Lecture 4: We began the study of the Simplex Method for solving Linear Programs.

We showed that every LP instance can be formulated as an instance of LP in standard form (for definition and proof see Chapter 1 of Chvatal).

We showed how the simplex method worked using the example of Chapter 2 of Chvatal although we did a different set of pivots(our calculations are set out below).

We defined slack and decision variables, dictionaries, feasible dictionaries, basic and non-basic variables, pivot. We implicitly set out the main steps of the Simplex Method

1) Find a feasible dictionary.
2) Determine if there is a non-basic variable with positive cost coefficient in the expression for z. If not, terminate the current solution is optimal
3) Choose a non basic variable x_e with positive cost coefficient in this expression (x_e is the entering variable).
4) Determine if there is a basic variable x_j whose nonnegativity gives an upper bound on the possible values of x_e if we increase it while leaving all other nonbasic variables at 0. If not, the problem is unbounded, return this fact. Otherwise let x_l be (one of) the basic variable(s) whose nonnegativity gives the most stringent upper bound on the possible values of x_e. x_l is the leaving variable.
5) Rewrite the equation for x_l in the dictionary as an equation for x_e.
6) Replace x_e by the RHS of this equation in all the other equations of the dictionary.
7) Go back to Step 2.

We asked ourselves what could go wrong. We were concerned with whether we could find a feasible dictionary as required in Step 1 and whether the method would always terminate with an optimal solution or if it could, e.g., cycle through the same set of dictionaries endlessly.

We have now finished Chapters 1 and 2 of Chvatal. Ignore the tableau section on pages 23-25.

Our Simplex Example.

Maximize: $5x_1+4x_2+3x_3$

Subject to:

$$
\begin{aligned}
2x_1+3x_2+x_3 &\leq 5\\
4x_1+x_2+2x_3 &\leq 11\\
3x_1+4x_2+2x_3 &\leq 8\\
x_1,x_2,x_3 &\geq 0
\end{aligned}
$$

Adding slack variables, and using z to denote the value of the objective function, we obtained the following (feasible) dictionary which expressed the slack variables in terms of the original decision variables:

$x_4 = 5-2x_1-3x_2- x_3$
$x_5=11-4x_1-x_2-2x_3$
$x_6= 8-3x_1-4x_2-2x_3$
$z = 5x_1+4x_2+3x_3$

If we try to increase x_3 to improve the objective function value then, the first equation and the nonnegativity of x_4 us that we can increase it to at most 5, the second constraint tells us that we can increase it to at most 11/2, and the third constraint tells us that we can increase it to at most 4. So we increase it to 4 and obtain the new solution:

$x_1=0,x_2=0,x_3=4, x_4=5-4=1,x_5=11-2(4)=3,x_6=8-2(4)=0$ with objective function value $3(4)=12$.

We rewrite the new non-zero variables as linear functions of the new zero valued variable as follows. We first rewrite the equation for x_6 as an equation for x_3:

$x_3=4- \frac{3x_1}{2}-2x_2-\frac{x_6}{2}$

We then substitute its right hand side for x_3 in the equations for z,x_4, and x_5 to obtain:

$x_3=4- \frac{3x_1}{2}-2x_2-\frac{x_6}{2}$
$x_4= 1 -\frac{x_1}{2} - x_2 + \frac{x_6}{2}$
$x_5= 3 -x_1 + 3x_2 + x_6$
$z =12 +\frac{x_1}{2} - 2x_2 - \frac{3x_6}{2}$

We now increase x_1, while leaving x_2 and x_6 at 0. Our first equation tells us we can increase x_1 to at most 8/3, our second that we can increase it to at most 2, and our third that we can increase it to at most 3. So, we increase it to 2, which yields the new solution:

$x_1=2,x_2=0,x_3=4-(3*2)/2=1,x_4=1-(2/2)=0,x_5=3-2=1$ with $z=12+(2/2)=13$.

We rewrite the equation for x_4 as an equation for x_1 and then substitute it into the equations for x_3,x_5, and z to obtain the new dictionary:

$x_3= 1 + x_2 + 3x_4 - 2x_6$
$x_1= 2 - 2x_2 - 2x_4 + x_6$
$x_5= 1 + 5x_2 + 2x_4$
$z =13 - 3x_2 - x_4 - x_6$

Since, the coefficients for z are all negative, and x_2,x_4, and x_6 are all negative this is an optimal solution and we can stop.