Some NP-Completeness Proofs

The notion of an NP-complete problem is introudced on pages 984-986 of Cormen. We prove problems are NP-complete using Lemma 34.8 (read the page beginning Section 34.4 and going to the heading Formula Satisfiability). We proved that 3SAT, 0-1 IP, and 3-Colouring are NP Complete Problems.

Our Proof that 3-SAT is NP-complete is found on pages 48-49 of the Garey and Johnson book (handed out during the March 27th class). Our proof that 3-colouring is NP-complete uses the reduction discussed in Exercise 34-3 of Cormen.

To prove that 0-1 IP is NP-complete, we first show that it is in NP and then show that we can reduce SAT to it. So, consider a 0-1 IP maximize $\sum_{i=1}^{n} c_i x_i$ subject to $\sum_{i=1}^{n} a_{i,j} x_i \leq b_j, \quad j = 1, ..., m.$ and $x_i \in \{0, 1\}$, and an integer k. This instance is a yes instance if there is some solution with objective function value at least k. Our certificate consists of n bits $z_1, .., z_n$ such that setting $x_i = z_i$ yields the desired solution. Clearly, we can determine the objective function value and check that all the constraints hold in polynomial time. So the problem is indeed in NP.

Next consider an instance of SAT with variables $\{x_1, ..., x_n\}$ and clauses $C_1, ..., C_m$. We construct an $0{-}1IP$ with variables $\{t_1, ..., t_n, f_1, ..., f_n, s_1, ..., s_m\}$. The objective function is $\sum_{i=1}^{m} s_i$, the value of k is m, for each variable x_i we have the constraint:

$$f_i + t_i = 1$$

and for clause C_j we have the constraint:

$$\sum_{x_i \in C_j} t_i + \sum_{\overline{x_i} \in C_j} f_i - s_j \geq 0$$

Now given a satisfying truth assignment, we can show that the IP has a solution with objective function value greater than k as follows. We set $t_i = 1$ if x_i is set to true by the assignment and 0 otherwise and set f_i to be 1 if x_i is set to false and 0 otherwise. We also set each $s_i = 1$. Clearly, each constraint corresponding to a variable is satisfied. Furthermore, since there is a true literal in every clause, the constraint for each clause is satisfied. Finally, the objective function value is indeed k as required.

Conversely, given a solution with objective function value k, we obtain a satisfying truth assignment by setting x_i to be true if $t_i = 1$ and false otherwise. Note that the constraints corresponding to the variables tell us that f_i is one precisely if $\overline{x_i}$ is true. The fact that the objective function value is $k = m$ tells us that each s_i is equal to 1 in our IP solution. It follows that the fact that the constraints corresponding to the clauses hold tells us that each clause contains a true variable. I.e we have indeed obtained a satisfying truth assignment.

Clearly, we can construct the IP corresponding to an instance of SAT in time which is polynomial in the number of variables and the sum of the sizes of the clauses. Thus, there is a Turing reduction from SAT to $0-1$-IP and hence we have shown that this latter problem is NP-Complete.