Lower Bounds for Line Stabbing

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Abstract

We present an $\Omega(n \log n)$ fixed order algebraic decision tree lower bound for determining the existence of a line stabber for a family of n line segments in the plane. We give the same lower bound for determining the existence of a line stabber for n translates of a circle in the plane. In proving this lower bound, we show that this problem is equivalent to determining if the width of a set of points is less than or equal to w. Through this transformation we can reexamine an old example by Hadwiger, Debrunner and Klee of a family of $k+1$ translates where every k translates have a line transversal but the entire family has no line transversal.

A line which intersects every member in a family of objects is known as a *line stabber* or *line transversal*. Recently, computer scientists have been very interested in algorithms for finding such line stabbers, (see [1, 2, 3, 5, 6, 7, 8, 14, 24, 25].) Mathematicians have also shown renewed interest in transversal problems, (see [11, 15, 16, 17, 18, 19, 26, 30, 31].)

In [8], Edelsbrunner et al. presented an $O(n \log n)$ algorithm for constructing a representation of the line stabbers of n line segments in the plane. This was generalized by Atallah and Bajaj to an $O(n\alpha(n) \log n)$ algorithm for line stabbing n simple objects in the plane, where $\alpha(n)$ is the inverse of Ackerman's function [1]. A simple object is an object which has an $O(1)$ storage description and for which common tangents and intersections can be computed in $O(1)$ time. Edelsbrunner, Guibas and Sharir showed how to construct a representation of the line stabbers of convex polygons with a total of n vertices in $O(n\alpha(n) \log n)$ time [7]. Recently Hershberger announced an $O(n \log n)$ algorithm for finding the lower envelope of a set of line segments in the plane [13]. His result gives an $O(n \log n)$ algorithm for line stabbing n simple objects in the plane and an $O(n \log n)$ algorithm for line stabbing convex polygons with a total of n vertices in the plane.

In their paper [8], Edelsbrunner et al. showed an $\Omega(n \log n)$ lower bound for constructing a representation of the line stabbers. However, this lower bound did not apply to the decision problem, determining if there existed a line stabber of n line segments. They conjectured that $\Omega(n \log n)$ was also a lower bound for the decision problem. We prove their conjecture.

In proving the lower bound for line stabbing of line segments we make use of line segments with many different slopes. In fact, a line stabber of n line segments all with the same slope can be found by linear programming in linear time [4, 22, 23]. There also exists an $O(n)$ algorithm by Edelsbrunner for finding a line stabber of n rectangles [5]. In [5] Edelsbrunner gave an $O(n \log n)$ algorithm for finding the line stabbers of n translates. One might, however, conjecture that there exists an $O(n)$ algorithm for finding a line stabber of n translates. Unfortunately, this is not the case. We show that there exists an $\Omega(n \log n)$ lower bound for line stabbing n translates of a circle in the plane.

The lower bound for line stabbing translates of a circle is based upon the equivalence of this problem to the *width problem*, determining if the width of a set of points in the plane is less than or equal to w for some fixed w. The *width* of a set of points is the minimum distance between two parallel lines of support of the set. Using this transformation to the width problem, we review an old example by Hadwiger, Debrunner and Klee of a family of $k+1$ translates of a circle where every k translates have a line stabber but the entire family does not [12, p. 9, fig. 3].

In 1958 Grünbaum conjectured that a family of pairwise disjoint translates of a compact convex set in the plane has a line transversal if and only if every five translates have a line transversal [10]. Katchalski proved a weaker form with the condition that every 128 translates have a line transversal [16], and Tverberg settled the problem by proving Grünbaum's conjecture [30]. A similar theorem by Hadwiger which applied to line transversals of families of pairwise disjoint compact convex sets was shown by Wenger to apply to all families of compact convex sets [12, 31]. However, the example by Hadwiger, Debrunner and Klee shows that the pairwise disjointness condition cannot be dropped from Tverberg's theorem. Through the width problem we give an easier way to view the construction by Hadwiger, Debrunner and Klee.

Families of pairwise disjoint compact convex sets do not have the nice transversal property that if every k have a transversal then the entire family has a transversal [12]. Even for families of pairwise disjoint line segments, Lewis showed that there exist examples of $k+1$ line segments with no line transversal where every k line segments have a line transversal [20]. Thus families of pairwise disjoint translates have a special mathematical property which does not apply in general. Egyed and Wenger recently proved line transversals of these families can be found in $O(n)$ time [9], showing that they also have special algorithmic properties.

Lower bounds for the fixed order algebraic decision tree do not imply lower bounds for other models of computation. In fact the max gap problem referred to above has an $O(n)$ algorithm when the floor function is permitted [27]. Is there an $O(n)$ algorithm for line stabbing using the floor function?

The *stabbing segments problem* is given a set of line segments in the plane determine if there exists a line which intersects all the segments. In the proof of $\Omega(n \log n)$ lower bounds for the stabbing problem, we also prove $\Omega(n \log n)$ lower bounds for two other problems. The *arc cover problem* is given a set of n open arcs on a circle determine if their union covers the circle. The *arc intersection problem* is given a set of n closed arcs on a circle determine if the intersection of all the arcs is non-empty. In both problems the arcs lie on a circle in the plane with endpoints p, q specified in Cartesian coordinates and with the arcs extending counter-clockwise from p to q.

Theorem 1 *Any fixed order algebraic decision tree which solves the stabbing segments problem, the arc cover problem or arc intersection problem has* $\Omega(n \log n)$ *depth.*

Proof:

We prove the theorem by a reduction of the max gap problem to line stabbing in the plane. The *max gap problem* is given a set of n points on the real line and a real number δ determine if the distance between two consecutive points is greater than or equal to δ. Lee and Wu [21] and Ramanan [28] gave an $\Omega(n \log n)$ fixed order algebraic decision tree lower bound for the max gap problem. The papers by Lee and Wu and Ramanan addressed a slightly different version of the max gap problem, determine if the distance between two consecutive points is greater than δ. However, their proofs of the lower bounds work equally well for our version of the problem.

We first reduce max gap to the arc cover problem. Let the leftmost and rightmost input point to the max gap problem be p_l and p_r respectively. For

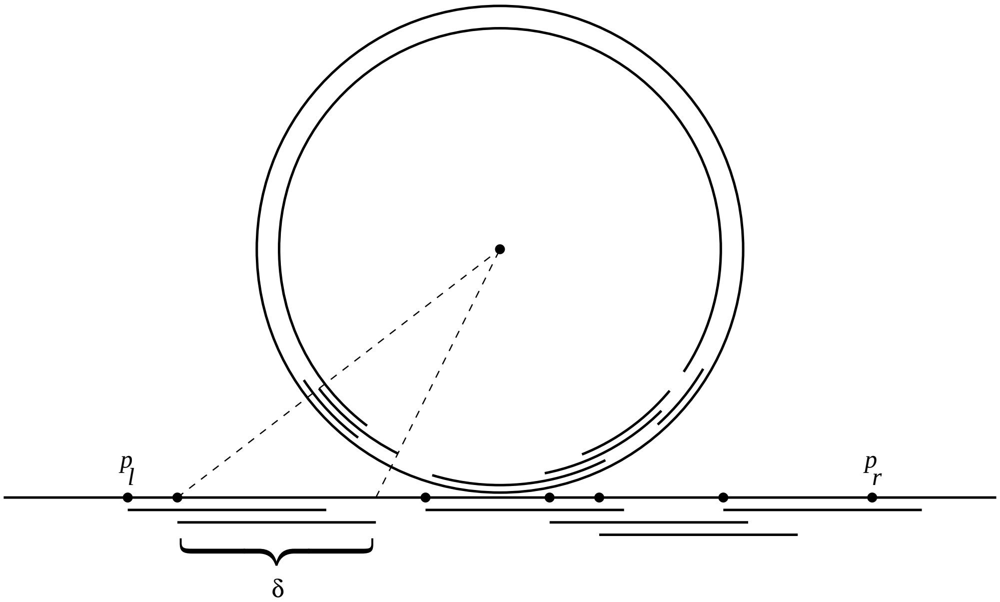

Figure 1: Transformation of the max gap problem to the arc cover problem.

each input point p other than p_r create an open line segment with endpoints p and $p + \delta$. Using a stereographic projection Φ, map the real line onto the circle, mapping open line segments onto open arcs on the circle. There is one point θ on the circle which is not the image of any point on the real line. Add one additional open arc which extends from $\Phi(p_r)$ to $\Phi(p_l + \delta)$ and passes through θ, (see Figure 1). The distance between two consecutive input points is greater than or equal to δ if and only if the open arcs do not cover the circle.

A fixed order algebraic decision tree can simulate the stereographic projection. Square roots are removed by repeated squaring with a constant increase in the degree of the polynomials in the tree and a constant factor increase in depth. Thus the $\Omega(n \log n)$ lower bound for the max gap problem extends to an $\Omega(n \log n)$ fixed order algebraic decision tree lower bound for the arc cover problem.

The arc intersection problem is merely the formulation of the arc cover problem in terms of the arc complements. A set of closed arcs intersect if and only if the set of complement arcs do not cover the circle. The same $\Omega(n \log n)$ lower bound applies to the arc intersection problem.

Finally we reduce the arc intersection problem to the stabbing segments problem. Instead of using Cartesian coordinates, we will describe the reduction using polar coordinates where the center of the circle lies on the origin and the circle has radius 1. All the transformations can be done on the Cartesian coordinates and can be simulated in a fixed order algebraic decision tree.

Every point is labelled by its counter-clockwise angle α, $0 \leq \alpha < 360$, from the x-axis. Each arc is labelled (α, β) and extends counter-clockwise from α to β. Map each arc (α, β) where $\alpha \leq \beta$ to an arc half the size with endpoints $(\alpha/2, \beta/2)$. Map each arc (α, β) where $\beta < \alpha$ to an arc with endpoints $(180 + \alpha/2, \beta/2)$.

We claim that the intersection of the original arcs is non-empty if and only if there exists some diameter, two antipodal points on the circle, which intersects all the new arcs. An arc (α, β) contains a point γ if and only if $\alpha \leq \gamma \leq \beta$ or $\gamma \leq \beta < \alpha$ or $\beta < \alpha \leq \gamma$. The first inequality describes an arc where $\alpha \leq \beta$ while the next two inequalities describe the two possible cases in which $\beta < \alpha$. In the case $\alpha \leq \beta$, $\alpha \leq \gamma \leq \beta$ if and only if $\alpha/2 \leq \gamma/2 \leq \beta/2$. Similarly in the case $\beta < \alpha$, $\gamma \leq \beta < \alpha$ if and only if $\gamma/2 \leq \beta/2 < 180 + \alpha/2$, and $\beta < \alpha \leq \gamma$ if and only if $\beta/2 < 180 + \alpha/2 \leq 180 + \gamma/2$. Thus γ lies in the intersection of each of the original arcs if and only if $\gamma/2$ or $180 + \gamma/2$ intersect each of the new arcs.

Add a line segment between the endpoints of each new arc. Add two horizontal line segments with endpoints at the center of the circle one extending to the right and one to the left. Any line which stabs these two line segments must go through the origin. There exists a line which stabs all the $n + 2$ line segments if and only if the intersection of the original arcs is non-empty. All the transformations can be simulated by a fixed order algebraic decision tree so there is an $\Omega(n \log n)$ fixed order algebraic decision tree lower bound for the stabbing problem.□

Theorem 2 *Any fixed order algebraic decision tree which determines the existence of a line stabber for circles of equal radius has depth* $\Omega(n \log n)$.

Proof:

We prove Theorem 2 by a reduction from the width problem. Lee and Wu [21] and later Ramanan [29] gave $\Omega(n \log n)$ fixed order algebraic decision tree lower bounds for solving the width problem.

We can formulate the width problem as a problem of stabbing translates of a circle. Around each point in S place a circle of radius $w/2$. It is easy to see that there exists a line stabber for this family of circles if and only if the width of S is less than or equal to w. Thus the lower bound for solving the width problem gives the same lower bound for the stabbing of circles problem, proving Theorem 2.$\Box$

Finally, we present a proof of the theorem of Hadwiger, Debrunner and Klee using the transformation to a width problem.

Theorem 3 *For any $k > 0$, there exists a family of $k + 1$ translates of a circle such that every subset of the family of size k has a line transversal but the family does not have a line transversal.*

Proof:

Let S' be the set of vertices of a regular $k+1$-gon and let S be the set S' with one point removed. Let w be the width of set S. It is a simple exercise to show that the width of S' is greater than w. Center $k+1$ circles with radius $w/2$ on each of the $k+1$ vertices of the $k+1$-gon. If any circle is removed, the width of the centers of the remaining sets is w. By the argument in Theorem 2, there exists a line stabber for the k circles. Since the width of S is greater than w, there is no line stabber for the entire family of $k+1$ circles.$\Box$

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