Quantum Correlations: From Bell inequalities to Tsirelson's theorem

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Abstract

The cut polytope and its relatives are good models of the correlations that can be obtained between events that can be well described by classical physics. Bell's Theorem and subsequent experiments demonstrate that correlations obtainable between events at the quantum level cannot be modelled in this way. This raises the question of whether a "good" mathematical characterization of quantum correlation vectors can be obtained. An important special case was completely solved by Tsirelson, who showed that a projection of the elliptope provides the desired body. (This parallels the well know semi-definite programming approach to approximating max-cut.) I will survey this material and present some new joint work with Hiroshi Imai and Tsuyoshi Ito on a possible direction for extending Tsirelson's theorem.

1 Classical Correlations

Let $A_1, ..., A_n$ be a collection of n 0/1 valued random variables that belong to a common joint probability distibution. For $1 \leq i < j \leq n$, we define new random variables $A_i \triangle A_j$ that are one when $A_i = A_j$ and zero otherwise. Denote by $\langle A \rangle$ the expected value of a random variable A. The *full correlation vector* x based on $A_1, ..., A_n$ is the vector of length $N = n + n(n-1)/2$ given by the expected values:

$$x = (\langle A_i \rangle, \langle A_i \triangle A_j \rangle) \equiv (\langle A_i \rangle_{1 \leq i \leq n}, \langle A_i \triangle A_j \rangle_{1 \leq i < j \leq n}).$$

Note that each element of the above vector lies between zero and one. Now consider any vector $x = (x_1, ..., x_n, x_{12}, ..., x_{n-1,n}) \in [0,1]^N$ indexed as above, which we will call an *outcome*. We consider two related computational questions:

Recognition. *When is an outcome x a full correlation vector?*

Optimization. *For any $c \in R^N$ what is the maximum value of $c^T x$ over all possible full correlation vectors x?*

It turns out that the recognition problem is NP-complete, and the optimization problem is NP-hard. This follows from the fact that the set of full correlation vectors is in fact the cut polytope CUT_{n+1} defined on the complete graph K_{n+1}. This polytope is defined as the convex hull of the 2^N full correlation vectors obtained by deterministically setting each random variable A_i to either zero or one. For details of the above and other facts about cut polytopes, see the book by Deza and Laurent [8]. A well-known characterization of the cut polytope leads to the following.

L_1-characterization of full correlation vectors.
An outcome $x \in [0,1]^N$ is a full correlation vector if and only if there exist vectors $u^i, v^j \in R^d$, $1 \le i,j \le n$, $d \le N$, for which

$$x_i = \|u^i\|_1, \qquad x_{ij} = \|u^i - u^j\|_1.$$

where $\|.\|_1$ is the L_1-norm.

Full correlation vectors provide an adequate model for correlations obtained in physical experiments at the classical level. Let us call the random variables observables. For example, with $n = 3$, A_1, A_2, A_3 could obtain the value one if a given McGill student has blond hair, weighs more than 80 kg or is more than 180cm high, respectively. We could obtain a full correlation vector by determining these three observables for all McGill students.

In a quantum setting, things are very different. Firstly, it is difficult to apply the above model directly since at the quantum level it may be impossible to measure directly different observables for a given particle. Therefore the above model is replaced by a bipartite setting where the 0/1 random variables (observables) are labelled $A_1, ..., A_m$ and $B_1, ..., B_n$ respectively.

The *(bipartite) correlation vector* x based on random variables $A_1, ..., A_m$ and $B_1, ..., B_n$ is the vector of length $M = m + n + mn$ given by the expected values:

$$x = (\langle A_i \rangle, \langle B_j \rangle, \langle A_i \triangle B_j \rangle) \equiv (\langle A_i \rangle_{1 \le i \le m}, \langle B_j \rangle_{1 \le j \le n}, \langle A_i \triangle B_j \rangle_{1 \le i \le m, 1 \le j \le n}). \qquad (1)$$

As we will be concerned only with the bipartite case, we will simply use the term correlation vector where no confusion arises. As before, we call any vector $x \in [0,1]^M$ indexed as in (1) an *outcome*. Again we may define a polytope by considering the convex hull of the 2^{m+n} correlation vectors formed by letting each of the $m + n$ random variables take value either zero or one. This polytope is called the Bell polytope $B_{m,n}$ and was apparently first considered by Froissart [9]. It turns out the membership and optimization problems given above are still NP-complete and NP-hard respectively (for references, see, e.g., [2]). The characterization theorem generalizes in a natural way.

L_1-characterization of bipartite correlation vectors.
An outcome $x \in [0,1]^M$ is a bipartite correlation vector if and only if there exist vectors $u^i, v^j \in R^d$, $1 \le i \le m$, $1 \le j \le n$, $d \le M$, for which

$$x_i = \|u^i\|_1, \qquad x_{m+j} = \|v^j\|_1, \qquad x_{ij} = \|u^i - v^j\|_1.$$

The Bell polytope has been much studied. Valid inequalities for the $B_{m,n}$ are often called *Bell inequalities*, although here we will reserve this term for the facets of $B_{m,n}$. These inequalities have been studied by many researchers, see for example [10], [14], [7]. The *CHSH inequality* is the only non-trivial facet of $B(2,2)$ and is given by

$$\langle A_1 \triangle B_1 \rangle - \langle A_1 \triangle B_2 \rangle - \langle A_2 \triangle B_1 \rangle - \langle A_2 \triangle B_2 \rangle \le 2$$

or equivalently

$$x_{11} - x_{12} - x_{21} - x_{22} \le 0. \qquad (2)$$

Although few Bell inequalities were known until recently, much is known about facets of the cut polytope, including several large classes of facets. In [1] a method is given to generate Bell inequalities from facets of the cut polytope, producing a large number of new inequivalent Bell inequalities.

The correlation vector with $m = 2, n = 2$ given by

$$x = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{2+\sqrt{2}}{4}, \frac{2-\sqrt{2}}{4}, \frac{2-\sqrt{2}}{4}, \frac{2-\sqrt{2}}{4}) \tag{3}$$

clearly violates the CHSH inequality (2), so it follows there is no joint distribution function for the four random variables. This correlation vector cannot arise as the result of an experiment for which the rules of classical physics apply. An outstanding prediction of quantum theory, apparently confirmed by numerous experiments, is that this correlation vector can arise from observations at the quantum level. This fact has lead to many surprising applications in quantum information theory, see for example Cleve et al. It raises the issue of whether there is a good characterization of such *quantum correlation vectors*, the topic of the rest of the paper.

2 Quantum Correlations

The postulates of quantum theory give a complete statistical description of the outcome of experiments at the quantum level. A two party quantum correlation experiment can be described by a quantum state and set of observables $A_1, ..., A_m, B_1, ...B_n$ on a bipartite Hilbert space. It is assumed the two parties are spatially separated and that the observations are performed simultaneously. For a given experimental outcome, the vector x defined by (1) is called a *quantum correlation vector.* The description given by the postulates does not appear to provide any tractable method to answer the recognition, optimization and characterization questions when applied to quantum correlation vectors. Such answers are provided, however, for one important case by a theorem of Tsirelson. A *quantum correlation function* is a vector $y \in R^{mn}$ defined by taking the last mn coordinates of a quantum correlation vector, i.e.,

$$y = (\langle A_i \triangle B_j \rangle) \equiv (\langle A_i \triangle B_j \rangle_{1 \le i \le m, 1 \le j \le n}). \tag{4}$$

Tsirelson's Theorem (0/1 version)[4] [13].
The following three statements are equivalent:

- $y = (\langle A_i \triangle B_j \rangle) \in [0, 1]^{mn}$ *is a quantum correlation function.*
- $x = (1/2, 1/2, ..., 1/2, \langle A_i \triangle B_j \rangle) \in [0, 1]^M$ *is a quantum correlation vector.*
- *There exist vectors* $u^i, v^j \in R^d$, $1 \le i \le m$, $1 \le j \le n$, $d \le m + n$, *for which*

$$x_i = \|u^i\| = \frac{1}{2}, \quad x_{m+j} = \|v^j\| = \frac{1}{2}, \quad x_{ij} = \|u^i - v^j\|.$$

where $\|u\| \equiv u^T u$.

We call an experimental outcome *unbiased* if for all i and j we have $\langle A_i \rangle = \langle B_j \rangle = 1/2$, otherwise it is *biased.* A remarkable result implied by this theorem is that the recognition and optimization problems for correlation functions and unbiased quantum correlation vectors can be solved in polynomial time by semi-definite programming(SDP). Using the theorem, we can verify that (3) is a quantum correlation vector by exhibiting the vectors:

$$u^1 = (\frac{1}{2}, 0, \frac{1}{2}), \quad u^2 = (0, \frac{1}{2}, \frac{1}{2}), \quad v^1 = (\frac{1}{2\sqrt{2}}, \frac{1}{2\sqrt{2}}, \frac{1}{2}), \quad v^2 = (\frac{1}{2\sqrt{2}}, \frac{-1}{2\sqrt{2}}, \frac{1}{2}). \tag{5}$$

Furthermore, it can be verified by SDP that this is the maximum violation of (2), although in this case Tsirelson [4] has provided an analytic proof. These maximum quantum violations have many interesting applications, see e.g. [6]. The maximum quantum violation of any Bell inequality (like CHSH) that does not have terms involving the expectations $\langle A_i \rangle$ or $\langle B_j \rangle$ can likewise be found by using SDP. Unfortunately, most of the Bell inequalities produced recently [1] do not satisfy these conditions. For these inequalities the maximum quantum violation may only be achieved by a biased quantum correlation vector, and the above method cannot be directly applied.

Tsirelson's theorem may not hold for experimental outcomes that are biased. Consider the outcome for $m = n = 1$ given by $x = (3/4, 3/4, 3/4)$. If we set $u^1 = (\sqrt{3}/4, 3/4)$ and $v^1 = (-\sqrt{3}/4, 3/4)$ then

$$x_1 = \|u^i\|, \quad x_2 = \|v^1\|, \quad x_{12} = \|u^1 - v^1\|,$$

and the corresponding vector $y = (3/4)$ is obviously a quantum correlation function. However x is not a quantum correlation vector because it violates the nosignalling condition. This condition derives from the fact that the expectations $\langle A_i \rangle, 1 \le i \le m$ should be the same regardless of which measurement j the other party decides to make, due to the spatial separation of the two parties. Similar conditions should hold for the expectations $\langle B_j \rangle$. It is shown in [2] that a vector x satisfies the *nosignalling condition* if and only if it belongs to the *rooted semimetric polytope* defined by the inequalities:

$$x_i + x_j + x_{ij} \le 2, \quad x_i + x_j - x_{ij} \ge 0, \quad x_i - x_j + x_{ij} \ge 0, \quad -x_i + x_j + x_{ij} \ge 0. \tag{6}$$

It is easy to see that unbiased quantum correlation vectors satisfy the no-signalling condition. However, the vector $x = (3/4, 3/4, 3/4)$ violates the first of these inequalities.

It is tempting to conjecture that an outcome x is a quantum correlation vector if it satisfies the nonsignalling conditions (6) and the corresponding vector y is a quantum correlation function. However, consider the vectors

$$x = (\frac{1}{2\sqrt{2}}, \frac{1}{2\sqrt{2}}, \frac{1}{2}, \frac{1}{2}, \frac{2+\sqrt{2}}{4}, \frac{2-\sqrt{2}}{4}, \frac{2-\sqrt{2}}{4}, \frac{2-\sqrt{2}}{4})$$

$$y = (\frac{2+\sqrt{2}}{4}, \frac{2-\sqrt{2}}{4}, \frac{2-\sqrt{2}}{4}, \frac{2-\sqrt{2}}{4}).$$

The outcome x satisfies (6), and y is a quantum correlation function, as shown by the vectors given in (5). Nevertheless, it is proved in [3] that x is not a quantum correlation vector. Perhaps even more surprising is an outcome they exhibit for the case $m = n = 3$:

$$\langle A_i \rangle = \langle B_j \rangle = \frac{1}{3} \quad 1 \le i, j \le 3$$

$$\langle A_1 \triangle B_1 \rangle = \langle A_2 \triangle B_2 \rangle = 0, \quad \langle A_i \triangle B_j \rangle = \frac{2}{3} \ \ \mathit{for\ all\ other}\ 1 \le i, j \le 3.$$

This gives an outcome x which satisfies (6) and for which the corresponding correlation function can even be obtained classically. For example with vectors

$$u^1 = v^1 = (0,0,0,1/3), \quad u^2 = v^2 = (0,0,1/3,0), \quad u^3 = (1/3,0,0,0), \quad v^3 = (0,1/3,0,0)$$

we have

$$\langle A_i \triangle B_j \rangle = \|u^i - v^j\|_1 \qquad 1 \le i < j \le 3.$$

and can use the L_1 characterization theorem given in the previous section.

Even though Tsirelson's theorem does not give a characterization of quantum correlation vectors, it can be extended to give a necessary condition that can be combined with the nosignalling condition.

Necessary conditions for quantum correlation vectors [2].
If $x = (\langle A_i \rangle, \langle B_j \rangle, \langle A_i \triangle B_j \rangle) \in [0,1]^M$ *is a quantum correlation vector then*

- x *must satisfy the nosignalling conditions (6), and*
- *There exist vectors* $u^i, v^j \in R^d$, $1 \le i \le m$, $1 \le j \le n$, $d \le m+n$, *for which*

  $$x_i = \|u^i\|, \quad x_{m+j} = \|v^j\|, \quad x_{ij} = \|u^i - v^j\|.$$

  where $\|u\| \equiv u^T u$.

Using this theorem, it can be shown that, in the two previous examples, the outcomes are not quantum correlation vectors. Although it is not known if the conditions above are sufficient, they do provide an efficient means of bounding the maximum quantum violation of general Bell inequalities by SDP. If the conditions are in fact sufficient, this bound would be tight, and the recognition and optimization problems for quantum correlation vectors would be solvable efficiently by SDP.

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