On Minimal 5-Chromatic Triangle-Free Graphs

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ABSTRACT

It is shown that the minimum number of vertices in a triangle-free 5-chromatic graph is at least 19.

Mycielski [5] has constructed a sequence of graphs that are triangle-free and k-chromatic ($k = 1, 2, 3, \ldots$). If we denote by $f(k)$ the minimum number of vertices in a triangle-free k-chromatic graph, then this construction yields the bounds.

$$f(k) \leqslant 2^k - 2^{k-2} - 1, \qquad k = 2, 3, 4 \ldots.$$

A nonconstructive theorem of Erdös [3] shows that in fact

$$f(k) < c(k \log k)^2.$$

For lower bounds, Chvátal [2] observes that Brook's theorem and the fact that $f(4) = 11$ implies

$$f(k) \geqslant \binom{k+2}{2} - 4.$$

He also shows that Mycielski's construction gives the unique 11 vertex graph with the required properties. For $f(5)$ these bounds give $17 \leqslant f(5) \leqslant 23$. In this note we show that $f(5) \geqslant 19$.

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Let $\mathcal{G}$ denote the set of edge-maximal triangle-free graphs with 18 vertices. We show that all graphs in $\mathcal{G}$ are 4-colorable. For a graph G, let $\alpha(G)$ denote the size of the largest stable set and let $\Delta(G)$ denote the maximum degree in G. The proof relies on the following result from Ramsey theory [4, p. 16]: every triangle-free graph with 18 vertices has a stable set of size at least 6. In addition, Brook's theorem [4, p. 128] implies that a necessary condition for a graph G in $\mathcal{G}$ to be 5-chromatic is that $\Delta(G) \geq 5$.

We begin by eliminating the cases $\alpha(G) \geq 7$ and $\Delta(G) \geq 6$. We will often make use of the above mentioned fact that the Mycielski graph, M, is the unique 4-chromatic triangle-free graph with 11 vertices. We will make use of the labeling and coloring of M given in Figure 1.

Lemma 1. If $G \in \mathcal{G}$ and $\alpha(G) \geq 7$, then G is 4-colorable.

Proof. Let A be a stable set of maximum cardinality in G. We may assume that $|A| = 7$ or else $G - A$ is 3-colorable and we are done. Thus $|G - A| = 11$ and again is either 3-colorable or M. In the latter case we use the coloring of Figure 1 and color A as follows: if $x \in A$ and x is not adjacent to v_1, then x gets color 1; otherwise, x is adjacent to at most two of $\{v_7, v_8, \ldots, v_{11}\}$ and none of $\{v_2, v_3, \ldots, v_6\}$ and it gets the unused color. ∎

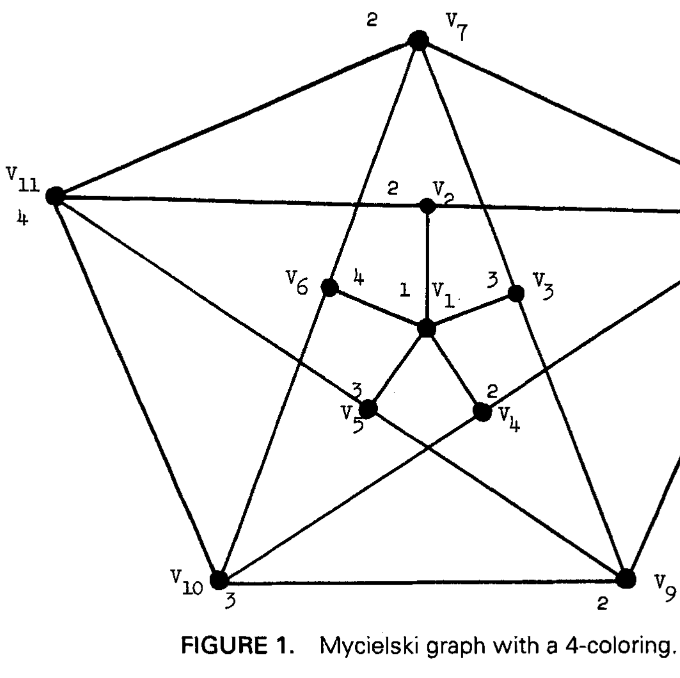

FIGURE 1. Mycielski graph with a 4-coloring.

Lemma 2. If $G \in \mathscr{G}$ and $\alpha(G) = \Delta(G) = 6$, then G is 4-colorable.

Proof. Let x be a vertex of degree 6 and let A denote the set of vertices given by x and its neighbors. As in Lemma 1, we need only consider the case when $G - A$ is M. Since $\alpha(G) = 6$, v_1 must be adjacent to a neighbor y of x. Now y can be adjacent to at most two of $\{v_7, v_8, \ldots, v_{11}\}$ and none of $\{v_2, \ldots, v_6\}$. Since $\Delta(G) = 6$, v_1 is adjacent to no other vertices in A. Thus y receives an unused color, the other neighbors of x all receive color 1, and x can be suitably colored. ∎

Theorem 1. If $G \in \mathscr{G}$, then G is 4-colorable.

Proof. By the preceding remarks and lemmas, we need only consider the case where $\alpha(G) = 6$ and $\Delta(G) = 5$. Let A be a stable set of cardinality 6 and let x be a vertex that is adjacent to the maximum number of vertices in A. It is easy to see that x must be adjacent to at least three vertices in A. Otherwise, since the edge-maximality of G implies that each pair of vertices in A has a common neighbor, G would have at least 21 vertices, a contradiction. Further, $G - A - \{x\}$ must be M. If it were not, it would be 3-colorable, we could assign the fourth color to A, and x could always be legally colored since degree considerations ensure that it is adjacent to at most two vertices of M. Let k be the number of vertices in A adjacent to x, and let T be the set of vertices in A not adjacent to x. Now x must have a common neighbor with each of the $6 - k$ vertices in T, but x can be adjacent to at most $5 - k$ additional vertices. Therefore there exists some neighbor y of x that is adjacent to at least two vertices of T. But y is also in M and so y has degree at least 6, a contradiction. ∎

We have shown that the smallest triangle-free 4-chromatic graph has at least 19 vertices. We note that Christofides [1, p. 60] gives an erroneous example of a 5-chromatic triangle-free graph with 16 vertices.

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References

[1] N. Christofides, *Graph Theory: An Algorithmic Approach.* Academic, New York (1975).

[2] V. Chvátal, The minimality of the Mycielski graph. *Graphs and Combinatorics*. Springer-Verlag, Berlin (Lecture Notes in Mathematics 406) (1973) 243–246.

[3] P. Erdös, Graph theory and probability. *Canad. J. Math.* 11 (1959) 34–38.

[4] F. Harary, *Graph Theory*. Addison-Wesley, Reading, Mass. (1969).

[5] J. Mycielski, Sur le coloriage des graphes. *Coll. Math.* 3 (1955) 161–162.