Solution to Assignment 3

K_5 has a bramble of order 5 where each vertex is an element of the bramble.

The vertices of L can be labelled as v_0 to v_7 so that two vertices are adjacent precisely if their indices differ by 1 or 4 mod 7. Then the set of edges between vertices whose indices differ by 1 along with the edges v_0v_4 and v_1v_5 form a bramble, as can be easily verified. Furthermore, this bramble has order greater than 4 (to hit all the edges whose indices differ by 1 with a set H of 4 vertices we need to take the vertices of odd index or the vertices of even index, but then H fails to intersect one of the other two bramble elements).

Consider next, the planar graph *Pyramid* obtained from a K_6 by removing a perfect matching. If it has a tree decomposition of width less than 4, then by the Helly property for subtrees of a tree, there must be two vertices x and y of Pyramid such that S_x fails to intersect S_y. By the definition of tree decomposition x and y are non-adjacent. Now, every other vertex v of Pyramid is adjacent to both x and y so S_v intersects both S_x and S_y. Thus, S_v contains both endpoints of the unique path from x to y, and it follows that if t is one of these endpoints then $|W_t|$ is 5, a contradiction.

Finally consider the graph *Cherry Blossom* obtained from two cycles C_1 and C_2 of length 5 by adding a matching of size 5 between their vertex sets so as to obtain a planar graph. The edges of C_1, along with the five paths obtained from C_2 by deleting a vertex form a bramble as can be easily verified. This bramble has order 5, as any hitting set for the edges of C_1 contains 3 vertices of C_1 and if H contains only one vertex of C_2 then deleting this vertex from C_2 yields an element of the bramble disjoint from H.

We prove below that letting $\mathscr{F}$ be the set {K_5,L,Pyramid,Cherry Blossom} we have:

Claim 1: If no graph in $\mathscr{F}$ is minor of G, then TW(G)<4.

By Wagner's Theorem K_5 is not a minor of L, Pyramid, or Cherry Blossom. Since L is not planar, L is not a minor of Cherry Blossom. If there were a model of pyramid in Cherry Blossom or L, every vertex image of the model would have to contain two vertices because Pyramid has minimum degree four while Cherry Blossom and L are 3-regular. But this means that there would be six edges within the images of the vertices of the model, leaving at most nine edges to be images of the edges. Since Pyramid has 12 edges, this is impossible. No graph is a minor of a smaller graph.

By the results set out in the last paragraph, none of the graphs in $\mathscr{F}$ is a minor of another graph in $\mathscr{F}$. It follows from Claim 1 that if H is a proper minor of a graph in $\mathscr{F}$ then TW(G)<4. Since every graph in $\mathscr{F}$ has bramble number exceeding 4 and hence tree width exceeding 3, we see that Claim 1 implies that $\mathscr{F}$ is the obstruction set for the property TW(G)>3. So, it remains to prove this claim.

Assume for a contradiction that there is a graph which contains no member of $\mathscr{F}$ as a minor but has tree width exceeding 3. Then, we choose G to be minor-minimal with this property so all of its proper minors have tree width at most 3. If G has a 1-cut, 2-cut, a (3,3)-cut X, or a 3-cut X such that two vertices of X are joined by an edge then for each component U of G-X, the graph obtained from the subgraph of G induced by (V(U) union X) by adding edges so that X is a clique is a proper minor of G and hence has tree width at most 3. But then, mimicking the use of Lemma 8.5 as in the proof of Corollary 8.6, the graph obtained from G by adding edges so X is a clique also has tree width at most 3, a contradiction. So, G has no cutset of any of the types discussed above. Hence, G is 3-connected, and by Wagner's theorem, planar.

Suppose next that X is a 3-cut of G and let U_1 and U_2 be the components of G-X. If U_i is not a single vertex then as discussed in the solution to assignment 2 it contains a cycle and so the graph obtained from the subgraph of G induced by (V(U) union X) by adding edges so that X is a clique is a proper minor of G and hence has tree width at most 3. So, mimicking the proof of Corollary 8,6 as we did in the solution to assignment 2, we see that one of U_1 or U_2 is a singleton.

Consider some planar drawing of a graph F. For any cycle C of G, the edges of C in a the drawing form a simple closed curve of the plane. We let inside(C)(respectively outside(C)) be the subgraphs of F induced by the vertices in the finite (respectively infinite) region obtained when we remove this curve. We remark that since G has a unique embedding, for any cycle C, this pair of subgraphs will be the same for every drawing of G (although we will have two possibilities for the infinite side).

Lemma 1: For every cycle C of G, one of inside (C) or outside(C) consists of a stable set all of whose members have degree 3 in G.

Proof: Otherwise, we an choose a component U_0 of outside(C) and component U_I of inside(C) each of which either contains an edge or is a vertex of degree 4 in G. We claim that there are four paths from $V(U_0)$ to $V(U_I)$ whose interiors are disjoint. If not, there is a cutset X of size 3 disjoint from $V(U_0)$ union $V(U_I)$ which separates these two sets. Since G is planar, G-X has two components. Each component has two vertices since if either of the two sets we are separating is a single vertex then this vertex has degree 4 in G and hence has a neighbour in G-X. This contradiction proves our claim.

Contracting U_0 and U_I into a single vertex, we see that G has as a minor a graph H which has a drawing such that for some cycle C, there is a vertex of v in inside(C) and a vertex w in outside(C) which are joined by 4 internally disjoint paths: P_1,P_2,P_3,P_4. We claim that any such H has Pyramid as a minor. Our claim contradicts the choice of G. Thus to prove the lemma it remains to prove the claim.

The claim and its proof are similar to one given in the solution to Assignment 2. So assume the claim is false and choose a minor-minimal counterexample H. Thus, for any minor of H, we cannot find a cycle, a pair of vertices and four paths satisfying the hypotheses of the claim.

So, if any edge of a P_i is also an edge of C, then contracting it contradicts the minimality of H.

If any edge e of P_i joins two vertices of C, then one of the two cycles formed by this edge and a path of C between its endpoints can be used in place of C in showing that H is a counterexample. But now, we can contract e to obtain a contradiction.

If any edge of a P_i has an endpoint which is not in C and is not v or w, then we can contract it and contradict the minimality of H.

So, every P_i has precisely two edges and is vx_iw for some x_i on C. Thus, together with the edges of C, the edges of the 4 paths yield a subdivision of the pyramid. This contradiction completes the proof of the claim and the lemma.

Lemma 2: A planar graph H with a universal vertex v has tree width at most 3.

Proof: Since H contains no K_5 model, H-v contains no K_4 model. Hence, by Theorem 8.7, H-v has a tree decomposition of tree width at most two, and so H has a tree decomposition of width 3 (obtaining from the tree decomposition of width 2 for H-v by setting S_v to be the underlying tree T and leaving all other S_w unchanged).

Lemma 3: G is 3-regular.

Proof: We know G is 3-connected and hence has minimum degree 3. Suppose for a contradiction that G contains a vertex v of degree 4. Consider an embedding of G and the embedding of G-v it contains. Let f be the face of the latter embedding which contains v (w.r.t. the embedding of G). Since G-v is 2-connected, the boundary of f is a cycle C. By Lemma 1, G-v-V(C) is a stable set S all of whose vertices have degree 3 in G.
Clearly, the graph G' obtained from G by adding edges from v to every vertex of C is planar (we can add these edges in f).
We claim that the graph G* obtained from G'-S by adding edges so that for each v in S, N(v) induces a clique is planar. Since v is universal in G'-S and hence G*, Lemma 2 would then imply that G* has a tree decomposition of width<4. This, would also be a tree decomposition of G'-S. By the Helly Property of subtrees of a tree, for each vertex v of S since N(v) is a clique of G*, there would be a node t(v) of the tree such that N(v) is contained in $W_{t(v)}$. We could obtain a tree decomposition of width<4 for G' by adding for each v in S, a leaf l(v) incident to t(v) and setting $W_{l(v)}$ to be v union

N(v). Since G is a subgraph of G', this implies TW(G)<4, a contradiction. So to prove the lemma it remains to prove our claim.

We show that if v is a vertex of degree 3 in a planar graph G then deleting v and making its neighbourhood a clique yields a new planar graph G_v. Since S is a stable set, if we apply this operation to a vertex v of S, then S-v is a stable set of vertices of degree 3 in G_v , so applying induction on |S|, the claim follows.

Assume for a contradiction that G_v is not planar. Then it must contain a K_5 or $K_{3,3}$ model. If this model only requires 2 of the new edges then it is a model in a proper minor of G, contradicting the planarity of the latter. So, it must use all 3 new edges. Mimicking a proof given in class, we see that (i) G_v must have a K_5 model such that every new edge is an edge image, and (ii) we can obtain a $K_{3,3}$ model in G using v and the five vertex images of this K_5 model as the vertex images, and using a subset of the images of the edges of the K_5 and the three edges from v as the edge images. But (ii) contradicts the fact that G is planar, completing the proof of the lemma. QED.

Lemma 4: G contains no triangle

Proof: Since G is 3-regular and contains no 3-cut X two of whose vertices are joined by an edge, the neighbourhood of every vertex of v is stable. QED.

Lemma 5: G contains 2 vertex disjoint cycles.

Proof: Let C be a shortest cycle of G. C is induced so does not span G, which has mininimum degree 3. If C has length at least 5 then no vertex of G-V(C) has more than 2 neighbours on C, so G-V(C) has minimum degree 2 and contains a cycle.
If G has length 4, then since G is 3-regular there are only 4 edges from V(C) to C, and hence counting degrees shows G-V(C) must contains a cycle unless G only has six vertices. But there is no 6-regular triangle free planar graph containing a C_4 (the vertices off the C_4 see disjoint pairs of diagonally opposite corners and are adjacent, giving a non-planar graph).

Lemma 6: G is the cube (planar graph consisting of 2 4-cycles joined by a matching of size 4).

Proof: By our connectivity condition on G there must be four vertex disjoint paths between the two disjoint cycles of Lemma 5 and hence G contains a subdivision of the cube. We choose such a subdivision with as few vertices as possible.

Suppose for a contradiction that this subdivision is not induced subgraph of G. Let xy be an edge joining two vertices of the subdivision which is not part of the subdivision. Since G is 3-regular, both x and y are on the interior of some path of the subdivision corresponding to an edge of the cube. Since we took a shortest subdivision the paths they are on correspond to distinct edges ab and cd of the cube. Since G is planar and does not have Cherry Blossom as a minor, we see that ab and cd must have a common endpoint, so WLOG d=a. Now, we can find a subdivision of the cube which has the same centres except that we replace a by x and uses the same edges except that we use xy and delete the subpath of the subdivision from y to a which goes through no centres. The minimality of the subdivision implies ay is an edge of G. Symmetrically, the minimality of the subdivision implies ax is an edge of G. But since xy is an edge of G, we contradict the fact that G has no triangles.

Thus G contains an induced subdivision of the cube. Suppose that G does not contain the cube as an induced subgraph. Then for some edge ab of the cube, there is a vertex x on the interior of the path of the subdivision corresponding to ab. Since the subdivision is induced, x has a neighbour y which is not in the subdivision. We let cd be the unique edge of the cube disjoint from the union of the closed neighbourhoods of a and b. Deleting c,d, and the edges and internal vertices of the paths corresponding to ab and all the edges leaving c or d from the subdivision leaves a cycle C which separates x from c and d. Now, there is a path from c to d on one side of the cycle, and the edge xy is on the other. This contradicts Lemma 1.
So G contains the cube as an induced subgraph and since it is 3-regular and 3-connected it is the cube.QED.

Now, for any bramble in G containing an element which has just one vertex v, has a hitting set of size 4 consisting of v and its neighbourhood (since every bramble element touches v). For any other bramble, we take a stable set S of size 4 in G and note that V-S is a hitting set (since any connected subgraph of G which is not a vertex must intersect V-S). QED.