Lecture 11: Some More NP-Complete Problems.

We noted that by the transitivity of the polynomial reduction relation, to prove a decision problem π complete, we need only to (a) show it is in NP and (b) prove that for some NP-complete problem π' we have $\pi' <_P \pi$.We used this to show that a number of problems were NP-complete.

To begin we presented a proof that the problem of determining if an IP is feasible is NP-complete. We noted that the certificate for such a problem was a feasible solution but that we needed the non-trivial result (which we will not even state or prove) that this solution had sized bounded by a polynomial in the input in order to show this certificate could be verified in polynomial time. To complete the proof we gave a reduction from SAT to IP-feasibility.
We had an integer variable y_i for each Boolean variable x_i. We had the constraints: $y_i \geq 0$, $y_i \leq 1$, and for each clause C_j in the formula:

$$\sum_{x_i \in C_j} y_i + \sum_{not(x_i) \in C_j} (1 - y_i) \geq 1$$

These constrains ensured that feasible solutions were in 1 to 1 correspondence with satisfying truth assignments.

We then showed that 3-SAT is NP-complete. Clearly 3-Sat is in NP (the certificate is a satisfying truth assignment- given this we just need to check if every clause contains a true literal which can be done in linear time). We then proved that SAT can be reduced to 3-SAT by replacing each clause with a set of clauses of size 3 in such a way that the new formula obtained will be satisfiable if and only if the original formula was.

We then showed the NP-completeness of the decision problem STABLE SET where an instance is a graph G and integer k, and we ask: does G contain a stable set of size k? STABLE SET is clearly in NP (given as a certificate the names of k vertices in a clique we need only ensure that this list really does have k elements and that there is no edge between any two of them- this can be done in $O(|V|^2)$ time). We showed that 3-SAT reduces to STABLE SET using the reduction of pages 460-462 of Kleinberg and Tardos.

The same proof, replacing edges by non-edges shows the NP-completeness of the decision problem CLIQUE where an instance is a graph G and integer k, and we ask: does G contain a clique of size k?

We next observed that the decision problem VERTEX COVER is NP-complete. In this decision problem the input is a graph G and an integer k, and we are asked if there is a vertex cover C of size k in G (i.e. a set C such that every edge of G has an endpoint in C). . A polynomially-verifiable certificate for this problem consists of a list of vertices in the cover. To check the cover exists, we then simply need to verify that this set of vertices has size k and that every edge has at least one endpoint in it. We can reduce an instance of stable set to an instance of vertex cover, since C is a vertex cover in G if and only if V-C is a stable set in G. So, we reduce an instance (G,k) of stable set to the instance (G,|V|-k) of vertex cover.

We next considered the decision problem 3-COL where an input is a graph G and the question is, can we label the vertices of G from {1,2,3} so that no edge has both endpoints the same colour. Clearly, the desired colouring provides a poynomially verifiable certificate for this problem so it is in NP. I asked you to think about how to reduce 3-SAT to 3-COL, thereby proving the latter is NP-complete