THE m-CORE PROPERLY CONTAINS THE m-DIVISIBLE POINTS IN SPACE

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ABSTRACT

A point x in R^d is an $m-$*divisible* point of a finite set of n points $S \subset R^d$ if x is contained in conv S_i , $i = 1, \ldots, m$, for some partition $S = S_1 \cup S_2 \cup \cdots \cup S_m$ of S. Let $D_m(S)$ denote the set of m-divisible points of S. We say x is in the $m-$*core* of S if each closed half space containing x also contains at least m points of S. Let $C_m(S)$ denote the set of points in the m-core. Reay, Sierksma and others have conjectured that conv$D_m(S) = C_m(S)$. In this note we give a counterexample for $n = 9$, $d = 3$, $m = 3$. The conjecture in known to be true for $d = 2$.

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Let S be a finite set of n points in R^d. A point x in R^d is an $m-$ *divisible* point of S if x is contained in convS_i , $i = 1, \ldots, m$, for some partion $S = S_1 \cup S_2 \cup \cdots \cup S_m$ of S. Such a partition is called a $m-$ *partition*, and when $m = 2$, a Radon partition. Tverberg[8] [9] proved that a m-partition exists for whenever $n = (m-1)(d+1)+1$. This settled a conjecture of Birch[1] who had proved it for the case $d = 2$. Let $D_m(S)$ denote the set of m-divisible points of S.

A point x is in the $m-$ *core* of S if each closed half space containing x also contains at least m points of S. Let $C_m(S)$ denote the set of points in the m-core. It is easily seen that $C_m(S)$ is the intersection of each closed half-space that contains at least $n-m+1$ points of S, and which is bounded by a hyperplane containing at least d points of S. It follows from Helly's theorem that the m-core is non-empty whenever $n = (m-1)(d+1)+1$.

It is easily seen that $D_m(S) \subset C_m(S)$. Reay[6], Sierksma[7] and others have conjectured that conv$D_m(S) = C_m(S)$. In this note we give a counterexample for $n = 9$, $d = 3$, $m = 3$. Reay[5] had already proved the conjecture for $d = 2$. Kolodziejczyk[4] has previously shown that the analogous conjecture for aligned spaces is false.

The counterexample consists of 9 points on the moment curve in R^3. For concreteness, we set

$$S = \{ (t, t^2, t^3) \mid t = 1, 2, \ldots, 9 \}.$$

The convex hull of S is known as a cyclic polytope. In order to enumerate the 3-partitions of S, we use a result of Breen[2], who has characterized primitive Radon partitions for cyclic polytopes. In R^3, Breen's result says that every primitive Radon partition of 5 points on the moment curve into sets A and B must alternate along the curve. That is the partition has the form A, B, A, B, A or B, A, B, A, B as we consider the points in the order they appear along the curve. Suppose we wish to find all 3-partitions of S into sets A, B, C. Clearly each pair of sets must contain a primitive Radon partition. We can therefore limit our consideration to all sequences of length 9 with elements chosen from $\{A, B, C\}$ satisfying Breen's condition. For example $A, B, C, B, A, B, C, A, B$ is a candidate since it contains the subsequences (A, B, A, B, A), (A, C, A, C, A) and (B, C, B, C, B). The sequence $A, B, A, C, B, A, C, B, C$ is not a candidate for a 3-partition since A and C do not form a primitive Radon partition.

A few simple observations greatly limit the number of candidate sequences that must be considered. Firstly, we need only consider sequences consisting of three each of A, B, C (3-3-3 candidate) or four A's, three B's and two C's (4-3-2 candidate). If a candidate is not a 3-partition, then neither is its reversal. In a 3-3-3 candidate which is a valid 3-partition, the sequence consists of three groups of three elements, each group being a permutation of A, B, C. The first three elements of a candidate can be assumed to be A, B, C in that order. If a candidate generates a 3-partition, then the reverse sequence does also, and may generate a distinct 3-divisible point.

From the list of candidates, it is a simple, but tedious, calculation to determine if they form a 3-partition. For this we used the symbolic computation language Maple. One of the advantages of Maple is that all calculations are performed exactly in rational arithmetic, and so there is no round off error. It turned out there were eight 3-partitions with seven distinct 3-divisible points. The partitions and their corresponding divisible points are as follows.

1. A, B, C, A, B, C, A, B, C	(5, 83/3, 165)
2. A, B, C, A, B, C, A, C, B	(77/17, 23, 2165/17)
3. A, B, C, B, A, C, B, C, A	(5, 28, 170)
4. A, B, C, B, A, C, B, A, C	(93/17, 551/17, 3465/17)
5. A, B, C, A, B, A, C, A, B	(9/2, 24, 291/2)
6. A, B, C, A, B, A, C, B, A	(5, 29, 185)
7. A, B, C, B, A, B, C, B, A	(5, 29, 185)
8. A, B, C, B, A, B, C, A, B	(11/2, 34, 449/2)

In order to compute the 3-core of S, we enumerate the planes containing 3 points of S for which the two open half-spaces determined by the plane contain 2 and 4 points of S respectively. The intersection of those closed half-spaces so obtained that contain 7 points of S generate the 3-core. The validity of this follows from the definition of 3-core above and the fact that all points on the moment curve are in general position. It is a simple matter using properties of the moment curve to list the required planes. Let $p_i = (i, i^2, i^3)$, $i = 1, \ldots, 9$. The plane through three such p_i partitions the moment curve into 4 parts. These parts lie in alternate half-spaces as we move along the curve. If we choose three points p_2, p_3, p_7 then p_1, p_4, p_5, p_6 lie in one open half-space and p_8, p_9 lie in the other. This plane is used to define the 3-core. The plane through p_2, p_5, p_8 is not used to define the 3-core, as the open half-spaces it bounds both contain 3 points of S. In this way 30 half-spaces were generated whose intersection is the 3-core of

S. Note that the plane through $p_s,\ p_t,\ p_u$ is given by:

$$(st + tu + su)\ x - (s + t + u)\ y + z = stu$$

Let l_{stu} (respectively, h_{stu}) denote the half-space generated by replacing $=$ with the inequality $\leq$ (respectively, $\geq$). Then the 3-core of S is defined by:

$$l_{127},\ l_{138},\ l_{148},\ l_{237},\ l_{248},\ l_{259},\ l_{347},\ l_{358},\ l_{369},\ l_{457},\ l_{468},\ l_{479},\ l_{567},\ l_{578},\ l_{589}$$

and

$$h_{125},\ h_{136},\ h_{147},\ h_{158},\ h_{169},\ h_{235},\ h_{246},\ h_{257},\ h_{268},\ h_{279},\ h_{345},\ h_{356},\ h_{367},\ h_{378},\ h_{389}$$

To disprove the conjecture, it suffices to give an extreme point of the 3-core that is not a 3-divisible point. We give three, together with the inequalities that are satisfied as equations at each of the extreme points:

(9/2, 91/4, 251/2)	$h_{356},\ h_{147},\ h_{279}$
(11/2, 131/4, 207)	$l_{369},\ l_{347},\ h_{158}$
(37/7, 31, 1357/7)	$h_{158},\ h_{279},\ h_{147}$

Again, all calculations were done using Maple.

The results given above were confirmed independently by Johanne Caron[3]. She wrote general purpose programs to compute the m-core and m-partitions for sets of points in R^3. In particular, for the point set S chosen above, she determined that the $3-core$ contains many redundant half-spaces. In fact only six half-spaces $h_{147},\ h_{158},\ l_{259},\ l_{347},\ h_{367},\ l_{369}$ define the 3-core.

Although the conjecture is false, perhaps something weaker is true. For example, is it true that at least one extreme point of the m-core is an m-divisible point? If not, is there any simple way to get an m-divisible point from the m-core? This could lead to an alternate proof of Tverberg's theorem. It would also be useful computationally. Currently there is no reasonably efficient way to compute a m-divisible point, even in R^3. The m-core on the other hand can be computed in a number of ways by generating its defining half-spaces, as above.

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