Duality

In order to find the multipliers for the constraints of a linear program, we defined a dual LP. For LPs in standard form, this dual is discussed on pages 54-60 of Chvatal and pages 11-13 of Professor Fukuda's notes. We proved the Weak Duality Theorem, which states that the objective function value of any feasible solution to a maximization problem is at most the objective function value of any feasible solution to its dual. We noted that this implies that if we have a solution to the original problem and a solution to the dual with equal objective function values then both are optimal (i.e. best possible) solutions.

We also discussed taking duals for LPs in which some constraints were equations and some variables were *free* to take either negaitve or non-negative values. We saw that in taking the dual, the dual variable corressponding to an equation is free, whilst that corresponding to an inequality is constrained to be non-negative. Conversely, the dual constraint coreesponding to a free variable is an equation whilst that corresponding to a variable which is constrained to be non-negative is an inequality. This is presented on page 16 of Professor Fukuda's note: (P3) and (D3).

We then took the dual of two of our LPS, presented in the last class. The Dual of the Shortest Path +Cycles LP was:

$max\ d_t$
$subject\ to$
$\forall e = uv \in E$:
$if\ s \notin \{u, v\}: \quad d_v - d_u \leq w_e$
$if\ s = u: \quad d_v \leq w_e$
$if\ s = v: \quad -d_u \leq w_e$

To simplify this we could use a dummy variable d_s and set it to 0 obtaining the following LP:

$max\ d_t$
$subject\ to$
$d_s = 0$
$\forall e = uv \in E, \quad d_v - d_u \leq w_e$

Which can be rewritten as:

$max\ d_t$
$subject\ to$
$d_s = 0$
$\forall e = uv \in E, \quad d_v \leq d_u + w(u, v)$

Thus, the certificate d function we have been using to certify that our solution ot the shortest path IP is optimal is a feasible solution to the dual LP.

Ie, we have for graphs with no negative cycles:

length of shortest path = opt solution to Shortest path + cycles IP
$\leq$ optimal solution to Shortest path +cycles LP
(since the optimal IP solution is a feasible solution to the LP)
$\leq$ optimal solution to the Dual LP
$\leq d_t$ for the certificate used when we studied the shortest path problem.

Since we showed that the first and last of these are equal, all of them are, so we have proved that, in this case, if both the original problem and its dual have a feasible solution then the optimal solutions are equal.

Furthermore, we have shown that the certificates we used are not just feasible solutuions to the dual problem but optimal solutions to the dual problem.

We also showed that the dual of the Max Flow LP was the Min Cut LP. As in the above example it was useful to introduce a dummy variable for s and t even though there are no corresponding constraints. The dual, with these variables added becomes.

The Max Flow Dual LP

$min \quad \sum_{e=uv} c_e y_e$
subject to
$\forall e = uv \in E, \quad y_e - z_v + z_u \geq 0$
$z_s = -1$
$z_t = 0$
$\forall e \in E, \quad y_e \geq 0$

Now, we noted that in any solution to this LP, if we reset z_v to be -1 for all v with $z_v \leq -1$ and reset z_v to 0 for all v with $z_v \geq 0$ then the solution remains feasible and has the same objective function value. Thus, the max flow dual has the same optimum objective function value as:

The Min Cut LP

$min \quad \sum_{e=uv} c_e y_e$
subject to
$\forall e = uv \in E, \quad y_e - z_v + z_u \geq 0$
$\forall v \in V, \quad z_e \geq -1$
$\forall v \in V, \quad z_e \leq 0$
$z_s = -1$
$z_t = 0$
$\forall e \in E, \quad y_e \geq 0$

Now, we have that:

Max flow LP optimum
$\leq$ Max Flow Dual LP optimum
$=$ Min Cut LP optimum
$\leq$ Min Cut IP optimum

The Max Flow Min Cut theorem states that the first and last of these are equal, so all four must be equal.

Finally we stated that as with these two examples, if both the original problem and the dual have a feasible solution then the optimal solutions of the two problems have the same objective function value. This is the Strong Duality Theorem which we will not prove.

Then I blathered on about how looking at the dual of the LP can help you solve an IP by giving some insight. Don't worry about retaining any of this. You can learn more about solving LPs and IPs in 308-566 and 308-567.