Lecture 12: Knapsack and Directed Hamilton Cycle are NP-Complete

We finished off the proof that 3-colouring is NP-complete. See pages 486-490 of Kleinberg and Tardos. We showed that Knapsack, Subset Sum, Splitting the Sum in Half(aka Partition), and Splitting Many Sums in Half (aka Multiway Partition) were all in NP, the certificate being the partition of the items into two sets. We then showed that these problems were all NP-complete by (i) reducing Subset Sum to Knapsack, (ii) reducing Partition to Subset Sum, (iii) reducing Multiway-Partition to Partition, and (iv) reducing 3-SAT to Multiway-Partition.

The first two of these reductions are straightforward. We give here a reduction from 3-SAT to Partition which combines the ideas of the last two reductions. You are responsible only for this reduction not the two given in class.

We considered an instance of 3-SAT with variable $x_1,\ldots,x_n$ and m clauses, $C_1,\ldots,C_m$, each of which contains three literals. . Our instance of PARTITION has 2n+2m+1 items. For i between 1 and n, item i corresponds to x_i and letting I(i) be $\{j|x_i$ is a literal of $C_j\}$ has value $7^{i+m-1}+\sum_{j \text{ in } I(i)} 7^{j-1}$ while item i+n corresponds to $\neg x_i$ and letting J(i) be $\{j| \neg x_i$ is a literal of $C_j\}$ has value $7^{i+m-1}+\sum_{j \text{ in } J(i)} 7^{j-1}$. For j between 1 and m, item 2n+j and2n+j+m correspond to clause j and have value 7^{j-1}. Finally the last item has value $\sum_{j=1}^{m} 7^{j-1}$.
Even though the numbers we construct are exponentially big in terms of the input SAT instance we can do this reduction in polynomial time because we only use O(log k) symbols to represent an integer k.
We note that our weights were chosen so that when we do addition in base 7, there is no carry over. This is crucial to the verification that the reduction is correct.
Given, a satisfying truth assignment for the 3-SAT instance, we will choose to put on one side of the partition all of the items which correspond to true literals under the assignment. On the other side of the partition we put the last element and all of the variables corresponding to false literals. Finally for each clause C_j, the number of items corresponding to clause C_j which are put on the same side as the true literals is equal to the number of false literals in the clause (which is between 0 and 2). The sum of the weights of these items is W.
Given a partition, each side of which has weight exactly half of the total weight of all the items, setting the literals corresponding to items on the same side of the partition as the last item to false yields a satisfying truth assignment.
We also showed that Directed Hamilton Cycle is NP-complete using the reduction given on pages 475-479 of Kleinberg and Tardos.