hence a path of $C_h(k)$ between the vertices of $Z$, and (iii) $v$ (since $Z$ is disjoint from $C_i$). Thus the vertex set of $K$ is precisely the interior of $Q$. We note that, since $K$ intersects $H$ precisely in the interior of $Q$, and there are no edges from $K$ to $H - C_h(k)$ replacing $Q$ by any path between the vertices of $Z$ through $K$ would yield a new set of protecting cycles. Since the set of protecting cycles is unique, $Q$ is the unique such path and it is induced.

We have shown that every 2-cut of $G$ has 2 components one of which is a path, all of whose vertices have degree 2 in $G$. Hence, $G$ is the subdivision of a 3-connected graph $G'$.

Suppose now that $G$ and hence $G'$ is not planar. $G'$ is clearly not $L$, as $G'$ contains too many nested circuits ($L$ does not contain 3 disjoint circuits). Thus, by Wagner’s theorem $G'$ has a $(3,3)$-cut which is also a $(3,3)$ cut of $G$. Let $K_1 = K_2 , K_3$ and $K_i$ be the $i > 2$ components of $G - Z$. Let $i$ be the unique index such that $Z$ contains at least two vertices of $C_i$. We have seen that for $j > 1$ $V(K_j)$ is contained in $V(C_j)$ and that $V(K_1)$ intersects $V(C_i)$. Hence every component of $G - Z$ contains the interior of a path of $C_i$ between two vertices of $Z$. Thus, all three vertices of $Z$ lie on $C_i$ and $G - Z$ has exactly three components, each of which intersects $C_i$ in the interior of a path between two vertices of $Z$. We note further that each component of $G - Z$ corresponds to a component of $G' - Z$ and hence has an edge to each vertex of $Z$. Now, if $i$ is not $h(k)$ then $K_3$ is connected not just in $G$ but also in $H$ because $C_h(k)$ is connected and lies in $K_3$ and all the edges of $G - H$ have both endpoints on $C_h(k)$. Furthermore, $K_1$ and $K_2$ are components of $H - Z$. But now contracting these three components into vertices we see that $H$ has a $K_{3,3}$ minor contradicting its minimality. So $i$ is $h(k)$.

Now, we note that the vertices of $K_1$ and $K_2$ are joined to none of $H - C_h(k)$. Thus, if we let $a$ and $b$ be the vertices of $Z$ which are joined via a path of $C_h(k)$ then replacing the path of $C_h(k)$ which joins them, uses the third vertex $c$ of $Z$ and is contained in $K_1$ union $K_2$ union $c$ by any other such path, would yield a new set of protecting cycles. So no such path can exist. But this is nonsense since all of $a,b$, and $c$ have a neighbour in both $K_1$ and $K_2$, QED.

Question 2: We use the fact discussed in the text that (1) We can read the solution of the instance $(G,X)$ of $|X|$-realizations of the solution to the instance $(G,X')$ of $|X'|$-realizations in $O(f(|X'|))$ time provided $X'$ contains $X$ (A partition of $X$ in $G$ if and only if it can be obtained from a realizable partition of $X'$ by deleting the elements of $X' - X$ in each element of the latter partition), and (2) if $Z$ is a subset of $X$ and $G - Z$ splits into $l$ graphs $U_1 , ... , U_l$ between any two of which there is no edge, then we can read off the solution to the instance $(G,X)$ of $|X|$-realizations in $O(f(|X|,l))$ time from the solution to the realization
instances \((G[Z], Z), (H_1,X_1), \ldots, (H_i,X_i)\) where \(H_i\) is the subgraph of \(G\) induced by the union of \(V(U_i)\) and the vertices of \(Z\) with an edge to \(V(U_i)\) and \(X_i\) is the intersection of \(V(H_i)\) and \(X\) (From a choice of a realizable partition of \(X_i\) in \(H_i\) for each \(i\) we can obtain a partition of \(X\) which is realizable in \(G\) by (i) adding edges so that each element of each of the partitions is a clique, and taking the partition of \(X\) given by the vertex sets of the resultant graphs. Conversely every realizable partition of \(X\) in \(G\) arises in this way. The number of choices we need to consider here is bounded by \(|X|\) and \(k\) and is independent of \(|G|\).

Applying this second fact when \(Z\) is the empty set shows that we can consider each component of \(G\) containing a vertex of \(X\) separately and then combine the solutions, so we can assume \(G\) is connected. We will now use these two facts in a more sophisticated way to show that it is enough to solve \(k\)-realizations for all \(k\) just on two-connected graphs, and in fact just on three connected graphs without \((3,3)\)-cuts in order to do so in general.

To begin, we assume we can solve \(k\)-realizations for all \(k\) on 2-connected graphs and consider some graph \(G\). We consider the block tree of \(G\). If it has only one node then we just apply our algorithm for 2-connected graphs. Otherwise we root it at any cutvertex \(x\). We traverse the block-tree in post-order, and when we come to a node corresponding to a block \(B\) which shares a vertex \(v\) with its parent, we delete \(B-v\) from \(G\), and this node from the tree if all of its children have been deleted and \(B-v\) is disjoint from \(X\). We delete a cutvertex node from the tree if we have deleted all its children. The resultant tree \(T\) has at most \(k\) leaves. We can assume it is non-empty as otherwise \(X\) contains no vertex other than \(x\), in which case the 1-Realization or 0-Realization problem is trivial.

For each cutvertex \(v\) of \(G\) such that the corresponding node \(n_v\) of the block tree was not deleted, we let \(G_v\) be the union of the block nodes in the maximal rooted subtree of \(T\) rooted at \(n_v\). We let \(Z_v\) consist of all of the cutvertices corresponding to the grandchildren of \(v\) in \(T\). We note that \(|Z_v|\) is at most \(k\). We let \(X_v\) be the union of \(v\) and \(\{\text{intersection} V(G_v)\}\). We will traverse \(T\) in postorder computing for each \(n_v\) in turn the solution to the instance \((G_v,X_v)\) of \(|X_v|\)-realizations. We read the solution to this problem off from the solution to the instance \((G_v,X_v\cup Z_v)\) of \(|X_v\cup Z_v|\)-realizations.

Apply Fact 1, we see that we need only solve the subproblem described in the statement of the fact for each graph \(U\) of \(G-Z_v\) such that \(U\) either (i) is \(G[Z]\), (ii) is \(G_{w-B}\) for some grandchild \(n_w\) of \(n_v\) and has edges to \(w\) but no edges to \(Z_v-w\), or (iii) is \(B-Z_v\) for some block which is a child of \(n_v\) in \(T\) and has edges exactly to those vertices of \(Z_v\) in \(B\). In the first case, we are considering a bounded size graph, in the second case we have already solved the subproblem corresponding to \(U\) in our postorder traversal. In the third case, we are considering a 2-connected block of \(G\). Since we consider each block at most once, the fact that we can solve \(k^{'}\)-realizations on 2-connected graphs in linear time for all \(k'<|X|+|Z_v|+1<2k+1\) implies that we can solve it in all graphs in linear time for \(k\).

To prove that it is enough to solve the problem on 3-connected graphs, we apply exactly the same dynamic programming approach to the Lipton-Tarjan decomposition tree of a 2-connected graph. Now however, when in constructing \(T\) we delete a block node \(B\) because it has become a leaf of the tree and for the cutset \((x,y)\) corresponding to the parent of \(B\) we have \(B-x-y\) is disjoint from \(X\), we will have to add the edge \(xy\) to \(G\). Furthermore, the size of the set of vertices corresponding to a cutnode of the graph and its grandchildren is bounded by \(2k+2\) rather than \(k+1\).
In exactly the same way, we can use the \((3,3)\)-block tree to show that it is enough to solve \(k\)-realizations in linear time for all \(k\) in graphs which are planar or \(L\). Again, when deleting a block \(B\) corresponds to a node which has become a leaf and such that for the cut \(Z\) corresponding to its parent in the tree, \(B-Z\) is disjoint from \(X\), we will add edges to make \(Z\) a 3-clique.

Since we can solve \(k\)-realizations in linear time on planar graph and \(|L|\) is 8, the desired result follows.

**Question 3:** We will set \(f(l)\) to be \(2g(2(l \text{ choose } 2))-3, 2(l \text{ choose } 2)) + 6.\) We will use the edges off the \(k\) by \(k\) grid as the edges of a \(K_4\) model and then apply our results on routing in the plane to the unique embedding of the grid (with the unique non-square face as the infinite face), to find the trees of the model. We can assume \(l\) is at least 5 because \(K_4\) is a minor of the 3 by 3 grid.

We consider a minimal counterexample to the problem. This implies that the first row, first column, \(k\)th row, and \(k\)th column of the grid all contain an endpoint of an off-grid edge. By hypothesis they both contain at most one such endpoint. We let \(X^*\) be the set of at most four such endpoints lying in these rows and columns.

We arbitrarily choose a bijection \(f\) from the edges of \(K_4\) to the edges off the grid. For each edge \(uv\) of \(K_4\) we choose one endpoint of \(f(uv)\) to correspond to \(u\), this is \(h(u,uv)\). The other endpoint of \(f(uv)\) corresponds to \(v\) and is \(h(v,uv)\). We make this choice arbitrarily, except that if we have that for some \(u,v,w\) the vertex on the top row of the grid is \(h(u,uv)\) and the vertex on the bottom row is \(h(u,uw)\) then we swap the choice we made for this second edge (leaving all other choices the same). We set \(X\) to be the union of all the endpoints of \(K_4\) and all the non-grid edges. We consider the partition \(\Delta\) of \(X\) into \(l\) sets, indexed by the vertices of \(K_4\). The element of \(\Delta\) corresponding to \(u\) consists of the \(l\) vertices which are \(h(u,e)\) for some edge \(e\) with \(u\) as an endpoint. To prove that \(G\) has a \(K_4\) model it remains to show that \(\Delta\) is realizable in the grid.

To this end for each vertex \(v\) of \(X-X^*\), we cut out a small disc intersecting the drawing of the grid only at \(v\). We also cut out a cuff \(C^*\) whose interior is contained in the infinite face which intersect \(G\) precisely in \(X^*\). So the grid is now embedded in an \((2(l \text{ choose } 2))-3\)-punctured plane \(\Sigma\). This will allow us to use Theorem ** to show that \(\Delta\) is realizable in the grid.

To begin we must show that \(\Delta\) is realizable in \(\Sigma\). I.e. that there is a planar embedding of the realizability graph which respects the clockwise orientation around each cuff vertex. Since every cuff vertex has degree 1 except for that corresponding to the cuff in the infinite face, it is enough to show that the realizability graph is planar, as we can always choose clockwise so that it agrees with the cyclic order of the edges around one vertex. Now the realizability graph is obtained from a forest \(F\) of trees by adding a cycle \(C\) of length four through four leaves in at least two trees and then adding a vertex \(x\) adjacent to the vertices of \(C\). We suppose that the realizability graph is non-planar and consider the smallest subforest \(F'\) of \(F\) such that adding \(C\) and \(x\) to \(F'\) yields a non-planar graph \(F^*\). It will have been formed in the same way. No tree of \(F\) has a leaf off \(C\), as otherwise we could delete it without affecting planarity. So, there are at most four leaves and two trees. If one of the trees has one leaf then it is a vertex, and it is easy to see that \(F^*\) is planar. If both leaves have two trees then they are paths and the
graph will be planar unless both paths join antipodal points of C. But by our clever choice of H, this is impossible.

Thus, to show that Δ is realizable in the grid, we need simply show that the schisms, O-arcs, and I-arcs are long enough. In doing so, we let \( C_1 \) be the unique cycle of \( G \) formed by subpath of rows \( f(I)/2-2 \) and \( k-f(I)/2+2 \) and columns \( f(I)/2-2 \) and \( k-f(I)/2+2 \) and let \( C_2 \) be the unique cycle of \( G \) formed by subpath of rows \( f(I) \) and \( k-f(I) \) and columns \( f(I) \) and \( k-f(I) \). For each vertex \( x \) in \( X-X^* \), we let \( C_x \) be the column of the grid containing \( x \).

Now for every face of the grid of length four, the indices of the vertices on that face differ by at most 1. Thus for any schism, I-arc, or O-arc \( J \) of length at most \( f(I)/2-3 \) we have: (I) if \( J \) does not intersect the infinite face, both the set of row indices and the set of column indices of the vertices of \( J \) form a consecutive sequence, and (II) if \( J \) does intersect the infinite face then every vertex on \( J \) has an index which is at most \( f(I)/2-3 \) or at least \( k-f(I)/2-3 \).

It follows that no linking I-arc can have length less than \( f(I)/2-3 \). Furthermore, every O-arc, schism \( J \) of length less than \( f(I)/2-3 \), or non-planar looping I-arc is either (i) contained in the infinite face of the embedding of \( C_2 \) which is a subembedding of our embedding of the grid, or (ii) is in the non-infinite face of the embedding of \( C_1 \) which is subembedding of our embedding of the grid. In the first case, the intersection of \( \Sigma \) with the non-infinite face of \( C_2 \) lies in one component of \( \Sigma - J \). Thus this component contains all the cuffs except possibly \( C^* \). So \( J \) must be an O-arc which is not a schism, or a near-planar I-arc with its endpoints on \( C^* \). In the second case, \( C_1 \) and \( C^* \) are contained in one component of \( \Sigma - J \) If the cuff containing \( x \) is separated from \( C^* \) by \( J \), then \( J \) must intersect \( C_x \). By the consecutiveness of the indices of the vertices of \( J \), we see that this can occur for at most one cuff and that if it does occur for a cuff then \( J \) cannot have endpoints on any other cuff. So \( J \) must be an O-arc which is not a schism, or a near planar looping I-arc.

So, it remains to show that if for some cuff \( C \), \( J \) is a near-planar looping I-arc with endpoints on \( C \) or an O-arc surrounding \( C \) then \( J \) satisfies the hypothesis of Theorem **.

If \( C \) is not \( C^* \) then if \( J \) surrounds the cuff containing \( X \) or cuts off a cuff component containing \( x \) then it must intersect \( C_x \) and hence contains a vertex and has the required length. If \( C \) is \( C^* \) then \( J \) has the required length unless it has length three or less. Since \( X-X^* \) has at least 16 elements, this implies that there is some \( x \) in \( X-X^* \) such that \( J \) is disjoint from \( C_x \). So, if \( J \) cuts off a vertex \( v \) of \( X \) from the cuffs other than \( C^* \) and thus from \( x \) and \( C_x \), then \( J \) must contain a vertex in the row of the grid which contains \( v \). By the distinctness of the indices, this vertex is not in \( X \), so if \( J \) is a looping I-arc then it is on the interior of \( J \). Thus, by the distinctness of the indices again, \( J \) satisfies the hypotheses of Theorem ** QED,