On the Complexity of Testing Hypermetric, Negative Type, k-gonal and Gap Inequalities

David Avis
Computer Science, Mcgill University and GERAD
3480 University, Montreal, Quebec, Canada H3A 2A7
avis@cs.mcgill.ca
May 28, 2003

Abstract

Hypermetric inequalities have many applications, most recently in the approximate solution of max-cut problems by linear and semidefinite programming. However, not much is known about the separation problem for these inequalities. Previously Avis and Grishukhin showed that certain special cases of the separation problem for hypermetric inequalities are NP-hard, as evidence that the separation problem is itself hard. In this paper we show that similar results hold for inequalities of negative type, even though the separation problem for negative type inequalities is well known to be solvable in polynomial time. We also show similar results hold for the more general k-gonal and gap inequalities.

1 Introduction

Let $b = (b_1, ..., b_n)$ be an integer vector, let $k = \sum_{1 \le i \le n} |b_i|$ and $s = \sum_{1 \le i \le n} b_i$. Let $x = (x_{ij}), 1 \le i < j \le n$ be a vector in $R^{\binom{n}{2}}$. We say that b defines a *k-gonal inequality*:

$$Q(b,x) = \sum_{1 \le i < j \le n} b_i b_j x_{ij} \le \lfloor \frac{s^2}{4} \rfloor \tag{1}$$

Since the vector $-b$ generates the same inequality as the vector b, we will assume throughout the paper that $s \ge 0$. The inequality (1) is called *hypermetric* if $s = 1$, in which case k is necessarily odd. It is called *negative type* if $s = 0$, in which case k is even. It is called *pure* if $b_i \in \{\pm 1, 0\}$. Inequalities of type (1) have been well studied and rediscovered many times. The hypermetric inequalities appear in Deza [3], and the negative type inequalities in the work of Cayley. The book of Deza and Laurent [5] collects a wealth of information about them and their applications.

For fixed n, it is easy to show that the cone formed by the negative type inequalities is not polyhedral. However, Deza, Grishukhin and Laurent [4] showed that the hypermetric inequalities do form a polyhedral cone. Furthermore, each $2k+2$-gonal inequality can be obtained by a non-negative combination of $2k+1$-gonal inequalities. This was proved by Deza [3] for negative type inequalities, and for the general k-gonal inequalities by Avis and Umemoto [2].

The k-gonal inequalities have application in both linear programming (LP) [2] and semidefinite programming (SDP) [8] relaxations of the well known MAX_CUT problem. The hypermetric inequalities form a good approximation of the cut polytope, and in fact many induce facets. The SDP relaxation essentially consists of optimizing over the feasible region of all k-gonal inequalities (1) where the floor function is omitted. The efficiency of the SDP relaxation depends on the fact that separation for this feasible region can be performed in polynomial time (see, e.g. Section 28.4.1 in [5]).

In order to use hypermetric inequalities effectively, it is necessary to be able to test whether a vector x satisfies all of them, and if not, to find a violated inequality. This is the *separation* problem for hypermetric inequalities. Its complexity status is unknown, although it is known that *membership* testing for hypermetric inequalities is in the class co_NP [1]. The paper [1] also contains the following complexity results:

P1. Hypermetric (2m+1)-gonality testing
Instance: A rational vector x of length $\binom{n}{2}$ and an integer m.
Question: Does x satisfy all $(2m+1)$-gonal hypermetric inequalities?
Complexity: Co-NP complete.
Comments: Remains co-NP complete for testing if x satisfies all pure $(2m+1)$-gon hypermetric inequalities.

P2. Strong hypermetricity
Instance: A rational vector x of length $\binom{n}{2}$.
Question: Is x hypermetric? If not, give smallest k such that x violates a $(2k+1)$-gonal hypermetric inequality.
Complexity: NP-hard.

These results may seem to give evidence to the claim that the separation problem for hypermetric inequalities is hard (see, eg. [5], p. 454). However, in the next section we prove that the following similar results apply to negative type inequalities.

P1N. Negative type 2m-gonality testing
Instance: A rational vector x of length $\binom{n}{2}$ and an integer m.
Question: Does x satisfy all $2m$-gonal negative type inequalities?
Complexity: co-NP complete.
Comments: Remains co-NP complete for testing if x satisfies all pure $2m$-gonal negative type inequalities.

P2N. Strong negative type
Instance: A rational vector x of length $\binom{n}{2}$.

Question: Is x of negative type? If not, give smallest m such that x violates a $2m$-gonal negative type inequality.

Complexity: NP-hard.

For fixed n, the cone of negative type inequalities is a linear transformation of the cone of positive semidefinite matrices, for which separation can be performed in $O(n^3)$ time (see, eg. [10]). Therefore separation for negative type inequalities can be performed in $O(n^3)$ time also. These results, therefore, cast some doubt on the evidence that hypermetric separation is hard. We can also show similar results for testing k-gonality.

P1K. Testing k-gonality

Instance: A rational vector x of length $\binom{n}{2}$ and an integer k.

Question: Does x satisfy all k-gonal inequalities?

Complexity: co-NP complete.

P2K. Strong k-gonality

Instance: A rational vector x of length $\binom{n}{2}$.

Question: Is x k-gonal? If not, give smallest m such that x violates a k-gonal inequality.

Complexity: NP-hard.

Laurent and Poljak [9] introduced a set of inequalities, called *gap inequalities*, that can be stronger than the k-gonal inequalities when $s \geq 2$. The gap $g = g(b)$ of an integer sequence $b_1, \ldots, b_n$ is defined by

$$g = g(b) = min_{S \subseteq \{1,2,\ldots,n\}} |\sum_{i \in S} b_i - \sum_{i \notin S} b_i|. \tag{2}$$

It is easy to see that for a negative type inequality, the b vector has gap zero and for a hypermetric inequality it has gap one. A gap inequality if formed by modifying the right hand side of (1):

$$Q(b,x) = \sum_{1 \leq i < j \leq n} b_i b_j x_{ij} \leq \frac{1}{4}(s^2 - g^2). \tag{3}$$

Since s and g have the same parity, the right hand side is always integral. We call a gap inequality k-gonal If $k = \sum_{i=1}^{n} |b_i|$, we call (3) a $k-$ *gonal gap inequality*.

The gap inequalities are also valid for the MAX_CUT problem. Very little is known about gap inequalities. For example, it is not known if they form facets of the cut polytope, if they form a polyhedron for each n, or if separation can be efficiently performed. Here we show that complexity results similar to those above can be obtained.

P1G. Testing k-gonal gap inequalities

Instance: A rational vector x of length $\binom{n}{2}$ and an integer k.
Question: Does x satisfy all k-gonal gap inequalities?
Complexity: co-NP complete.

P2G. Strong gap testing
Instance: A rational vector x of length $\binom{n}{2}$.
Question: Does x satisfy all gap inequalities. If not, give smallest m such that x violates a k-gonal gap inequality.
Complexity: NP-hard.

2 Preliminary Results

The proofs follow similar lines to the proofs of P1 and P2 in [1], see also Section 28.3 of [5]. We make use of the following problems known to be hard.

P3. Complete Bipartite Subgraph
Instance: Graph G on n vertices and an integer m.
Problem: (a) Does G contains an induced complete bipartite subgraph $K_{m,m}$?
(b) Does G contains an induced complete bipartite subgraph $K_{m,m+1}$?
Complexity: Both NP-complete. Reduction from independent set [6].

P4. Largest Complete Bipartite Subgraph
Instance: Graph G on n vertices.
Problem: (a) Find largest m such that G contains an induced $K_{m,m}$ subgraph.
(b) Find largest m such that G contains an induced $K_{m,m+1}$ as an induced subgraph.
Complexity: Both NP-hard. Reduction from independent set [6].

Let G be an undirected graph with vertices $V = V(G)$ and edges $E = E(G)$. Construct the edge weights

$$x^t_{ij}(G) = \begin{cases} 1 & \text{if } (ij) \in E \\ 1+t & \text{if } (ij) \notin E \end{cases} \tag{4}$$

Let b be an integer vector of length n and define

$$k = k(b) = \sum_{i=1}^{n} |b_i|, \qquad s = s(b) = \sum_{i=1}^{n} b_i, \tag{5}$$

which implies that

$$\sum_{i \in V_+(b)} b_i = (s+k)/2, \sum_{i \in V_-(b)} b_i = (s-k)/2. \tag{6}$$

Let $Q_G(b, x^t)$ be the left hand side of the inequality (1) with $x^t = x^t(G)$. We calculate $Q_G(b, x^t)$. We set

$$V_+(b) = \{i : b_i > 0\},\ V_-(b) = \{i : b_i < 0\},\ V(b) = V_+(b) \cup V_-(b)$$

$$n_+ = |V_+(b)|,\ n_- = |V_-(b)|,\ n_b = n_+ + n_-.$$

We denote by $G(b)$ the subgraph of G induced on the set $V(b)$. Let K_{n_+,n_-} be the complete bipartite graph on the set $V(b)$ with the partition $(V_+(b), V_-(b))$. Let

$$E_b(G) = E(G(b)) \bigtriangleup E(K_{n_+,n_-}),$$

where $\bigtriangleup$ denoted the symmetric difference of 2 sets.

Lemma 2.1 *Let G be a graph and let b and $E_b(G)$ be defined as above. Then*

$$Q_G(b, x^t) = \frac{s^2}{2} + \frac{t}{4}(s^2 + k^2) - \frac{1+t}{2}\sum_{i=1}^{n} b_i^2 - t \sum_{(ij)\in E_b(G)} |b_i||b_j| \tag{7}$$

Proof. Suppose at first that $E_b(G) = \emptyset$, i.e. the set $V(b)$ induces a complete bipartite graph $K(b)$. From the definitions we have

$$Q_G(b, x^t) = Q_{K_{n_+,n_-}}(b, x^t) = \sum_{i\in V_+, j\in V_-} b_i b_j + (1+t)\Big(\sum_{i,j\in V_+, i<j} b_i b_j + \sum_{i,j\in V_-, i<j} b_i b_j\Big).$$

Since for any set X

$$\sum_{i,j\in X, i<j} b_i b_j = \frac{1}{2}\Big(\big(\sum_{i\in X} b_i\big)^2 - \sum_{i\in X} b_i^2\Big),$$

using (6) we obtain

$$Q_{K_{n_+,n_-}}(b, x^t) = \frac{(s+k)(s-k)}{4} + \frac{1+t}{2}\left(\frac{(s+k)^2}{4} + \frac{(s-k)^2}{4} - \sum_{i=1}^{n} b_i^2\right).$$

After simplification, we obtain

$$Q_{K_{n_+,n_-}}(b, x^t) = \frac{s^2}{2} + \frac{t}{4}(s^2 + k^2) - \frac{(1+t)}{2}\sum_{i=1}^{n} b_i^2$$

which are the first 3 terms of (7).

If $E_b(G) \neq \emptyset$, then it may contain two types of edges: those in G between vertices both in either V_+ or V_-, or edges in K_{n_+,n_-} not in G. In both cases, the right hand side of the equality (2) obtains additional negative summand

$$-t \sum_{(i,j)\in E_b(G)} |b_i||b_j|$$

and we are done. ∎

We remark that for a pure b vector, with $k \leq n$, the equality (7) takes the simple form

$$Q_G(b, x^t) = \frac{s^2}{2} + \frac{t}{4}(s^2 + k^2) - \frac{1+t}{2}k - t|E_b(G)|. \tag{8}$$

We also have the immediate:

Lemma 2.2 *Let b be an integer vector of length n, and let k and s be defined as in (5). Then*

$$Q_G(b, x^t) \leq \frac{s^2}{2} + \frac{t}{4}(s^2 + k^2) - \frac{1+t}{2}k \tag{9}$$

with equality if and only if $G(b) = K_{n_+, n_-}$.

We first consider the k-gonal inequalities of negative type, ie. the situation when $s = 0$, k is even, and the right hand side of (1) is zero.

Lemma 2.3 *Let $n \geq 2m \geq 6$. Set $x^t = x^t(G)$, where $t = \frac{m^2+1}{m^2(m-1)}$. Then*

(a) x^t satisfies all k-gonal negative type inequalities with $k < 2m$.

(b) x^t satisfies all $2m$-gonal negative type inequalities, except when G contains $K_{m,m}$ as an induced subgraph. In this case only the pure $2m$-gonal negative type inequality is violated.

(c) $\frac{x^t}{n^2}$ satisfies all k-gonal inequalities with $k \leq n$ and $s \geq 2$.

Proof. If $k < 2m$, then it is easy to verify that $k \leq \frac{2}{t}$, hence $\frac{k^2 t}{4} \leq \frac{k}{2}$. Using Lemma 2.2 with $s = 0$, we see that $Q_G(b, x^t) \leq 0$, proving (a).

For part (b), first consider the case where $G = G(b) = K_{m,m}$, $k = 2m$, $s = 0$, and b is a pure $2m$-gonal inequality. From Lemma 2.2 we have that

$$Q_G(b, x^t) = m^2 t - mt - m = (m^2 - m)\frac{m^2+1}{m^2(m-1)} - m = \frac{1}{m} > 0. \tag{10}$$

Therefore this pure negative type inequality is violated by x^t. It is easy to check that $t > \frac{1}{m}$. If G is not $K_{m,m}$, then $|E(b)| \geq 1$, and as we saw in the proof of Lemma 2.1, $Q_G(b, x^t)$ will be reduced by at least t so it becomes negative. Therefore the inequality holds in this case.

Now let $G = K_{m,m}$, and assume b is $2m$-gonal but not pure. In this case $|b_i| \geq 2$ for some i, and so $\sum_{i=1}^n b_i^2 \geq 2m + 2$. Now from (7) with $s = 0$ and $k = 2m$ we obtain

$$Q_G(b, x^t) \leq m^2 t - \frac{1+t}{2}(2m+2) = m^2 t - mt - m - t - 1 < 0.$$

Therefore x^t satisfies all $2m$-gonal negative type inequalities that are not pure.

For part (c), we observe that when $k \leq n$

$$\sum_{1 \leq i < j \leq n} |b_i||b_j| \leq \frac{1}{2}(\sum_{i=1}^{n} |b_i|)^2 = \frac{k^2}{2} \leq \frac{n^2}{2},$$

Now $s \geq 2$, and since $m \geq 3$, $x^t \leq 2$, so

$$Q_G(b, \frac{x^t}{n^2}) = \sum_{1 \leq i < j \leq n} b_i b_j \frac{x^t_{ij}}{n^2} \leq \sum_{1 \leq i < j \leq n} |b_i||b_j| \frac{2}{n^2} \leq 1 \leq \lfloor \frac{s^2}{4} \rfloor.$$

This completes the proof of the lemma. ∎

The following similar lemma applies to hypermetric inequalities and parts (a) and (b) appeared in [1] (see also [5], Section 28.3.) Part (c) can be proved as in Lemma 2.3 (c).

Lemma 2.4 *Let $n \geq 2m+1 \geq 5$. Set $x^t = x^t(G)$, where $t = \frac{m^2+1}{m^3}$. Then*

(a) x^t satisfies all k-gonal hypermetric inequalities with $k < 2m+1$.

(b) x^t satisfies all $(2m+1)$-gonal hypermetric inequalities, except when G contains $K_{m,m+1}$ as an induced subgraph. In this case only the pure $(2m+1)$-gonal negative type inequality is violated.

(c) $\frac{x^t}{n^2}$ satisfies all k-gonal inequalities with $k \leq n$ and $s \geq 2$.

∎

We now turn to gap inequalities. Recall $g = g(b)$ defined in (2) is the gap of an integer vector b.

Lemma 2.5 *Let b, k and s be defined as in (5), such that $g = s \geq 2$. Then*

$$\sum_{i=1}^{n} b_i^2 \geq k + s^2$$

.

Proof Assume $b_1 \geq b_2$ are the two largest integers in b. We may also assume without loss in generality that no integer b_i is zero. Since $g = s$ we observe that the set $S = \{1, 2, ..., n\}$ realizes the minimum gap. Assume first that $b_2 > 0$. It is easily seen that $b_1 \geq b_2 \geq g = s \geq 2$, since otherwise a smaller gap could be formed by removing b_2 from S. Therefore

$$\sum_{i=1}^{n} b_i^2 \geq b_1^2 + b_2^2 + \sum_{i=3}^{n} |b_i| = b_1^2 + b_2^2 + k - b_1 - b_2 \geq 2s^2 - 2s + k \geq k + s^2.$$

Otherwise, suppose $b_2 < 0$. From (1), $b_1 = (k+s)/2$. Furthermore, since b has positive and negative components, $k = s + u$, for some $u \geq 2$. Therefore

$$\sum_{i=1}^{n} b_i^2 \geq \frac{(k+s)^2}{4} + \frac{k-s}{2} \geq \frac{(2s+u)^2}{4} + \frac{u}{2} = s^2 + su + \frac{u^2}{4} + \frac{u}{2} \geq s^2 + s + u = k + s^2.$$

∎

3 Proofs of Main Results

In the proofs that follow, we can and do assume that $m \geq 3$.

Proof of P1N.

For a graph G on n vertices, set t as in Lemma 2.3, and define $x^t(G)$ as in (4). $x^t(G)$ satisfies all $2m$-gonal inequalities if and only if G does not contain $K_{m,m}$ as an induced subgraph. Hence P1N is co-NP complete. Part (b) of Lemma 2.3 shows that the problem remains co-NP complete even when restricted to testing whether x^t satisfies all pure $2m$-gonal negative type inequalities. ∎

Proof of P2N.

Let G be a graph with n vertices, and let m be the largest integer such that G contains $K_{m,m}$ as an induced subgraph. For each $u = \lfloor n/2 \rfloor, \lfloor n/2 \rfloor - 1, ..., 2, 1$ we ask a question of type P2N. Set $t = \frac{u^2+1}{u^2(u-1)}$ and let $x^u(G)$ be defined as in (4). Consider the answers given to question P2N. If $u > m$, G does not contain a $K_{u,u}$, so by the lemma either the answer to P2N is x^u is of negative type, or x^u is not of negative type but violates a $2p$-gonal negative type inequality for $p > u$. When $u = m$, G contains an induced $K_{u,u}$. By Lemma 2.3 the answer to P2N must be that x^u is not hypermetric and the minimum hypermetric inequality violated is $2u$-gonal. This answer to P2N gives us the value $m = u$ that answers P4. Therefore P2N is NP-hard. ∎

Proof of P1K

Suppose first $k = 2m$ for some integer m. For a graph G on n vertices, set t as in Lemma 2.3, and define $x = \frac{x^t(G)}{n^2}$. Since the negative type inequalities form a cone, parts (a) and (b) of the lemma apply to x also. Therefore x satisfies all $2m$-gonal negative type inequalities with except if G contains an induced $K_{m,m}$, in which case it violates only the pure $2m$-gonal inequality.

Now consider any k-gonal inequality that is not of negative type. Since k is even, s is even also, and so we can apply part (c) of Lemma 2.3. This shows that x satisfies this k-gonal inequality also. Hence the answer to P1K can be used to determine if G contains an induced $K_{m,m}$.

Next suppose $k = 2m + 1$ for some integer m. We now make use of Lemma 2.4. Arguing as above, we show that P1K can be used to determine if G contains an induced $K_{m,m+1}$. It follows that P1K is co-NP complete. ∎

Proof of P2K

Let G be a graph with n vertices, and let m be the largest integer such that G has $K_{\lfloor \frac{m}{2} \rfloor, \lceil \frac{m}{2} \rceil}$ as an induced subgraph. For $u = n, n-1, ..,$ we ask a question of type P1K for the vector $x = \frac{x^t(G)}{n^2}$, where $x^t(G)$ is defined in Lemma 2.3 if u is even or Lemma 2.4 if u

is odd. If $u > m$, G does not contain an induced $K_{\lfloor \frac{m}{2} \rfloor, \lceil \frac{m}{2} \rceil}$ and so, using the appropriate lemma, we see that the answer is either that x satisfies all k-gonal inequalities or that x violates a k-gonal inequality for $k > u$. When $u = m$, we get the answer that the smallest k-gonal inequality violated by x is u-gonal. Therefore we have determined m and P2K is NP-hard. ∎

Proof of P1G

We proceed as in the proof of P1K. Suppose first $k = 2m$ for some integer m. For a graph G on n vertices, set t as in Lemma 2.3, and define $x = \frac{x^t(G)}{n^2}$. As before, we see that x satisfies all $2m$-gonal negative type inequalities, except if G contains an induced $K_{m,m}$, in which case it violates only the pure $2m$-gonal inequality.

We show that x satisfies all other $2m$-gonal gap inequalities. First assume the gap $g = 0$. In this case the gap inequality coincides with the $2m$-gonal inequality (1). We proved in P1K that x satisfies all such inequalities. Next we assume that $g = s \geq 2$. From Lemmas 2.1 and 2.5 we have

$$Q_G(b, x^t) \leq \frac{s^2}{2} + \frac{t}{4}(s^2 + k^2) - \frac{1+t}{2}(k + s^2) = \frac{k^2 t}{4} - \frac{k}{2} - \frac{tk}{2} - \frac{s^2 t}{4}.$$

$$= m^2 t - mt - m - \frac{s^2 t}{4} = \frac{1}{m} - \frac{s^2 t}{4} < 0.$$

In the second line we used (10), $s \geq 2$ and $t > 1/m$. By homogeneity it follows that $x = \frac{x^t(G)}{n^2}$ also satisfies all gap inequalities with $g = s \geq 2$.

It remains to check that x satisfies all gap inequalities with $s > g$. Since s and g have the same parity, we have $s \geq g + 2$ and so $\frac{1}{4}(s^2 - g^2) \geq 1$. But the proof of Lemma 2.3(c) showed that

$$Q_G(b, \frac{x^t}{n^2}) \leq 1,$$

so x satisfies these inequalities also.

The proof for $k = 2m + 1$ is similar and omitted. ∎

Proof of P2G

Let G be a graph with n vertices, and let m be the largest integer such that G has $K_{\lfloor \frac{m}{2} \rfloor, \lceil \frac{m}{2} \rceil}$ as an induced subgraph. We proceed as in the proof of P2K to use answers to questions of type P2G to find m. ∎

4 Acknowledgements

I would like to thank an anonymous referee for remarks leading to several clarifications of the original manuscript.

References

[1] D. Avis and V.P. Grishukhin, *A Bound on the k-gonality of Facets of the Hypermetric Cone and Related Complexity Problems*, Computational Geometry: Theory and Applications 2 (1993) 241-254.

[2] D. Avis and J. Umemoto, *Stronger Linear Programming Relaxations of Max-Cut*, Les cahiers du GERAD G-2002-48, September, 2002.

[3] M. Deza (M.E. Tylkin), *Realizablility of Distance Matrices in Unit Cubes(in Russian)*, Problemy Kybernetiki 7 (1962) 31-42.

[4] M. Deza, V.P. Grishukhin and M. Laurent, *The Hypermetric Cone is Polyhedral*, Combinatorica 13 (1993) 397-411.

[5] M. Deza and M. Laurent, *Geometry of Cuts and Metrics*, Springer, 1997.

[6] Garey M.R., Johnson D.S., *C*omputers and Intractability, W.H.Freeman and C^o 1979.

[7] M. Goemans and D. Williamson, *0.878-Approximation Algorithms for MAX CUT and MAX 2SAT*, Proc 26th STOC, (1994) 422-431.

[8] C. Helmberg and F. Rendl, *Solving Quadratic (0,1)-Problems by Semidefinite Programs and Cutting Planes*, Math. Prog. 82 (1998) 291-315.

[9] M. Laurent and S. Poljak, *Gap Inequalities for the Gap Polytope*, Europ. J. Combinatorics 17 (1996) 233-254.

[10] B. Reed, *A Gentle Introduction to Semi-Definite Programming*, in Pefect Graphs, ed. J. Alfonsín and B. Reed, Wiley, (2001).