Flows

By a *network*, we mean a directed graph D to each edge uw of which we have associated a non-negative *integer* capacity c_{uw}. By an s-t flow f in a network, we mean a set $\{x_{uw}|uw \in E(D)\}$ of real numbers with $0 \leq x_{uw} \leq c_{uw}$ such that the following *flow conservation constraint* is satisfied for every vertex w of $V - s - t$:

$$\sum_{uw \in E(D)} x_{uw} = \sum_{wu \in E(D)} x_{wu}$$

Now,

$$\sum_{uw \in E(D)} x_{uw} = \sum_{w \in V(D)} \sum_{uw \in E(D)} x_{uw} = \sum_{w \in V(D)} \sum_{wu \in E(D)} x_{wu}.$$

Furthermore, the flow conservation constraints tell us that:

$$\sum_{w \in V(D)-s-t} \sum_{uw \in E(D)} x_{uw} = \sum_{w \in V(D)-s-t} \sum_{wu \in E(D)} x_{wu}.$$

So,

$$\sum_{ut \in E(D)} x_{ut} + \sum_{us \in E(D)} x_{us} = \sum_{su \in E(D)} x_{su} + \sum_{tu \in E(D)} x_{tu}.$$

Or written another way

$$\sum_{su \in E(D)} x_{su} - \sum_{us \in E(D)} x_{us} = \sum_{ut \in E(D)} x_{ut} - \sum_{tu \in E(D)} x_{tu}.$$

The common value of the LHS and RHS of this constraint is called the *volume* of the flow, and denoted $v = v(f)$.

We are interested in finding s-t flows of maximum volume through an input network. Sometimes we want to find an *integer* flow, i.e. a flow such that each x_{uw} is an integer.

By an *s-t cut* of the network we mean a subset $\mathcal{S}$ of the vertices containing s but not t. The *capacity* of the cut, denoted $c(\mathcal{S})$ is $\sum_{uw \in E(D), u \in \mathcal{S}, w \notin \mathcal{S}} c_{uw}$.

For any s-t cut $\mathcal{S}$ and s-t flow f, the flow conservation constraints for the vertices of $\mathcal{S} - s$ tell us that:

$$\sum_{u \in S} \sum_{uw \in E(D)} x_{uw} - \sum_{u \in S} \sum_{wu \in E(D)} x_{wu} = v(f).$$

Clearly,

$$\sum_{u \in S} \sum_{uw \in E(D)} x_{uw} = \sum_{u \in S} \sum_{w \in \mathcal{S}, uw \in E(D)} x_{uw} + \sum_{u \in S} \sum_{w \notin \mathcal{S}, uw \in E(D)} x_{uw}$$

and

$$\sum_{u \in S} \sum_{wu \in E(D)} x_{wu} = \sum_{u \in S} \sum_{w \in \mathcal{S}, wu \in E(D)} x_{wu} + \sum_{u \in S} \sum_{w \notin \mathcal{S}, wu \in E(D)} x_{wu}$$

Finally,

$$\sum_{u \in S} \sum_{w \in \mathcal{S}, wu \in E(D)} x_{wu} = \sum_{u \in S} \sum_{w \in \mathcal{S}, uw \in E(D)} x_{uw}.$$

Combining these four equations we obtain:

$$v(f) = \sum_{u \in S} \sum_{w \notin \mathcal{S}, uw \in E(D)} x_{uw} - \sum_{u \in S} \sum_{w \notin \mathcal{S}, wu \in E(D)} x_{wu}.$$

Now, $x_{uw} \leq c_{uw}$ for each arc uw, so

$$\sum_{u \in S} \sum_{w \notin \mathcal{S}, uw \in E(D)} x_{uw} \leq \sum_{u \in S} \sum_{w \notin \mathcal{S}, uw \in E(D)} c_{uw}.$$

Also, $x_{wu} \geq 0$ for each arc wu so $\sum_{u \in S} \sum_{w \notin \mathcal{S}, wu \in E(D)} x_{wu} \geq 0$.
Combining the last three inequalities yields: $v(f) \leq c(\mathcal{S})$.
Thus, we have:

(*) the maximum volume of an s-t flow is at most the minimum capacity of an s-t cut.

We saw in class that if we interpreted the network as a multigraph then we could use our results on edge disjoint paths to prove that there is an integer s-t flow whose volume is equal to the capacity of some s-t cut. Now, (*) implies that the flow is a maximum volume s-t flow and the cut is a minimum capacity s-t cut and so we have proven:

The Max Flow - Min Cut Theorem (Ford and Fulkerson 1956): *The maximum volume of an s-t flow is equal to the minimum capacity of an s-t cut. Furthermore, we can find a maximum volume flow which is integer.*

We could treat our network as a multigraph and use our edge disjoint path algorithm to find such a flow but this might take a long time if the capacities are large (since the input size of an integer is the logarithm of its value). Instead we modify the algorithm so it runs more quickly. The details will be discussed in the next lecture.