A family of polytopal digraphs that do not satisfy the shelling property

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March 30, 2009

Abstract: A polytopal digraph $G(P)$ is an orientation of the skeleton of a convex polytope P. The possible non-degenerate pivot operations of the simplex method in solving a linear program over P can be represented as a special polytopal digraph known as an LP digraph. Presently there is no general characterization of which polytopal digraphs are LP digraphs, although four necessary properties are known: acyclicity, unique sink orientation(USO), the Holt-Klee property and the shelling property. The shelling property was introduced by Avis and Moriyama (2009), where two examples are given in $d = 4$ dimensions of polytopal digraphs satisfying the first three properties but not the shelling property. The smaller of these examples has $n = 7$ vertices. In this paper for each $d \geq 4$ and $n \geq d + 2$, we construct a polytopal digraph for a polytope P in dimension d with n vertices which is an acyclic USO that satisfies the Holt-Klee property, but does not satisfy the shelling property. It is known that such examples cannot exist for other values of n and d.

Keywords: polytopal digraphs, simplex method, shellability, polytopes

*Research is supported by NSERC and FQRNT.
†Research is supported by KAKENHI.

1 Introduction

Let P be a d-dimensional convex polytope (d-polytope) in $\Re^d$. We assume that the reader is familiar with polytopes, a standard reference being [13]. The vertices and extremal edges of P form an (abstract) undirected graph called the skeleton of P. A *polytopal digraph* $G(P)$ is formed by orienting each edge of the skeleton of P in some manner. In the paper, when we refer to a polytopal digraph $G(P)$ we shall mean the pair of both the digraph and the polytope P itself, not just the abstract digraph.

We can distinguish four properties that the digraph $G(P)$ may have, each of which has been well studied:

- *Acyclicity*: $G(P)$ has no directed cycles.
- *Unique sink orientation (USO)* (Szabó and E. Welzl [10]): Each subdigraph of $G(P)$ induced by a face of P has a unique source and a unique sink.
- *Holt-Klee property* (Holt and Klee [8]): $G(P)$ has a unique sink orientation, and for every k-dimensional face (k-face) H of P there are k vertex disjoint paths from the unique source to the unique sink of H in the subdigraph $G(P,H)$ of $G(P)$ induced by H.
- *LP digraph*: There is a linear function f and a polytope $P^{'}$ combinatorially equivalent to P such that for each pair of vertices u and v of $P^{'}$ that form a directed edge (u,v) in $G(P^{'})$, we have $f(u) < f(v)$. (LP digraphs are called polytopal digraphs in Mihalsin and Klee [9].)

Interest in polytopal digraphs stems from the fact that the simplex method with a given pivot rule can be viewed as an algorithm for finding a path to the sink in a polytopal digraph which is an LP digraph. Research continues on pivot rules for the simplex method since they leave open the possibility of finding a strongly polynomial time algorithm for linear programming. For example, Zadeh's [11][12] history based rules still have not been analyzed. An understanding of which polytopal digraphs are LP digraphs is therefore of interest. The other three properties are necessary properties for $G(P)$ to be an LP-digraph. We note here that Williamson Hoke [7] has defined a property called *complete unimodality* which is equivalent to a combination of acyclicity and unique sink orientation. LP digraphs are completely characterized when $d = 2, 3$. Their necessary and sufficient properties in dimension $d = 2$ are a combination of acyclicity and unique sink orientation, i.e. complete unimodality, and those in $d = 3$ are a combination of acyclicity, unique sink orientation and the Holt-Klee property [9]. On the other hand, no such characterization is known yet for higher dimensions.

Another necessary property for $G(P)$ to be an LP digraph is based on *shelling*, which is one of the fundamental tools of polytope theory. A formal definition of shelling is given in Section 2. Let $G(P)$ be a polytopal digraph for which the polytope P has n vertices labelled $v_1, v_2, ..., v_n$. A permutation r of the vertices is a *topological sort* of $G(P)$ if, whenever (v_i, v_j) is a directed edge of $G(P)$, v_i precedes v_j in the permutation r. Let $L(P)$ be the face lattice of P. A polytope P^* whose face lattice is $L(P)$ "turned upside-down" is called a *combinatorially polar* polytope of P. Combinatorial polarity interchanges vertices of P with facets of P^*. We denote by r^* the facet ordering of P^* corresponding to the vertex ordering of P given by r.

- *Shelling property* (Avis and Moriyama [2]): There exists a topological sort r of $G(P)$ such that the facets of P^* ordered by r^* are a shelling of P^*.

Results relating these properties of polytopal digraphs have been obtained by various authors. For further information, see [2] which contains the following theorem.

Theorem 1 *For polytopal digraphs $G(P)$ based on a d-polytope P, the relationships between acyclicity, USO, the Holt-Klee property, LP-digraph and the shelling property are as shown in Figure 1, where the regions A,B,...,J,X,Y are non-empty.*

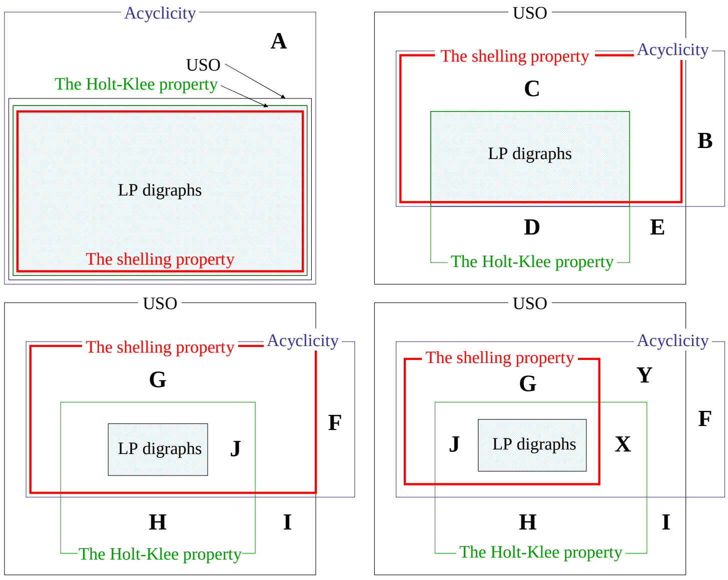

Figure 1: The relationships when $d = 2$[upper left] and $d = 3$[upper right] and the relationships when for $d \geq 4$, P is simple[lower left] and general[lower right] [2]

The existence of the non-empty region X shows the importance of the shelling property, namely that there exist polytopal digraphs satisfying the three existing necessary properties for LP digraphs, but not the shelling property. Two such examples are shown in Figure 2. Develin's example [5] is a polyhedral digraph on the skeleton of a 4-dimensional crosspolytope with eight vertices, and the example proposed by Avis and Moriyama [2] is a polyhedral digraph on a 4-dimensional polytope with seven vertices.

We now present the main results of this paper.

Theorem 2 *There exists a polytope P in $d = 4$ dimension with $n = 6$ vertices for which there is a polytopal digraph $G(P)$ which is an acyclic USO satisfying the Holt-Klee property, but which does not satisfy the shelling property. P is minimal with respect to both the dimension d and the number of vertices n.*

Using this polytope and the operations of truncation and forming a pyramid, we can prove the following result.

Theorem 3 *For every $d \geq 4$ and every $n \geq d + 2$, there exists a polytope P in d dimension with n vertices for which there is a polytopal digraph $G(P)$ which is an acyclic USO satisfying the Holt-Klee property, but which does not satisfy the shelling property.*

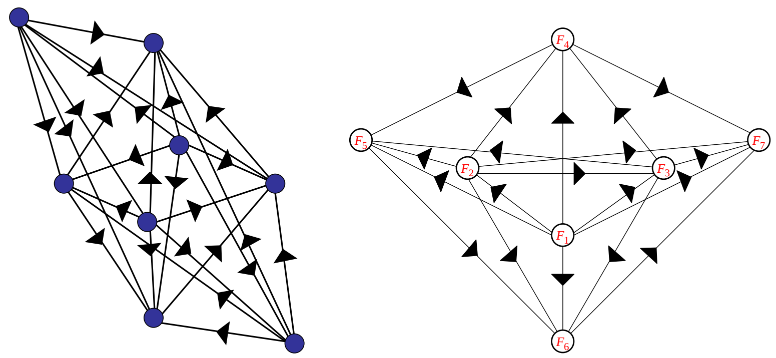

Figure 2: Two X-type graphs by Develin [5] [left] and Avis and Moriyama [2] [right]

For brevity in the rest of the paper we call a polytopal graph $G(P)$ an *X-type graph* if it an acyclic USO satisfying the Holt-Klee property, but not the shelling property.

2 Preliminaries

We use the definition of shelling given in Ziegler [13, Definition 8.1] which is slightly more restrictive than the one used by Brugesser and Mani [3]. Let P be a d-polytope in $\Re^d$. A *shelling* of P is a linear ordering $F_1, F_2, \cdots, F_s$ of the facets of P such that either the facets are points, or it satisfies the following conditions [13, Definition 8.1]:

(i) the first facet F_1 has a linear ordering of its facets which is a shelling of F_1.
(ii) For $1 < j \leq m$ the intersection of the facet F_j with the previous facets is non-empty and is a beginning segment of a shelling of F_j, that is,

$$F_j \cap \bigcup_{i=1}^{j-1} F_i = G_1 \cup G_2 \cup \cdots \cup G_r$$

for some shelling $G_1, G_2, \cdots, G_r, \cdots, G_t$ of F_j, and $1 \leq r \leq t$. (In particular this requires that *all maximal faces included in* $F_j \cap \bigcup_{i=1}^{j-1} F_i$ *have the same dimension* $d-2$.)

Any polytope has at least one shelling because of the existence of *line shellings* [3], described below. Hence the condition (i) is in fact redundant [13, Remark 8.3 (i)].

In this paper we will sometimes identify a facet of P with the set of extreme points of P that it contains.

Let P be a d-polytope with m facets in $\Re^d$. A directed straight line L that intersects the interior of P and the affine hulls of the facets of P at distinct points is called *generic* with respect to P. We choose a generic line L and label a point interior to P on L as x. Starting at x, we number consecutively the intersection points along L with facets as $x_1, x_2, ..., x_m$, wrapping around at infinity, as in Figure 3. The ordering of the corresponding facets of P is the *line shelling* of P generated by L. Every line shelling is a shelling of P (see, e.g., [13]).

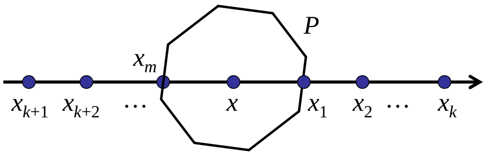

Figure 3: The intersection points along a directed straight line L

vertex	(x_1,x_2,x_3,x_4)	vertex	(x_1,x_2,x_3,x_4)	vertex	(x_1,x_2,x_3,x_4)	vertex	(x_1,x_2,x_3,x_4)
1	(−2, 1, 0, 1)	2	(2, 1, 0, 1)	3	(0, −2, 0, 1)	4	(−4, 2, −2, −1)
5	(4, 2, −2, −1)	6	(0, −4, −2, −1)	7	(0, 0, 2, −1)		

Table 1: The coordinates of the seven vertices of the 4-dimensional polytope Λ^*

3 Construction of an infinite family of X-type graphs

In this section, we give the proofs of the two main theorems.

3.1 Proof of Theorem 2

Let Λ^* be the 4-dimensional polytope with seven vertices $1, 2, ..., 7$ and six facets facets $F_1, F_2, ..., F_6$ shown in Figure 4. The coordinates of the seven vertices are given in Table 1 and the supporting hyperplanes of the six facets of Λ^* are given as $a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 \le b$ with the coefficients in Table 2.

The correctness of this V and H-reprsentation can be checked by standard software such as *cdd* [6], *lrs* [1] or PORTA [4]. Let Λ be a combinatorial polar of Λ^*. A polytopal digraph $G(\Lambda)$ on Λ is also shown in Figure 4. We have the following result.

Proposition 4 *$G(\Lambda)$ is an acyclic USO that satisfies the Holt-Klee property, but not the shelling property.*

Proof: In $G(\Lambda)$ all edges are directed from smaller index to larger index, so it satisfies acyclicity. The proper faces of the polytope Λ consist of i-dimensional simplices for $i \le 3$ except for the facet $\{F_1, F_2, F_3, F_4, F_6\}$. It is easy to check that this non-simplicial facet has a unique source and sink by referring to Figure 4. If the orientation of the skeleton of a simplex is acyclic, it also satisfies both the USO property and the Holt-Klee property. There exist three vertex-disjoint paths from F_1 to F_6 in the non-simplicial facet, and four vertex disjoint paths from F_1 to F_6 in $G(\Lambda)$ (see Figure 4). It follows that the graph $G(\Lambda)$ is an acyclic USO that satisfies the Holt-Klee property. We prove that the graph $G(\Lambda)$ does not satisfy the shelling property.

facet	vertices	(a_1, a_2, a_3, a_4, b)	facet	vertices	(a_1, a_2, a_3, a_4, b)
F_1	1, 2, 4, 5, 7	(0, 2, 1, 0, 2)	F_2	1, 3, 4, 6, 7	(−3, −2, 2, 0, 4)
F_3	2, 3, 5, 6, 7	(3, −2, 2, 0, 4)	F_4	1, 2, 3, 7	(0, 0, 1, 1, 1)
F_5	1, 2, 3, 4, 5, 6	(0, 0, −1, 1, 1)	F_6	4, 5, 6, 7	(0, 0, 0, −1, 1)

Table 2: The six facets of the 4-dimensional polytope Λ^* in Figure 4 and the coefficients of their supporting hyperplanes for the coordinates of the seven vertices in Table 1

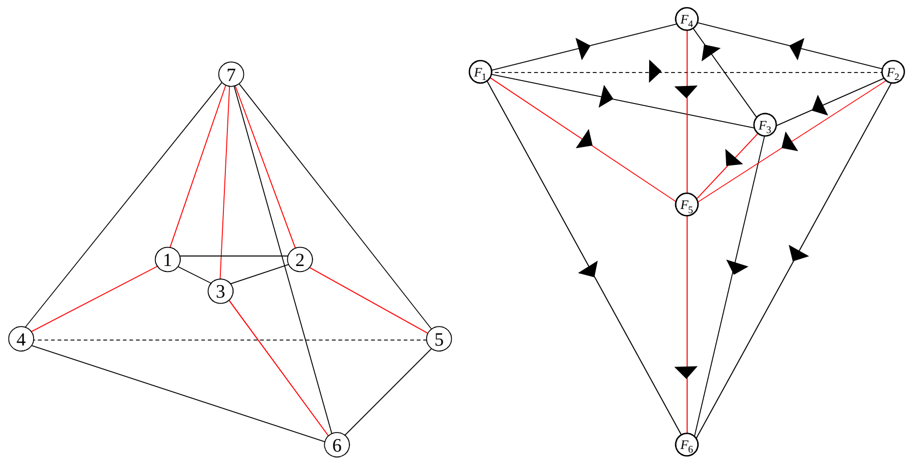

Figure 4: The polytope Λ^* and the polytopal digraph $G(\Lambda)$

There is a path $F_1, F_2, ..., F_6$ of all the six vertices of $G(\Lambda)$ in order of their indices, and so this ordering is the unique topological sort of the graph. By referring to Table 2 and Figure 4, we see that the first three facets of Λ^* in this order are: $F_1 = \{1, 2, 4, 5, 7\}$, $F_2 = \{1, 3, 4, 6, 7\}$, and $F_3 = \{2, 3, 5, 6, 7\}$. Therefore $F_3 \cap \bigcup_{i=1}^{2} F_i$ is the union of two 2-faces of Λ^*: one with vertices $\{2, 5, 7\}$ and one with vertices $\{3, 6, 7\}$. These two faces intersect at a single vertex and so cannot be the beginning of a shelling of F_3. Hence the unique topological sort of $G(\Lambda)$ is not a shelling of Λ^*. This completes the proof. $\square$

By Theorem 1 [2], there can only exist an X-type graph $G(P)$ if P is non-simple and the dimension of P is at $d \geq 4$. Since P cannot be a simplex, it must have at least $d + 2$ vertices. Thus $G(\Lambda)$ is a minimal X-type graph with respect to the dimension d and the number of vertices of P. Therefore we obtain Theorem 2.

3.2 Proof of Theorem 3

To generate a family of polytopal digraphs with the required properties we make use of two operations. The first is the *truncation* of a 4-dimensional polytope.

Definition 5 *(Truncation) Let P be a 4-dimensional polytope in $\Re^4$ containing a simple vertex v, i.e. a vertex v with exactly four neighbors, $\{v_i \ : \ i = 1, 2, 3, 4\}$ the vertices adjacent to v, and $\{u_j \ : \ j = 1, 2\}$ points in the relative interior of an edge (v, v_j). A truncated polytope $tr(P, v)$ is a 4-dimensional polytope $P \cap (H \cup H^+)$ where H is the hyperplane determined by u_1, u_2, v_3 and v_4, and H^+ is the open halfspace of H containing all vertices except v, see Figure 5.*

Using truncation we build a family of polytopes and polytopal digraphs in four dimensions starting from Λ and $G(\Lambda)$. Set $\Lambda_{4,6} := \Lambda$. Note vertex F_6 of Λ, is adjacent to the four vertices F_1, F_2, F_3 and F_5. we generate the truncated polytope $\Lambda_{4,n} := \mathrm{tr}(\Lambda_{4,n-1}, F_{n-1})$ repeatedly by adding a new vertex F_n (resp. F_{n-1}) in the relative interior of the edge (F_{n-1}, F_3) (resp. (F_{n-1}, F_{n-2})) of $\Lambda_{4,n-1}$. Note that $\Lambda_{4,n}$ has the n vertices $F_1, \cdots, F_n$. As with $G(\Lambda)$ in Figure 4, we orient all edges of the skeleton of $\Lambda_{4,n}$ from smaller index to larger index, and denote the directed graph by $G(\Lambda_{4,n})$. Figure 6 shows how to generate $G(\Lambda_{4,7})$. As we see in Figure

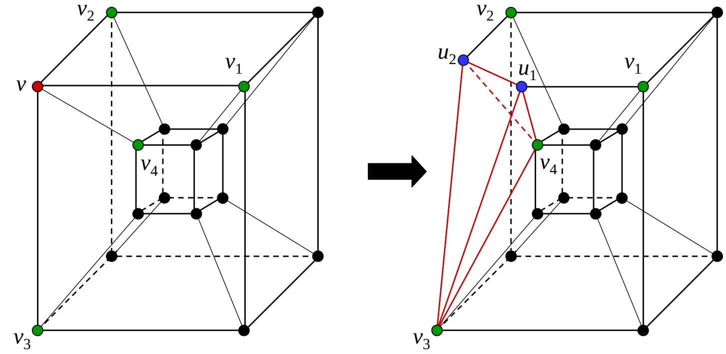

Figure 5: A truncation operation for a 4-dimensional polytope

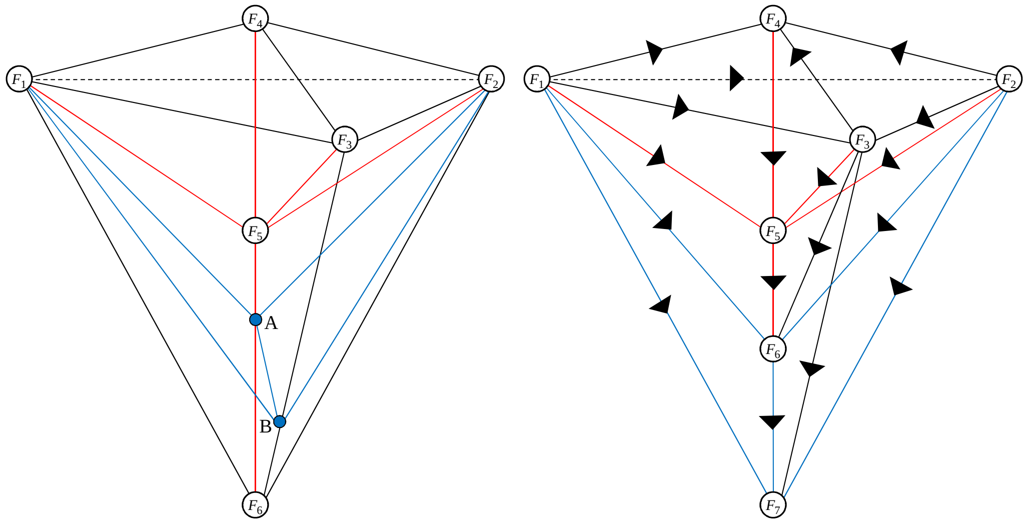

Figure 6: Generation of the graph $G(\Lambda_{4,7})$

6, the truncation from $\Lambda_{4,6}$ to $\Lambda_{4,7}$ generates the new facet $\{F_1, F_2, F_6, F_7\}$ of $\Lambda_{4,7}$, changes the facets $\{F_1, F_3, F_5, F_6\}$, $\{F_2, F_3, F_5, F_6\}$ $\{F_1, F_2, F_3, F_4, F_6\}$ of $\Lambda_{4,6}$ into $\{F_1, F_3, F_5, F_6, F_7\}$, $\{F_2, F_3, F_5, F_6, F_7\}$ and $\{F_1, F_2, F_3, F_4, F_7\}$ respectively, and preserves the other facets of $\Lambda_{4,6}$. In the next truncation from $\Lambda_{4,7}$ to $\Lambda_{4,8}$, one simplex is added to $\Lambda_{4,8}$, and the new vertex F_8 is added to the two facets of of $\Lambda_{4,7}$. Therefore, we obtain the graph $G(\Lambda_{4,n})$ for $n \geq 6$ as in Figure 7.

Lemma 6 $\Lambda_{4,n}$ *consists of the following* $n+1$ *facets:*
$\{F_1, F_3, F_4, F_5\}$, $\{F_2, F_3, F_4, F_5\}$, $\{F_1, F_2, F_i, F_{i+1}\}$ (*for* $4 \leq i \leq n-1$),
$\{F_1, F_3, F_5, F_6, F_7, ..., F_n\}$, $\{F_2, F_3, F_5, F_6, F_7, ..., F_n\}$, $\{F_1, F_2, F_3, F_4, F_n\}$.

We prove the following proposition.

Proposition 7 *For every* $n \geq 6$, $G(\Lambda_{4,n})$ *is an X-type graph.*

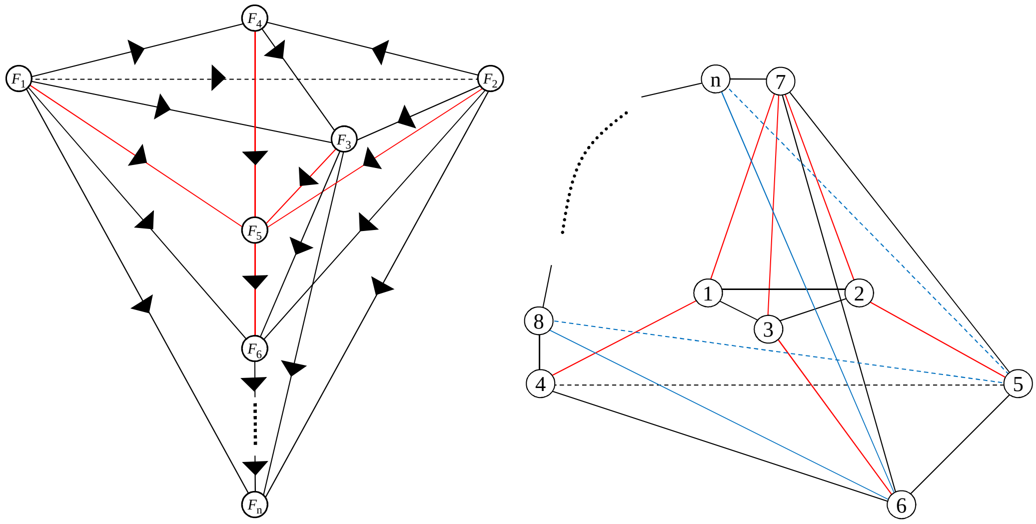

Figure 7: The graph $G(\Lambda_{4,n})$ [left] and its combinatorial polar polytope $\Lambda^*_{4,n}$ [right]

Proof: All edges of $G(\Lambda_{4,n})$ are directed from smaller index to larger index, so $G(\Lambda_{4,n})$ satisfies acyclicity. The polytope $\Lambda_{4,n}$ consists of i-dimensional simplices for $i \leq 3$, except for the three facets: $\{F_1, F_3, F_5, F_6, F_7, ..., F_n\}$, $\{F_2, F_3, F_5, F_6, F_7, ..., F_n\}$ and $\{F_1, F_2, F_3, F_4, F_n\}$. We can check that the induced subgraphs on the three facets satisfy both the USO property and the Holt-Klee property. There exist four vertex-disjoint paths from F_1 to F_6 in $G(\Lambda) = G(\Lambda_{4,6})$, as shown in Figure 4. Therefore there is a vertex disjoint path to each of the four neighbours of F_6. In $G(\Lambda_{4,7})$ each of these neighbours is connected to the new sink F_7. Therefore there are four vertex disjoint paths from F_1 to F_7 in $G(\Lambda_{4,7})$. Proceeding by induction, we see that each graph $G(\Lambda_{4,n})$ satisfies the Holt-Klee property. We prove that the graph $G(\Lambda_{4,n})$ does not satisfy the shelling property.

We consider the facets of a combinatorial polar of $\Lambda_{4,n}$, denoted by $\Lambda^*_{4,n}$, as in Figure 7. There is a path $F_1, F_2, ..., F_n$ through the n vertices of $G(\Lambda_{4,n})$ in order of their indices, and so this ordering is the unique topological sort of the graph. For $n \geq 6$, the first three facets of $\Lambda^*_{4,n}$ in this order are: $F_1 = \{1, 2, 4, 5, 7, \cdots, n+1\}$, $F_2 = \{1, 3, 4, 6, 7, \cdots, n+1\}$, $F_3 = \{2, 3, 5, 6, 7\}$. Therefore, as in the proof of Proposition 4, $F_3 \cap \bigcup_{i=1}^{2} F_i$ is the union of two 2-faces of Λ^* that intersect at a single vertex: one 2-face with vertices $\{2, 5, 7\}$ and the other with vertices $\{3, 6, 7\}$. This union cannot be the beginning of a shelling of F_3. This completes the proof. $\square$

Next we prove that given an X-type graph $G(\Lambda_{4,k+2})$ for $k \geq 2$, there exists an X-type graph on the skeleton of a d-dimensional polytope with $k+d$ vertices for every $d \geq 5$. For this purpose, we use another well known operation, the *pyramid* of a d-dimensional polytope.

Definition 8 *(Pyramid) Given a d-dimensional polytope P in $\Re^d$, its pyramid polytope $py(P, v)$ is a $(d+1)$-dimensional polytope in $\Re^{d+1}$ which is the convex hull of $P \times \{0\}$ and a point $v \in \Re^{d+1}$ not on the d-dimensional subspace containing P. A canonical choice is to set $v_{d+1} = 1$, see Figure 8.*

Starting with any X-type graph $\Lambda_{4,k+4}$ for $k \geq 2$, we generate the pyramid polytope $\Lambda_{d,k+d} :=$ py$(\Lambda_{d-1,k+d-1}, v)$ recursively. As with $G(\Lambda)$, we orient all edges of the skeleton of $\Lambda_{d,k+d}$ from

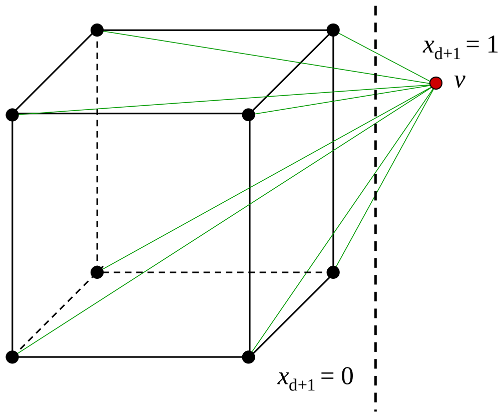

Figure 8: A pyramid polytope of a d-dimensional polytope

smaller index to larger index forming the polytopal digraph $G(\Lambda_{d,k+d})$. Figure 9 shows the graph $G(\Lambda_{5,7})$ generated from $G(\Lambda_{4,6})$, and its combinatorial polar polytope $\Lambda^*_{5,7}$.

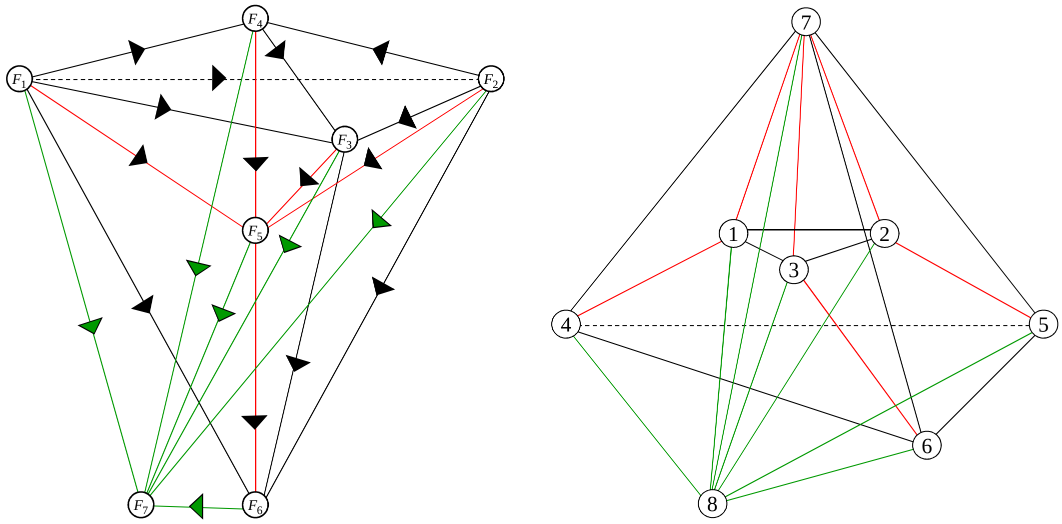

Figure 9: The graph $G(\Lambda_{5,7})$ and its combinatorial polar polytope $\Lambda^*_{5,7}$

Lemma 9 $\Lambda_{d,k+d}$ *consists of the following* $k+d-3$ *facets:*
$\{F_1, F_3, F_4, F_5, F_{k+1}, F_{k+2}, ..., F_{k+d-4}\}$, $\{F_2, F_3, F_4, F_5, F_{k+2}, ..., F_{k+d-4}\}$,
$\{F_1, F_2, F_i, F_{i+1}, F_{k+2}, ..., F_{k+d-4}\}$ (*for* $4 \leq i \leq k-1$), $\{F_1, F_3, F_5, F_6, F_7, ..., F_k, F_{k+2}, ..., F_{k+d-4}\}$,
$\{F_2, F_3, F_5, F_6, F_7, ..., F_k, F_{k+2}, ..., F_{k+d-4}\}$, $\{F_1, F_2, F_3, F_4, F_k, F_{k+2}, ..., F_{k+d}\}$,
$\{F_1, F_2, ..., F_k, ..., \hat{F}_{k+j}, ..., F_{k+d-4}\}$ (*for* $1 \leq j \leq d-4$).

For $d \geq 5$ and $k \geq 2$, let $F[d-1, k+d-1]_i$ denote the i-th facet of $\Lambda^*_{d-1,k+d-1}$. By Lemma 9, the $k+d$ facets of $\Lambda^*_{d,k+d}$ consist of $F[d, k+d]_{k+d} = \Lambda^*_{d-1,k+d-1}$ itself, and

$$F[d, k+d]_i = F[d-1, k+d-1]_i \cup \{k+d+1\} \quad \text{for } 1 \leq i \leq k+d-1. \qquad (1)$$

We prove the following proposition.

Proposition 10 *For every $d \geq 5$ and $k \geq 2$ there exists an X-type graph $G(\Lambda_{d,k+d})$.*

Proof: Given a polytopal digraph $G(P)$ we form the polytopal digraph $G(\mathrm{py}(P,v))$ by using the same edge orientations as in $G(P)$ and orienting the additional edges (u,v) from u to v. We first prove that if $G(P)$ is an acyclic USO satisfying the Holt-Klee property, $G(\mathrm{py}(P,v))$ is also. Let s (resp. t) be the global source (resp. sink) in $G(P)$. The vertex v is adjacent to all vertices of P, hence v becomes the global sink of $G(\mathrm{py}(P,v))$ and s is also the global source of $G(\mathrm{py}(P,v))$. Therefore $G(\mathrm{py}(P,v))$ is an acyclic USO. Moreover, from the assumption, there exist at least d vertex-disjoint paths from s to t in $G(P)$. Here we note that the vertices before t on the paths are adjacent to v in $G(\mathrm{py}(P,v))$, hence there also exist at least d vertex-disjoint paths from s to v in $G(\mathrm{py}(P,v))$. It follows that if for some d, $G(\Lambda_{d,k+d})$ is an acyclic USO satisfying the Holt-Klee property, then so is $G(\Lambda_{d+1,k+d+1})$. Since for $d=4$ this hypothesis is true, the conclusion holds for all $d \geq 4$ by induction.

Finally, we prove that $G(\Lambda_{d,k+d})$ does not satisfy the shelling property. It is immediate from the construction that there is a path $F_1, F_2, ..., F_{k+d}$ through all of the $k+d$ vertices of $G(\Lambda_{d,k+d})$. This ordering is the unique topological sort of the graph. Let $\Lambda^*_{d,k+d}$ be a combinatorial polar of $\Lambda_{d,k+d}$, and $F[d,k+d]_i := F_i$ be the i-th facet of $\Lambda^*_{d,k+d}$ for $1 \leq i \leq k+d$. The d-dimensional polytope $\Lambda^*_{d,k+d}$ is a pyramid on $\Lambda^*_{d-1,k+d-1}$. For $d \geq 5$ and $k \geq 2$ it follows from (1) that for $i = 1, \cdots, k+d-1$,

$$F[d,k+d]_i = F[4,k+4]_i \cup \{k+6, \cdots, k+d+1\}.$$

The first three facets of $\Lambda^*_{4,k+4}$ are given in Proposition 7. This implies that for $d \geq 5$ and $k \geq 2$ the first three facets of $\Lambda^*_{d,k+d}$ are defined by the following vertex sets:

$$F_1 = \{1,2,4,5,7,\cdots,k+d+1\}, F_2 = \{1,3,4,6,7,\cdots,k+d+1\}, F_3 = \{2,3,5,6,7,k+6,\cdots,k+d+1\}.$$

Therefore, $F[d,k+d]_3 \cap \bigcup_{i=1}^{2} F[d,k+d]_i$ is the union of two $(d-2)$-faces of $\Lambda^*_{d,k+d}$: one with vertices $\{2,5,7,k+6,\cdots,k+d+1\}$ and one with vertices $\{3,6,7,k+6,\cdots,k+d+1\}$. These $(d-2)$-faces intersect in a face of dimension at most $d-4$. This cannot be the beginning of a shelling of $F[d,k+d]_3$, hence the unique topological sort of $G(\Lambda_{d,k+d})$ is not a shelling of $\Lambda^*_{d,k+d}$. This completes the proof. $\square$

By combining Proposition 7 with Proposition 10, we obtain Theorem 3.

4 Concluding remarks

In this paper, for $d \geq 4$ and $n \geq d+2$, we constructed an infinite family of polytopal digraphs $G(P)$ in dimension d with n vertices which are acyclic USOs satisfying the Holt-Klee property, but not the shelling property. Previously only two 4-dimensional examples were known, one by by Develin [5] and one by Avis and Moriyama [2]. It is known that no such examples can exist if $d \leq 3$ or $n = d+1$. Our examples show that the shelling property has significance as a necessary property of LP digraphs in higher dimensions.

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