ALGORITHMS FOR LINE TRANSVERSALS IN SPACE

Preliminary Report

David Avis

Rephael Wenger

School of Computer Science
McGill University
805 Sherbrooke St. W.
Montreal, Canada, H3A 2K6

ABSTRACT

Algorithms are developed for determining if a set of polyhedral objects in R^3 can be intersected by a common transversal (stabbing) line. It can be determined in $O(n)$ time if a set of n lines in space has a line transversal, and such a transversal can be found in the same time bound. For a set of n line segments, the complexity of finding such a transversal becomes $O(nlogn)$. Finally, for a set of polyhedra with a total of n vertices, we give a $O(n^5)$ algorithm for determining the existence of, and computing, a line transversal. Helly-type theorems for lines and segments are also given. In particular, it is shown that if every six of a set of lines in space are intersected by a common transversal, then the entire set has a common transversal.

1. Introduction

Let $S = S_1, \ldots, S_m$ be a set of geometric objects in R^d. We say that the objects have a $k-transversal$ if they are simultaneously intersected by a k-dimensional flat. The question of the existence of a k-transversal has received considerable attention in the mathematical literature, see for example [10] and[4]. The corresponding computational issues have been considerably less studied. However, the computation of 1-transversals, or 'stabbing lines' as they are called in the computing literature, have appeared in the contexts of hidden line problems[6], set partitioning[1] and updating triangulations[8].

When $k = 0$, the problem reduces to that of deciding whether the given sets have a point in common. On the theoretical side, when the sets S_i are convex, the following theorem provides an answer to this question.

Helly's Theorem. A family of n convex sets in R^d have a common intersection point if and only if every $d + 1$ of the sets have a common intersection point.

On the computational side, when the sets are polyhedral, we can find an intersection point or determine that none exist by solving a linear programming problem. In low dimensions this can be done efficiently using Megiddo and Dyer's technique in time proportional to the number of half-spaces required to describe the input, see[13].

When $k = 1$ we call the 1-transversal a *stabbing line* and the problems become considerably more difficult. On the theoretical side there is no analogue of Helly's theorem. In R^2 there exist configurations of m line segments with stabbing lines for each $m - 1$ of the line segments but no stabbing line for all m line segments[10]. Hadwiger has found an additional condition which gives a Helly-like theorem in the plane. By an *oval* we mean a bounded, closed convex set.

Hadwiger's Theorem. A finite or countably infinite family of disjoint ovals in R^2 admits a stabbing line if and only if there is an ordering on the ovals such that each three ovals admit a line stabber in the given order.

As it stands, Hadwiger's theorem does not provide a 'good' characterization since if there is no stabbing line, the theorem does not provide an efficient way to demonstrate this, even for a finite family of sets. The theorem can however, be turned into a 'good' characterization. Consider our finite family of sets S. Any stabber for this family induces an ordering $\{i_1, i_2, \ldots, i_m\}$ and a reverse ordering $\{i_m, i_{m-1}, \ldots, i_1\}$ on the indices $1, .., m$. Out of the *a priori* factorial number of stabbers, Katchalski, Lewis and Zaks have shown that there can be at most $\binom{m}{2}$ such orderings[11]. This result has been improved by Wenger, who reduced the upper bound to $6m + 6$[14] and further reduced to $2m - 2$ by Edelsbrunner and Sharir[7]. In the case of polygonal objects, these potential orderings can be efficiently computed. For each ordering, it can be determined by linear programming if in fact a stabber exists which intersects the sets in the specified order[2]. The extension of Hadwiger's theorem to higher dimensions seems a challenging task. Some results on hyperplane stabbers have been obtained by Katchalski[12] and Goodman-Pollack[9]

On the computational side very little is known about stabbing lines. In the case of n line segments in the plane, Edelsbrunner et al.[6] have given an $O(n \log n)$ algorithm. Edelsbrunner gives algorithms for stabbing translates of a convex polygon[5]. Avis and Doskas have given a general approach for finding $d-1$-stabbers of line segments and polyhedra in R^d, using linear programming[2].

In dimension $d \geq 3$, no algorithms for line stabbing have previously been presented. In this paper, we develop a theory for line stabbing in R^3. We begin by considering line stabbers of lines in R^3 and then specialize the results to line segments. Somewhat surprisingly in view of the examples mentioned above, we find a Helly-type theorem in R^3 for lines, and for line segments providing they satisfy a 'non-degeneracy' assumption. Degenerate lines or segments will be called ruled. Naturally, if all the segments lie on a 2-flat they are ruled, but there are also other situations where the segments are ruled. We give efficient algorithms for finding line stabbers of lines and line segments. Next we consider convex polyhedra in R^3. We prove an extremal theorem that reduces the line stabbing problem for polyhedra to a number of line stabbing problems for segments and points. This gives rise to a polynomial time algorithm for finding a line stabber for polyhedra.

2. Theoretical Results

Any two lines in R^3 which can be embedded in the same plane are called $co-planar$. Two lines which are not co-planar are called *skewed*. Trivially, any intersecting lines are co-planar and not skewed. Two lines in R^3 which are co-planar but do not intersect are called *parallel*. Let $S = \{L_1, \ldots, L_m\}$ be a set of m lines in R^3. If every two lines in the set S are skewed, then we call the set S a *skewed* set.

We use the symbol $\times$ for the cross product and the symbol $\cdot$ for the dot product. For a square matrix M, $|M|$ is the determinant of M. $det(u, v, w)$, $u, v, w \in R^3$, is the determinant of the matrix with u, v, w as column vectors. If M is a 3x3 real matrix and $v \in R^3$, $M \cdot v$ is the matrix product of M and v as a column vector.

Let l, l' *be two distinct lines parametrized by* $ut + p$ *and* $vt + q$, $u, p, v, q \in R^3$, $t \in R$, $u \cdot u = v \cdot v = 1$, *respectively.* l *is parallel to* l' *if and only if* $u \times v = 0$. l *and* l' *are co-planar if and only if* $(u \times v) \cdot (p - q) = 0$. *Thus* l *intersects* l' *if and only if* $u \times v \neq 0$ *and* $(u \times v) \cdot (p - q) = 0$, *i.e.* l *and* l' *are not parallel but are co-planar.*

Lemma 1. Four distinct lines in R^3 admit either at most two stabbers or an infinite number of stabbers.

Proof:

Let $S = \{L_1, L_2, L_3, L_4\}$ be a set of four lines in R^3 where L_i is parametrized by $u_i t + p_i$, $u_i, p_i \in R^3$, $t \in R$, $u_i \cdot u_i = 1$. If S is not a skewed set, then, by relabeling the elements of S if necessary, assume that L_4 is co-planar with some other line in S. Let l parametrized by $vt + q$, $v, q \in R^3$, $t \in R$, $v \cdot v = 1$, be some stabbing line of L_1, L_2, L_3, L_4. If l intersects L_i, then l is co-planar with L_i. Thus $(u_i \times v) \cdot (p_i - q) = 0$ or equivalently $(u_i \times (q - p_i)) \cdot v = 0$. l intersects L_4 so we can assume q lies on L_4 and find an $\bar{s} \in R^3$ such that $q = u_4 \bar{s} + p_4$. Then,

$$(u_i \times (q(\bar{s}) - p_i)) \cdot v = 0, \quad i = 1, 2, 3.$$

For each value of s, define the matrix $M(s)$ by

$$M(s) = \begin{pmatrix} u_1 \times (q(s) - p_1) \\ u_2 \times (q(s) - p_2) \\ u_3 \times (q(s) - p_3) \end{pmatrix}.$$

where $q(s) = u_4 s + p_4$. Each row i of $M(s)$ is either the zero vector or the normal to the plane containing L_i and $q(s)$. If l is a stabbing line of L_1, L_2, L_3, L_4, l parametrized by $vt + q(\bar{s})$, then $M(\bar{s}) \cdot v = 0$. $x = (0, 0, 0)$ is a trivial solution to $M(\bar{s}) \cdot x = 0$. If $v \neq (0, 0, 0)$ is also a solution, $M(\bar{s})$ must have rank less than three and $|M(\bar{s})| = 0$. We consider four cases based on the rank of $M(s)$.

Case 1: $M(s)$ has rank zero for some s.

Each row of $M(s)$ must be the zero vector. This means that $q(s)$ must lie on lines L_1, L_2, L_3. An infinite number of different lines contain q stabbing S.

Case 2: $M(s)$ has rank one for some s.

Some row of $M(s)$, say $u_1 \times (q(s) - p_1)$, must be non-zero and all other rows of $M(s)$ are scalar multiples of this row. Let H be the plane containing L_1 and $q(s)$. H has normal $u_1 \times (q(s) - p_1)$. If a row i of $M(s)$ is the zero vector, then L_i contains $q(s)$. Otherwise, the normal to the plane H' *containing L_i and $q(s)$ is a scalar multiple of the normal to the plane H. Since $q(s)$ lies in H and H', $H' = H$ and L_i lies in H. An infinite number of different lines lie in plane H, contain q and are not parallel to L_1, L_2 or L_3. These lines will all stab S.*

Case 3: $M(s)$ is rank two for all s.

For each value s_r there is some vector $v_r \in R^3$, $v_r \cdot v_r = 1$, such that $M(s_r) \cdot v_r = 0$. Let l_r be the line parametrized by $v_r t + q(s_r)$. l_r intersects or is parallel to every line in S. If any other line parametrized by $v't + q(s_r)$ *intersected or was parallel to every line in S, then $M(s_r) \cdot v' = 0$ and $M(s_r)$ would have rank less than two. Since $M(s)$ has rank two for all s, l_r is the unique line through $q(s_r)$ intersecting or parallel to every line in S.*

Case 3.1: $M(s)$ is not a skewed set.

By assumption, L_4 is co-planar with some other line, say L_1. Let H be the plane containing L_4 and L_1. Let $q(s_1)$ and $q(s_2)$ be two distinct points on L_4 which do not lie on L_1. Let l_1 and l_2 be the unique lines through $q(s_1)$ and $q(s_2)$ respectively such that l_1 and l_2 intersect or are parallel to every line in S.

If l_1 and l_2 intersect in a line, then l_1 and l_2 are the line L_4. Thus L_4 intersects or is parallel to every line in S. $q(s)$ lies on L_4 for all s, so L_4 is the unique line containing $q(s)$ and intersecting or parallel to every line in S. Then, the only possible stabber of S is L_4 and S will have one or zero stabbers depending on whether L_4 stabs S.

Assume l_1 intersects l_2 at point $\bar{q}$. If line L_2 does not lie in H, the only way L_2 can intersect or be parallel to l_1 and l_2 is to contain point $\bar{q}$. Similarly, if L_3 does not lie in H, L_3 must contain $\bar{q}$. All lines L_i must lie in H or contain $\bar{q}$. There are an infinite number of lines in H which contain $\bar{q}$ and are not parallel to L_1, L_2, L_3, L_4, all of which are stabbers of S.

Finally, assume l_1 is parallel to l_2. Any line L_i which does not lie in H must be parallel to l_1 and does not intersect H. For each s_r the line which is a translate of l_1 through $q(s_r)$ either intersects or is parallel to each line in S. Thus this translate of l_1 is the unique line l_r through $q(s_r)$ which intersects or is parallel to each line in S. Since rank of $M(s)$ is two for each s, some line L_i must not lie in H. l_r lies in H for each r so there are no stabbers of S.

Case 3.2: Matrix $M(s)$ is a skewed set.

We claim that l_r is not parallel to $l_{r'}$ *for* $r \neq r'$. *For assume, to the contrary, that* l_r *is parallel to* $l_{r'}$. *Let* H *be the plane containing* l_r *and* $l_{r'}$. l_r *and* $l_{r'}$ *are stabbers of each* L_i, *so* H *contains* L_4 *and* L_1. L_4 *and* L_1 *are co-planar, contrary to the assumption. Thus* l_r *and* $l_{r'}$ *must not be parallel and* v_r *is not a scalar multiple of* $v_{r'}$.

There are at most four values of r such that $v_r \times u_i = 0$. For all other values of r, $v_r t + q(s_r)$ is a stabbing line of S. Since l_r is not parallel to $l_{r'}$ *for* $r \neq r'$, *all these stabbing lines are different. There are an infinite number of stabbers of* S.

Case 4: $M(s)$ has rank two or three for all s and rank three for some s.

$|M(s)|$ is a polynomial in s and is not the zero polynomial or it would be of rank less than three for all s. The coefficient of s^3 is

$$det(u_1 \times u_4, u_2 \times u_4, u_3 \times u_4) = 0,$$

so the degree of polynomial $|M(s)|$ is at most two. Thus $|M(s)|$ has at most two real roots, s_1 and s_2. By the same argument as in case 3, there is a unique line for each s_j, $j = 1, 2$, which intersects or is parallel to S. Thus there are at most two stabbers of S. Q.E.D.

We refer to a set of lines that admit an infinite number of stabbers as *ruled*. The terminology derives from the fact that the set of stabbing lines forms a doubly ruled surface. A set of three skewed lines is trivially ruled. The study of these ruled surfaces is most easily performed in projective space. The surface can be partitioned into two sets of lines: every pair of lines from the same set is skewed; every pair of lines from different sets is intersecting. There is thus an obvious duality between the stabbing lines and the lines to be stabbed. This doubly ruled surface is a quadric surface. A problem arises when a stabber in projective space stabs a line at a so-called ‘improper point’, since this corresponds to an intersection ‘at infinity’: ie. in the Euclidean case, the lines do not intersect. This situation is discussed in the next paragraph. The interested reader is referred to the book by Borsuk[3] for a very readable treatment of quadrics in projective space. We will confine our discussion to the more familiar Euclidean space.

Consider three skewed lines L_1, L_2, L_3. Then there is a unique plane, up to translates, that can be translated to contain both L_2 and L_3. Consider the intersection point x, if it exists, of L_1 with the translate of this plane containing L_2. Any stabbing line of L_2 through x lies in this plane and hence cannot intersect L_3. The point x is called a *critical point* of L_1. Similarly the intersection of L_1 with the translate of the plane containing L_3 is also a critical point of L_1. We denote by C_i the set of critical points on L_i. The following results about skewed, ruled lines will be used in the sequel. The proofs, which will be omitted, follow most easily from results about quadrics in projective space referred to above.

Lemma 2. Let $S = \{L_1, L_2, \cdots, L_m\}$ be a skewed, ruled set of lines, $m \geq 3$.

(a) For each line $L_i \in S$, there exist at most $m-1$ critical points on L_i, where each critical point $c_{ij} \in C_i$ corresponds to some line L_j, $j \neq i$.

(b) For every non-critical point x on each line there is a unique stabbing line $STAB(x)$ that intersects the other lines.

(c) Every stabber of $L_i, L_j, L_k \in S$, i, j, k distinct, which does not intersect a critical point on L_i, stabs all the lines in S.

(d) For each $i = 1, \ldots, m$, the critical points c_{ij}, $j = 1, 2, \ldots, i-1, i+1, \ldots, m$ are collinear.

As a consequence of Lemma 2, we can define the following functions. Let S be as in the lemma and choose any $L_i, L_j \in S, i \neq j$. Let

$$F_{ij}\colon L_i - c_{ij} \to L_j - c_{ji}$$

where

$$F_{ij}(x) = STAB(x) \cap L_j \qquad x \in L_i - C_i$$

and

$$F_{ij}(c_{ik}) = c_{jk} \qquad\qquad k \neq i, j.$$

Note that F_{ji} is the inverse of F_{ij}.

In order to get efficient algorithms for the ruled case, some further results are needed. The first result says that the stabbing lines in the ruled case behave in a ‘continuous’ way.

Lemma 3. F_{ij} is a one-to-one continuous and onto function.

Proof:

Lemma 2 implies that F_{ij} is a one-to-one onto function. We must merely show that F_{ij} is continuous. We construct F_{ij}. Let L_k, $k \neq i$, $k \neq j$, be some line in S other than L_i and L_j. Parametrize L_i, L_j, L_k by $u_i t + p_i$, $u_j t + p_j$, $u_k t + p_k$. Let q be some point on L_i and let $vt + q$ be the parametrization of the unique line through q stabbing L_j and L_k.

The plane containing L_j and q contains v. Since L_i and L_j are skewed, u_j is not a scalar multiple of $q - p_j$. We can write v as a linear combination of u_j and $q - p_j$, where

$$v = \alpha u_j + \beta(q - p_j), \qquad \alpha, \beta \in R.$$

Substituting for v in the equation

$$(u_k \times (q - p_k)) \cdot v = 0,$$

we find that

$$\alpha\, det(u_j, u_k, q - p_k) = -\beta\, det(q - p_j, u_k, q - p_k).$$

If q is not the critical point c_{ij} on L_i, $det(u_j, u_k, q - p_k) \neq 0$. If $\beta = 0$, then $\alpha = 0$ and $v = 0$, so $\beta \neq 0$. Thus,

$$-\frac{\alpha}{\beta} = \frac{det(q - p_j, u_k, q - p_k)}{det(u_j, u_k, q - p_k)},$$

for $q \neq c_{ij}$.

For some $t, s \in R$, $vt + q = u_j s + p_j$. Then, $(u_j \times v)s = (q - p_j) \times v$.

$$\begin{aligned} u_j \times v &= u_j \times (\alpha u_j + \beta(q - p_j)) \\ &= \beta(u_j \times (q - p_j)) \\ (q - p_j) \times v &= (q - p_j) \times (\alpha u_j + \beta(q - p_j)) \\ &= -\alpha(u_j \times (q - p_j)). \end{aligned}$$

Substituting into the equation with s we get

$$\beta(u_j \times (q - p_j))s = -\alpha(u_j \times (q - p_j)).$$

Since L_i and L_j are skewed, $u_j \times (q - p_j) \neq 0$. Therefore,

$$s = -\frac{\alpha}{\beta} = \frac{det(q - p_j, u_k, q - p_k)}{det(u_j, u_k, q - p_k)}.$$

For $q \in L_i - c_{ij}$,

$$F_{ij}(q) = u\,\frac{det(q - p_j, u_k, q - p_k)}{det(u_j, u_k, q - p_k)} + p_j.$$

Since the determinant function is continuous, F_{ij} is continuous. Q.E.D.

The next result is a consequence of Lemma 3 and tells us that a ruled set of segments has an ‘extremal’ stabber.

Lemma 4. If a set S of skewed, ruled line segments admits a stabber, then there exists a stabber of S which intersects one of the endpoints of the line segments.

The above results give rise to efficient algorithms for finding stabbing lines for lines and segments as will be described in the next section. Also, as a consequence of Lemmas 1 and 2 we have the following Helly-like theorem.

Theorem 1. A set of lines in R^3 have a stabbing line if and only if every six of the lines has a stabbing line.

In Theorem 1, lines may be co-planar or even intersect. The theorem generalizes to all dimensions. We call a set of line segments *ruled* if the set of lines containing the line segments is *ruled*. Theorem 1 has the following corollary.

Corollary 1. A set of segments in R^3 that are not ruled have a stabbing line if and only if every six segments has a stabbing line.

We now turn to the problem of stabbing lines for convex polyhedra in R^3. Let $P_1, \ldots, P_m$ be a set of disjoint polyhedra in R^3. Using the previous lemmas, we can prove the following 'extremal' theorem for polyhedra. For points x and y in space, let $l(x, y)$ denote the line through x and y.

Theorem 2. $P_{1,} \cdots, P_m$ have a stabbing line if and only if they have a stabbing line through:

(a) Two vertices in two distinct P_i; or

(b) One vertex and two skewed edges in three distinct P_i; or

(c) Four non-ruled edges in four distinct P_i.

Proof: We begin with the following

Observation: Let $Q_1, \ldots, Q_m$ be a set of m convex polygons in the plane that admit a stabbing line. Then there exist indices $1 \le r < s \le m$ and vertices $x_r \in Q_r$ and $x_s \in Q_s$ such that $l(x_r, x_s)$ is a stabbing line for $Q_1, \ldots, Q_m$.

To see this, let l be a stabber for $Q_1, \ldots, Q_m$. By translating l if necessary, we can assure that it goes through some vertex x_r of some polygon Q_r. We now rotate l about x_r, maintaining the property that it is a stabber, until it passes through a vertex x_s of some other polygon Q_s.

To prove the theorem, let l_1 be a stabbing line for $P_1, \ldots, P_m$. Let H be any plane containing l_1, and let $Q_i = H \cap P_i,\ i = 1, \ldots, m$. Clearly l_1 stabs the polygons $Q_1, \ldots, Q_m$ and so we can apply the observation to find a stabbing line l_2 through vertices $x_r \in Q_r$ and $x_s \in Q_s$. There are three cases.

Case 1. x_r is a vertex of P_r and x_s is a vertex of P_s.

We have part (a) of the theorem.

Case 2. x_r is a vertex of P_r and x_s lies in the interior of an edge $a_s b_s$ of P_s.

First note that if either $l(x_r, a_s)$ or $l(x_r, b_s)$ is a stabber for $P_1, \ldots, P_m$ we are back to Case 1. We assume that this is not the case. Let H be the plane through x_r and $a_s b_s$. Let

$$Q_i = H \cap P_i \qquad i \ne r, s$$

$$Q_r = x_r$$

$$Q_s = a_s b_s.$$

Applying the observation, we obtain a stabbing line l_3 that passes through a vertex x_t of some polygon Q_t. By assumption, x_t cannot be a_s or b_s, and so $t \ne r, s$. If x_t is a vertex of

P_t we are again back to Case 1. Otherwise, x_t lies on some edge $a_t b_t$ of P_t. Now if the edges $a_s b_s$ and $a_t b_t$ were coplanar, then they would both lie in H implying that x_t is an endpoint of the edge $a_t b_t$ and a vertex of P_t. As we have excluded this possibility, the edges are skewed and we have part (b) of the theorem. (There is also a symmetrical case with the roles of x_r and x_s reversed).

Case 3. x_r lies in the interior of an edge $a_r b_r$ of P_r and x_s lies in the interior of an edge $a_s b_s$ of P_s.

Case 3.1. $a_r b_r$ and $a_s b_s$ lie in a plane H.

Let

$$Q_i = H \cap P_i \qquad\qquad i \neq r, s$$

$$Q_r = a_r b_r$$

$$Q_s = a_s b_s.$$

Applying the observation, we may get a stabbing line through one or more of the vertices a_r, b_r, a_s, b_s. In this case we revert to Case 1 or 2. Otherwise, the stabbing line goes through vertices $x_t \in Q_t$ and $x_u \in Q_u$. We revert to an earlier case if either x_t is a vertex of P_t or x_u is a vertex of P_u. So x_t lies on an edge $a_t b_t$ of P_t that is not coplanar with H. Similarly x_u lies on an edge $a_u b_u$ of P_u that is not coplanar with H. These edges are therefore not ruled, and we have part (c) of the theorem.

Case 3.2. $a_r b_r$ and $a_s b_s$ are skewed.

If one or more of the lines $l(x_r, a_s)$, $l(x_r, b_s)$, $l(x_s, a_r)$, $l(x_s, b_r)$ are stabbing lines for $P_1, \ldots, P_m$ we revert to Case 2, making the appropriate change in r or s. We therefore assume this is not the case. We define the plane H and $Q_1, \ldots, Q_m$ as in Case 2. Applying the observation we obtain a vertex x_t of some polygon Q_t such that $l(x_r, x_t)$ is a stabbing line for $Q_1, \ldots, Q_m$. If x_t is also a vertex of P_t we again revert to Case 2 setting $r = t$. Otherwise x_t lies on some edge $a_t b_t$ of P_t. Now $a_t b_t$ cannot be coplanar with $a_s b_s$, but it could be coplanar with $a_r b_r$. If this is the case, we set $s = t$ and revert to Case 3.1. Let L_r, L_s, L_t be the skewed lines through the edges $a_r b_r, a_s b_s, a_t b_t$. As noted after Lemma 1, this set of lines is trivially ruled. For points z_1 and z_2 in space, let $[z_1, z_2]$ be the closed segment joining them, and let $(z_1, z_2]$ denote the half-open segment joining them. For point z in $[a_r, b_r]$ let $STAB(z)$ be the unique stabbing line of L_s, L_t that contains z (cf. Lemma 2.) Note that $STAB(x_r) = l(x_r, x_t)$. By assumption, $STAB(b_r)$ is not a stabbing line for $P_1, \ldots, P_m$. So there exists a point y_r in $[x_r, b_r]$ such that $STAB(z)$ is a stabbing line for $P_1, \ldots, P_m$ for all $z \in [x_r, y_r]$, but is not a stabbing line for $z \in (y_r, b_r]$. Therefore $STAB(y_r)$ intersects some edge $a_u b_u$ of a polyhedron P_u which is distinct from P_r, P_s, P_t. Furthermore for $z \in (y_r, b_r]$, $STAB(z)$ does not intersect the edge $a_u b_u$. If $a_u b_u$ lies in the ruled surface of stabbers of L_r, L_s, L_t, then $STAB(x_r)$ must pass through either a_u or b_u and we revert to Case 2 with $x_r = a_u$ or b_u respectively, and with P_u replacing P_r. Otherwise we have part (c) of the theorem. Q.E.D.

3. Algorithms

The algorithms follow quite naturally from the theory developed in the preceding section. Given any four skewed lines we can determine in $O(1)$ time whether they are ruled. If they are not ruled, then the four lines have at most two stabbers which we can also find in $O(1)$ time. If they are ruled, then we can find the critical point on L_1 corresponding to L_i in $O(1)$ time. We direct the line L_1 from 'left' to 'right' so that we can order points on L_1.

In Figure 1 we present an algorithm to determine if a set $S = (L_1, L_2, \cdots, L_n)$ of n lines in R^3 admits a stabber.

STABBER_CO_PLANAR is a subroutine which determines a stabber through S given two co-planar lines in S. It runs in $O(n)$ time. The implementation of this subroutine contains no new concepts and will be given in the full paper. The entire algorithm for line stabbing in R^3 runs in $O(n)$ time.

The algorithm for line segments works in a similar way. Let $T = \{M_1, M_2, \cdots, M_n\}$ be a set of n line segments in R^3. We begin by considering the set of lines $S = \{L_1, L_2, \cdots, L_n\}$, where line L_i contains line segment M_i. If any lines L_i are co-planar with lines L_1, L_2 or L_3, then we handle them by a special subroutine STABBER_SEG_CO_PLANAR. This subroutine handles several special cases and makes use of the algorithm of Edelsbrunner et al.[6] for stabbing line segments in a plane in $O(n \log n)$ time. If any skewed set of lines $\{L_1, L_2, L_3, L_i\}$, $1 \le i \le n$, is non-ruled, then at most two stabbers are found for this set as in the algorithm for stabbing lines. As above, it is a simple $O(n)$ time procedure to check if these two stabbers stab all line segments in T.

Finally, if every set of four lines $\{L_1, L_2, L_3, L_i\}$, $1 \le i \le n$, is skewed and ruled, then for each point on L_1 there is at most one line which stabs S. For $i = 2, \ldots, n$, recall the function $F_{i1}(x)$, for non-critical points $x \in L_i$, gives the point on L_1 that together with x determines a stabber for S. Using this function, we can 'project' the segment M_i onto the line L_1. The projection $\bar{M}_i$ of M_i is either a closed segment, a half-line or the complement of an open segment on L_1. We set $\bar{M}_1 = M_1$. It can be shown that there is a stabbing line of the segments T if and only if

$$\bigcap_{i=1}^{n} \bar{M}_i \neq \theta.$$

Each projection takes $O(1)$ time. To determine if the projections intersect is a routine application of line sweep and can be performed in $O(n \log n)$ time. Therefore, the complexity of finding a stabber of line segments in R^3 is $O(n \log n)$.

To find a line stabber for a set of polyhedra we test for the three possible cases in theorem 2:

For every four line segments in four different polyhedra, do:

> If the line segments are not ruled, find any stabbers (maximum 2) of the four line segments and check if they stab all the polyhedra.

For every vertex and two line segments in three different polyhedra, do:

> If the line segments do not lie on the same plane, find any stabber (maximum 1) of the vertex and two line segments and check if it stabs all the polyhedra.

For every two vertices in two different polyhedra, do:

> Check if the stabber of the two vertices stabs all the polyhedra.

The total time of this algorithm is dominated by step 2, which takes $O(n)$ time and is executed $O(n^4)$ times for a total of $O(n^5)$ time complexity.

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If line L_1 is co-planar with some line L_i ∈ S, then:
        Call STABBER_CO_PLANAR(L_1, L_i, S).
else if line L_2 is co-planar with some line L_i ∈ S, then:
        Call STABBER_CO_PLANAR(L_2, L_i, S).
else if line L_3 is co-planar with some line L_i ∈ S, then:
        Call STABBER_CO_PLANAR(L_3, L_i, S).
else

begin
        Let c̄ be any point on L_1.
        For i = 4 to m do:
                If (L_1, L_2, L_3, L_i) are not ruled, then:
                begin
                        Find stabbers T_1, T_2 of (L_1, L_2, L_3, L_i), if they exist.
                        If T_1 or T_2 stabs all lines L_j, then:
                                Report stabber and stop.
                        else
                                Report 'No Stabber' and stop.
                end

                else
                { (L_1, L_2, L_3, L_i) are ruled. }

                begin
                        Find critical point c_1i corresponding to L_i on L_1, if it exists.
                        If c_1i is to the 'right' of c̄, let c̄ = c_1i.
                end

        {S is ruled}
        Find some point p on L_1 which is to the right of c̄.
        {p is not a critical point.}
        Line T_3 through p, L_2, L_3 is a stabber of S.
end.
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Figure 1. Algorithm for Line Stabbing of Lines.