ON THE PARTITIONABILITY OF POINT SETS IN SPACE

(Preliminary Report)

David Avis

School of Computer Science
McGill University
805 Sherbrooke St. W.
Montreal, Canada, H3A 2K6

ABSTRACT

We consider the problem of partitioning sets of n points in d dimensions by means of k intersecting hyperplanes. We collect known results on this problem and give some new results. In particular, for $d=k=3$ it is known that a set in general position can be split into equal parts given any initial bisecting plane and two other carefully chosen planes. We show that this result does not extend to the case $d=k=4$. We also give bounds on the smallest integer $h(k)$ such that sets in $h(k)$-space can be partitioned by k hyperplanes into 2^k subsets of equal cardinality, partially answering a question raised by Paul Erdős.

1. Introduction

A fundamental problem in the area of database systems is the retrieval of data satisfying certain criteria. The simplest such problem is the retrieval of data based on a single key, and a comprehensive treatment of this problem is contained in Knuth[1]. A more difficult problem is the retrieval of data based on criteria for several keys, the so-called multidimensional search problem. Suppose that each of n records of a *file* contains a fixed number, d, of *keys*. A *query* asks for all records for which the d keys satisfy certain criteria. For example, upper and lower bounds may be specified for some or all keys, yielding the so-called *orthogonal range query problem*. This query problem can interpreted as finding all points in a given *d-dimensional* hyperrectangle. Suppose we are given upper bounds, u_i, and lower bounds l_i for each of the $i=1,2,\cdots,d$ keys. Then we are asked to find every record whose key $(x_1,\cdots,x_d)$ satisfies:

$$l_i \le x_i \le u_i \qquad i=1,\ldots,d.$$

A more general query could consist of a linear function: $a_1x_1+a_2x_2+\cdots+a_dx_d+a_{d+1}$. The *half-space query problem* is to find all records whose keys satisfy:

$$a_1x_1+a_2x_2+\cdots+a_dx_d+a_{d+1}\le 0.$$

Even more generally, we may be given a set of k such linear functions, which we represent in matrix form:

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1d} \\ \vdots & & \vdots \\ a_{k1} & \cdots & a_{kd} \end{bmatrix}, \quad b = \begin{bmatrix} a_{1,d+1} \\ \vdots \\ a_{k,d+1} \end{bmatrix}$$

Letting $x=(x_1, x_2, \ldots, x_d)$ be the *d-vector* of keys for a record, we search for all such vectors satisfying

$$Ax + b \le 0.$$

Such a query is called a *polyhedral range query*. For these types of queries, a new space partitioning approach has been introduced by Willard [2] for 2-dimensions, and generalized by Yao [3] for 3-dimensions. These results have been improved by Dobkin and Edelsbrunner[4]. For the half-space query problem, these results lead to sublinear algorithms, that is, algorithms that run in time $O(k+n^t)$ with $t<1$, where k is the number of points in the output. Avis [5] has shown that the necessary geometric results do not hold in dimension 5 and higher. In this paper we collect all of the relevant geometric results on the partitionability of sets of points in space. The proofs of theorems 6, 7 and 8 appear here for the first time.

2. Geometrical Results

In this section we discuss the geometrical results which are the basis for fast search algorithms that use polyhedron trees. We begin with some terminology.

Let S be a set of n points in R^d and let $0 < \alpha < 2^{-d}$. We say that S is *α-partitionable* if there exist a set of d intersecting hyperplanes that partition S so that no open region contains more than αn points of S. We say that S is *non-partitionable* if for every partition of S by d intersecting hyperplanes, some set of $2^d - 1$ open regions together contain all the points of S. A point x in space is called a *centre* of S if every hyperplane H through x contains at least $\frac{n}{(d+1)}$ points in each half-space bounded by H. Two classical results are the Centre Theorem and the Ham Sandwich Theorem.

Theorem 1. Centre Theorem[6]

Every set S in R^d has a centre.

Theorem 2. Ham-Sandwich Theorem[7]

Given d disjoint sets of points in R^d, there is a hyperplane which simultaneously divides each point set evenly.

From theorem 2, we get the immediate corollary:

Corollary 2.1[2]

Every set S in the plane can be $\frac{1}{4}$ - partitioned. Furthermore one of the partitioning lines may be chosen to be an arbitrary bisector of S.

Proof: Choose a line l that bisects S. This can easily be found by choosing the median point in one of the coordinate directions and taking a line parallel to the appropriate coordinate axis. This divides the set S into two disjoint sets, to which theorem 2 may be applied.

Corollary 2.1 thus shows that a perfectly balanced polygon tree can always be constructed. This justifies the sublinear half-plane algorithm described in the previous section. Before we move to three dimensions, another 2-dimensional result is required.

Lemma 1[3]

Let R and B be two n point sets in the plane with a common centre. Then R and B can be simultaneously $\frac{1}{12}$ - partitioned by two lines.

It can now be shown that all sets in R^3 can be partitioned.

Theorem 3[3]

Every set S in 3-dimensions can be $\frac{1}{24}$ - partitioned. Furthermore, one of the partitioning planes may be chosen to be any bisector of S.

Proof: Let H be any bisecting plane through S, splitting S into sets S_1 and S_2. Let x_1 be a centre for S_1 and x_2 be a centre for S_2. Map every point of S onto H by projecting parallel to the line through x_1 and x_2. This gives rise to sets R and B of size $\frac{n}{2}$ that satisfy the conditions of lemma 1. Therefore R and B can be $\frac{1}{12}$ - partitioned by two lines l_1 and l_2. Let H_i be the plane containing l_i, x_1 and x_2, $i = 1, 2$. Then H, H_1, H_2 form a $\frac{1}{24}$ - partition of S.

This result is sufficient to get an $O(n^{.98} + k)$ algorithm for half-space queries in three dimensions. To get the faster algorithm described in section 2, we require the following stronger and more difficult result.

Theorem 4[8] [4]

Every set S in 3-dimensions can be $\frac{1}{8}$ - partitioned.

The proof of theorem 4 is rather technical and is not included here. Apparently theorem 4 can be strengthened to allow one hyperplane to be chosen as an arbitrary bisector of S [8].

It is natural to ask whether the above results generalize to higher dimensions. The following result shows that in dimension 5 and higher the answer is negative. Let $\delta_d = d \ mod \ 2$.

Theorem 5[5]

For any integer n, and $d \geq 5$, there exist non-partitionable sets of cardinality n. Furthermore, sets exist for which every partition by d hyperplanes leaves at least $2^d - d^2 - \delta_d$ open regions with no points of S.

Proof. We outline the proof here as the point set constructed will be used in the proof of later results. For any integer t, let $s_t = (t, t^2, ..., t^d)$. For any integer n we set

$$S = \{ s_t : 1 \leq t \leq n \}. \qquad (1)$$

These sets play a fundamental role in the theory of convex polyhedra, as they form the vertices of the *cyclic polytopes*. An accessible introduction to these fascinating polytopes is the paper by Gale[9]. Let

$$f(x_1, ..., x_d) = \sum_{i=1}^{d} a_i x_i + a_{d+1}.$$

We denote by f the hyperplane $f(x_1, x_2, ..., x_d) = 0$ in d-space. The set S is partitioned by f according to the sign of the function

$$g(t) = \sum_{i=1}^{d} a_i t^i + a_{d+1}.$$

Integers t for which $g(t)$ is positive correspond to points s_t in S that lie in one of the open half-spaces bounded by f, and integers t for which $g(t)$ is negative correspond to points in the other open half-space. Let H and L, respectively denote these point sets. Integers for which $g(t) = 0$ correspond to points of S on f. Such points are denoted by M. Since g is a polynomial of degree d, it has some number k, which is at most d, of real roots. Therefore f can contain at most d points of S. If k is zero, then all points of S lie on the same side of f. If k is positive, let $t_1 \leq t_2 \leq \cdots \leq t_k$ denote the real roots of g. Then

$$M = \{ t_1, ..., t_k \},$$

and H and L contain $k + 1 \leq d + 1$ strings of consecutive points of S. If we consider the d partitioning hyperplanes together, it can be shown using this fact[5] that at most $d^2 + 1$ of the 2^d open regions contain points of S. In fact, if d is even, only d^2 regions can contain points. This outlines the proof of theorem 5.

In four dimensions, it is not known whether an analogue of corollary 2.1 and theorem 4 holds. In the strengthened form they definitely do not hold.

Theorem 6

For any $\alpha > 0$ and any n, there exists a set S in R^4, of cardinality n, that cannot be α - partitioned by a prescribed bisecting hyperplane and any three other hyperplanes.

Proof. We observe that $d = 4$ implies $d^2 = 2^d$, and so theorem 5 does not yield anything of interest. However the situation is very "tight" in the following sense. If we follow the argument of theorem 5 we see that the set S defined by (1) with $d = 4$ can be partitioned if and only if there exists a binary matrix of dimension 4×16 with the following properties:

P1: Every column of A is distinct.

P2: Each row of A consists of at most 5 strings of zeroes and ones.

P3: Two consecutive columns of A differ in exactly one row.

Properties P_1 and P_3 define a Gray code. Gilbert[10] has published a list of all Gray codes of length 16. Of the 9 such codes, one satisfies P_2. It is:

$$A = \begin{bmatrix} 0&1&1&0&0&0&1&1&1&1&1&1&0&0&0&0 \\ 0&0&1&1&1&1&1&1&0&0&0&1&1&0&0&0 \\ 0&0&0&0&1&1&1&1&1&1&0&0&0&0&1&1 \\ 0&0&0&0&0&1&1&0&0&1&1&1&1&1&1&0 \end{bmatrix}.$$

This matrix determines how the four intersecting hyperplanes must partition S. Suppose S contains 16 points. Row 1 of A can be interpreted as saying that points $s_1, s_4, s_5, s_6, s_{13}, s_{14}, s_{15}$, and s_{16} must lie above hyperplane H_1. Row 2 can be similarly interpreted for H_2, etc. Following this prescription, a separating set of four hyperplanes can be contructed. But suppose we begin with a bisecting hyperplane H such that $s_1, s_3, s_5, ..., s_{15}$ lie above H and $s_2, s_4, ..., s_{16}$ lie below H. Such a pattern does not appear in A (or any cyclic permutation of the columns of A). Since A is essentially unique, this shows that H cannot be used in any partition of S by 4 hyperplanes. A similar argument can be shown for arbitrary n, demonstrating theorem 6.

As a final result of this type, we show that lemma 1 cannot be generalized to 3-dimensions.

Theorem 7

There exist two n point sets R and B in 3 dimensions, with a common centre, such that R and B are not simultaneously partitionable.

Proof: (Outline)

Consider the set S and hyperplane H described in the proof of theorem 6. Let S_1 and S_2 be the sets separated by H. Let x_1 and x_2 be points in the centres of S_1 and S_2. Project S_1 and S_2 onto H parallel to the line through x_1 and x_2 giving R and B respectively. Now suppose that R and B could be simultaneously partitioned by three planes. Then as in lemma 1, this partition could be extended to a partition of S in 4-dimensions by 4 hyperplanes, one of which was H. But by the remarks preceding the theorem, this is impossible. Hence R and B may not be simultaneously partitioned.

The preceding results are negative and tell us that the algorithms for 2 and 3 dimensions may not be extended easily to high dimensions. Theorem 6 tells us that sets in 5-dimensional space may not be separable by 5 hyperplanes. Paul Erdős asked the following question[11]. Let k be any positive integer, and let $h(k)$ be the smallest dimension such that every general position point set of cardinality n in $h(k)$ - dimensional space can be partitioned by k hyperplanes into 2^k regions each containing at least $\lfloor \frac{n}{2^k} \rfloor$ points of S. Is $h(k)$ finite, and if so, what is $h(k)$? The following theorem gives a partial answer to this question.

Theorem 8

$$\frac{2^k - 1}{k} \leq h(k) \leq 2^{k-1} .$$

Proof:

For the lower bound, we argue as in the proof of theorem 5. Let d, S, f, g be as defined there, and suppose we can partition S by k hyperplanes into t non-empty regions. Then since each polynomial g has at most d real roots, we have

$$dk + 1 \geq t \qquad (2)$$

setting $t = 2^k$ gives the lower bound.

For the upper bound we appeal to theorem 2 (Ham Sandwich Theorem). The bound is true for $k=2$ by Corollary 2.1. We assume inductively that it it true for all integers up to some k. Set $d = 2^k$. By induction, every set in d-dimensions can be partitioned into d regions each containing at least $\lfloor \frac{n}{d} \rfloor$ points, by k intersecting hyperplanes. By theorem 2, these d regions can be simultaneously bisected by a $k+1$-st hyperplane. Hence the bound holds for $k+1$.

3. Final Remarks

Many open problems remain. Firstly, can every set of points in general position in 4 dimensions be partitioned by 4 intersecting hyperplanes. Secondly, the non-partitionable data set constructed in dimensions 5 and above is highly pathological. One wonders if in practice, most data sets in dimensions 5, or 6 may be partitionable. There also remains the problem of finding fast partitioning algorithms, and the problem of updating the data structure. Perhaps there is a role for heuristics here. Finally, there is the theoretical question of obtaining better bounds on the function $h(k)$ described at the end of the last section.

The author has recently learned that Yao and Yao[12] have proved that point sets in d-space can be partitioned into 2^d equal cardinality regions by $O(2^d)$ hyperplanes, such that no query hyperplane intersects all of the regions. For fixed d, this gives a sublinear algorithm for the half-space query problem. An application of inequality (2) shows that at least $\frac{2^d-1}{d}$ hyperplanes are necessary to perform such a partition. The author has also learned that Michael Patterson has another argument to show that point sets in 5-space cannot be partitioned by 5 hyperplanes into equal cardinality regions.

4. Acknowledgements

Many of the results reported in this paper were obtained whilst the author was a visiting researcher at the University of Tokyo, supported by a grant from the Japanese Society for the Promotion of Science. The author expresses his thanks to Professor Masao Iri for providing such a stimulating research environment. The author is also indebted to David Dobkin and Frances Yao for helpful information.

References

1. D. E. Knuth, *The Art of Computer Programming, Vol 3: Sorting and Searching*, Addison-

Wesley, Reading, MA, 1973.

2. D. E. Willard, "Polygon Retrieval," *SIAM Journal on Computing*, vol. 11, pp. 149-165, 1982.

3. F. Yao, "3-Space Partition and its Applications," in *Proceedings of the 15th STOC*, pp. 258-263, A.C.M., Boston, Massachusets , 1983.

4. D.P. Dobkin and H. Edelsbrunner, "Space Searching for Intersecting Objects," in *Proceedings of the 25th FOCS*, pp. 387-391, IEEE, Ocober, 1984.

5. D. Avis, "Non-Partitionable Point Sets," *Information Proc. Letters*, vol. 19, pp. 125-129, 1984.

6. I.M. Yaglom and V.G. Boltyanskii, *Convex Figures*, Holt, Rinehart and Winston, 1961.

7. B. Mendelson, *Introduction to Topology*, Allyn & Bacon, Boston, 1962.

8. F.F. Yao, D. Dobkin, and H. Edelsbrunner, *Manuscript in preparation*, 1985.

9. D. Gale, "Neighborly and Cyclic Polytopes," *Proceedings of the Symp. in Pure Mathematics*, vol. 7, pp. 225-232, 1963.

10. E. N. Gilbert, "Gray Codes and Paths on the n-cube," *Bell System Technical Journal*, pp. 815-826, 1958.

11. P. Erdős, *Private Communication*, Kyoto, 1984.

12. F.F. Yao and A. C. Yao, *Proc. 17th S.T.O.C.*, to appear, 1985.