Lecture 15: Introducing Tree Width

We started to talk about tree decompositions of graphs. This is covered in Section 10.4 of the text. We defined a tree decomposition as a tree T and a subset W_t of vertices for each node t of the tree (the text uses V_t instead) so that Node Coverage and Edge Coverage on page 575 of the text were satisfied and so that for every vertex v, the set $S_v = \{t | W_t \ni v\}$ is a tree. This is equivalent to saying that S_v is connected which is exactly the coherence condition of the text: for every t_1 and t_3 in S_v the vertices on the path of T between them must also be in S_v.

We then proved (10.13) and (10.14) in the text. We also gave the definition of tree width given on page 578 of the text. We next considered the following:

k-Good Property: For every subset S of vertices, there is a set X of at most k+1 vertices such that every component U of G-S satisfies $|U \cap S| \leq |S|/2$.

Lemma: If G has tree width at most k then G has the k-Good Property.

Proof: Consider a tree decomposition $[T, \{W_t \mid t \in V(T)\}]$ of width at most k.
For each edge xy of T, defining G_x and G_y as in 10.14 in the text we note that 10.14 states that every component of $G-(W_x \cap W_y)$ is either completely in $G_x-(W_x \cap W_y)$ or completely in $G_y-(W_x \cap W_y)$ (if there were a component intersecting both there would be a path with an endpoint in each and hence an edge with an endpoint in each). So, if neither $V(G_x)$ nor $V(G_y)$ contain more than half the vertices of S we are done with $X = W_x \cap W_y$. So we can assume this is not the case. If $V(G_x)$ contains more than half the vertices of S we direct xy from y to x, otherwise we direct xy from x to y. Since a tree has one more vertex than edge there is a vertex t which has no edge directed out of it. If T-t has components T(1),…,T(d) then by 10.13, every component U of $G-W_t$ is contained in $G_{T(i)}$ for some i. But then, our choice of direction on the edge from t to T(i) tells us U contains less than half of the vertices of S. QED.

We claimed that if a graph has the k-good property then it has tree width at most 3k+3. We will prove this on Wednesday. I gave some examples of tree decompositions. I will post these on Wednesday.