Lecture 14: DRP on Trees and 2-Trees

We began our study of the DRP problem. An instance of the problem consists of an integer k, a graph G=(V,E) and two subsets of V: $S=\{s_1,\ldots,s_k\}$ and $T=\{t_1,\ldots,t_k\}$. We are asked to determine if G has k vertex disjoint paths $P_1,\ldots,P_k$ such that P_i has endpoints s_i and t_i. This problem is NP-complete, as you were asked to show on Assignment 3. The problem k-DRP where k is fixed and the input is simply (G,S,T) can be solved in polynomial time. This is a seminal result of Robertson and Seymour. The proof of correctness of the algorithm requires about 600 pages of mathematics.
We will discuss solving DRP and k-DRP on some tree-like graphs. We warmed up in class by showing how to solve DRP on trees and 2-trees.
We adopted a dynamic programming approach which involved traversing the rooted tree (or the rooted tree certifying that G is a 2-tree) in post order, and using the table of partial solutions created for the children of a node s to compute the partial solution for s.
We omit the details as this is just a warm up for a similar approach on a wider class of graphs, which we discuss next week.

We also showed that a connected graph is a tree precisely if for every set S of vertices there was a vertex x such that no component of G-x contained more than half the vertices of S.

One direction of this result is trivial. If G is not a tree then it contains a cycle C. Letting S be the vertices of the cycle we see that for every vertex x, C-x is either C itself or a path P on |C|-1 vertices and hence is contained in a component of G-x containing more than half the vertices of S.

To prove the other direction, consider a tree T and a subset S of its vertices. For each edge *yz* of T, if both components of T-*yz* contain exactly |S|/2 vertices then we can use either y or z as the desired x. Otherwise, we orient *yz* towards the vertex in the component of T-*yz* containing more than |S|/2 vertices of S. Since a tree has one more vertex than edge, there is a node x of the tree such that no edge of the tree incident to x is directed away from x. This tells us precisely that the components of T-x all contain less than half of the vertices of S.