1 Lightest Paths

We began our study of the following problems:

Single Pair Lightest Path
Input: A digraph G. A rational weight $w(u, v)$ for each edge uv. Two vertices s and t
Output: The weight of a lightest $s - t$ path.

Single Source Lightest Path
Input: A digraph G. A rational weight $w(u, v)$ for each edge uv. A vertex s.
Output: The weight of a lightest $s - t$ path for every vertex t of G.

All Pairs Shortest Path
Input: A digraph G. A rational weight $w(u, v)$ for each edge uv.
Output: The length of a lightest $s - t$ path, for every pair s, t of vertices of G.

Single Pair Lightest Simple Path
Input: A digraph G. A rational weight $w(u, v)$ for each edge uv. Two vertices s and t
Output: The weight of a lightest simple $s - t$ path.

Single Source Lightest Path
Input: A digraph G. A rational weight $w(u, v)$ for each edge uv. A vertex s.
Output: The weight of a lightest simple $s - t$ path for every vertex t of G.

All Pairs Shortest Path
Input: A digraph G. A rational weight $w(u, v)$ for each edge uv.
Output: The length of a lightest simple $s - t$ path, for every pair s, t of vertices of G.

(The weight $w(P)$ of a path P is the sum of the weights of the edges on it. If there is no $s - t$ path then the weight of the lightest s-t path is defined to be ∞.)

If there are negative cycles then there may be no lightest path, as we could traverse a negative cycle as often as we liked, thereby making paths of lower and lower weights. So the first three problems above are not well-defined in general. Furthermore, the Hamilton Cycle Problem is a special case of the Single Pair Lightest Simple Path problem in which all edges have weight -1. So the last three problems above are likely to be difficult in general.

We decided to restrict our attention to directed graphs without negative cycles in the hopes that this would make the problems well-defined and easier.

We proved that for such weighted graphs:

1. Lightest Simple Paths are Lightest Paths.

Proof: Consider a Lightest Simple s-t Path P and, to obtain a contradiction, an s-t path P' with $w(P') < w(P)$ chosen to have minimum length over all such paths. We know that P' has a repeated vertex, so its edge set can be split into a shorter s-t path P^* and a cycle C. By assumption, $w(C) \geq 0$ and by our choice of P', $w(P^*) \geq w(P)$. So, $w(P') = w(P^*) + w(C) \geq w(P)$, a contradiction. QED.

Thus, all six problems above are well defined.

We obtain the following for graphs without negative weight cycles as an immediate corrollary of (1).

2. The Lightest s-t Path with at most $|V| - 1$ edges is a Lightest s-t Path.

This fact will help us to develop algorithms for finding Lightest Paths.

A simple exchange argument proves that in graphs without negative weight cycles:

3. Subpaths of Lightest Paths are Lightest Paths.

This fact will be useful in proving the correctness of our algorithms.