A Package for Triangulations*

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1 Introduction

Triangulations of a point set have been intensively studied in computational geometry in these years. Triangulations are very important not only from the viewpoint of their applications to computer graphics, numerical analysis but also from a fact that triangulations are yet another fundamental structure in discrete geometry like hyperplane arrangements.

Our group has developed many systems concerning triangulations. Specifically,

- minimum weight triangulation of a planar point set
- enumerating all triangulations of a planar point set
- uniform generation of triangulations of a planar point set
- enumeration of all regular triangulations of a point set in general dimensions

In this video, we demonstrate our results of the above-mentioned problems, mainly using animation.

2 Minimum weight triangulation

Kyoda [2] presents a branch-and-cut algorithm for computing the minimum weight triangulation of a planar point set. The branch-and-cut paradigm can solve hard integer programming problems by clever enumeration with powerful cutting planes. It power has been demonstrated for large traveling salesman problems.

In the branch-and-cut algorithm, first the minimum weight triangulation problem is formulated as finding a minimum weight set among all maximal independent (vertex) set of an intersection graph of edges connecting pairs of points. This is an integer linear programming problem, whose corresponding matrix is an incidence matrix of the intersection graph. Then, the β-skeleton is used to find many edges used in the minimum weight triangulation and to delete edges intersecting with them in the beginning. Then, cutting planes are produced by cliques of several vertices and dominating set conditions for maximal independent sets based on graph theory. Also, a cut produced by a geometric cluster is used.

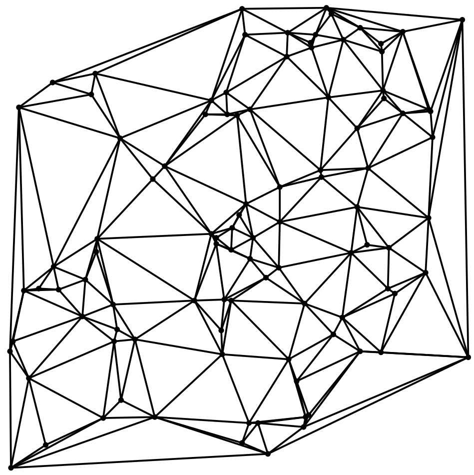

Figure 1: A minimum weight triangulation of 100 points

It can solve problems of size about 100 for points uniformly distributed in the square. A minimum weight triangulation of 100 points is shown in Fig.1. In the video, we show how this algorithm proceeds.

3 Enumerating all planar triangulations by reverse search

It is rather well know that, for planar triangulations of a given set of points, starting from any triangulation we can obtain another triangulation by flipping edges (diagonal flip) as in Fig.2, and by continuing to perform in a direction so that the flipped triangulation of related four points is the Delaunay triangulation of the four points, the Delaunay triangulation is obtained at the end. This shows that a graph formed by the adjacency relation induced by diagonal flips among planar triangulations is connected, and can be locally oriented so that it becomes a acyclic graph with a single root corresponding to the Delaunay triangulation.

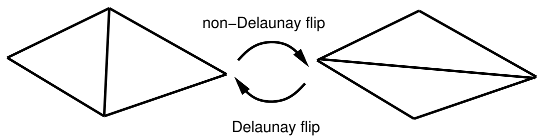

Figure 2: Diagonal flip: Delaunay and non-Delaunay flips

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Using this fact, together with reverse search technique by Avis and Fukuda [1], we can enumerate all the planar triangulations in an output-size sensitive and work-space efficient manner [2]. See Fig.3. In the video, we show the reverse search enumeration by animation.

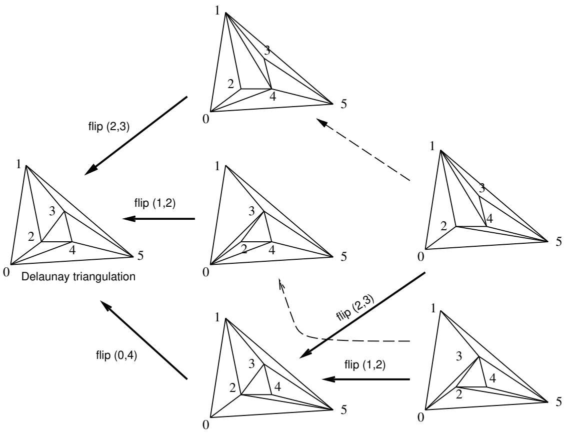

Figure 3: Reverse search for planar triangulations

4 Uniform generation of planar triangulations by a Markov chain

Adjacency of planar triangulation by diagonal flips as in Fig.3 can be used to generate a triangulation uniformly by a Markov chain. In the example there, by setting a transition probability to 1/6 for edges connecting different triangulations and a transition probability for a loop at each triangulation to 1 minus the summation of probabilities of incident edges, we obtain a doubly stochastic and aperiodic Markov chain. See Fig.4. It has a steady state probability which is uniform among triangles. Thus, this adjacency graph can be used to generate triangulations of a planar point set.

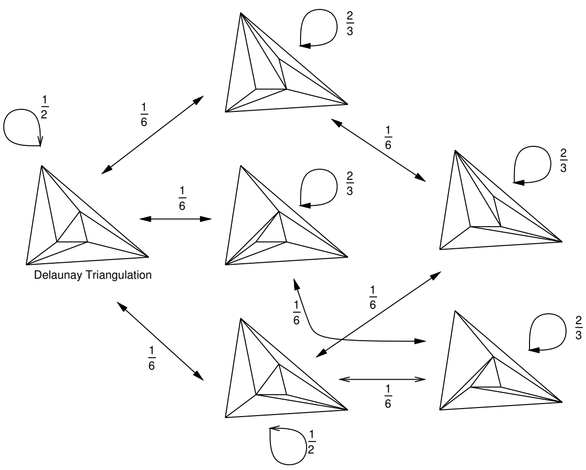

Figure 4: Markov chain for uniform generation

In Ono [4], the conductance of this Markov chain is partially investigated, especially for a convex polygon case, and it experimentally conjectures that by sampling one triangulation after $O(n^3)$ transitions a uniform generation is realized. For the convex case, by counting the number of triangulations by dynamic programming, a similar generation is possible, but it can not be generalized to non-convex general cases. Experimental investigations are done in [4].

5 Regular triangulations, secondary polytope and reverse search

Regular triangulations form a meaningful wide subclass of triangulations of points in general dimensions. By restricting our attention to this meaningful subclass of regular triangulations, algorithms for the planar case can be extended to the higher-dimensional case. Regular triangulation can be defined as a natural extension of the Delaunay triangulation as follows. For n points $v_1, ..., v_n$ given in an affine space $\mathbf{R}^{d-1}$, give a height w_i to each point v_i, lift up every point by the assigned height, and construct a convex hull of lifted points $(v_i, w_i) \in \mathbf{R}^d$. Then we obtain a regular triangulation induced by an assignment of heights $(w_1, ..., w_n)$ by projecting only lower facets of the convex hull into the original space $\mathbf{R}^{d-1}$ if all lower facets are simplices.

Masada, Imai and Imai [3] propose an output-size sensitive and work-space efficient algorithm to enumerate all regular triangulations by reverse search. The algorithm makes full use of the existing results on the so-called secondary polytope whose vertices correspond to regular triangulations (see [3]).

In the video, we present an animation of enumerating regular triangulations of 7 points in the 3-dimensional space. Although this is still a very limited example, it is seen that the number of triangulations (simplices) change unlike in the planar case. The secondary polytope in this case is a 3-dimensional polytope.

References

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