Metric Extensions and the L^1 Hierarchy

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ABSTRACT

A finite semimetric is L^1 − embeddable if it can be expressed as a nonnegative combination of Hamming semimetrics. A finite semimetric is called *hypermetric* if it satisfies the $(2k+1)$ − *gonal* inequalities which naturally generalize the triangle inequality. It is known that all L^1 − embeddable semimetrics are hypermetric and the metric induced by $K_7 - P_3$ is hypermetric but not L^1 − embeddable. In the first part of the paper we show that there are infinite metrics that are hypermetric but are not L^1 − embeddable, answering a question of M. Deza. We introduce the r − *extension* of a semimetric: this is the addition of new points from which all of the distances are r. We show that all 2-extensions of $K_7 - P_3$ are hypermetric. Unfortunately, 2 − *extensions* of hypermetrics in which all of the distances are either one or two may not be hypermetric in general. We illustrate this by showing that the metric induced by the star $K_{1,5}$ (which is hypermetric) has a nonhypermetric 2-extension involving three additional points.

In the second part of the paper, we show that 1-extensions of the metric induced by $K_4 - P_3$ form a family that span a hierarchy of metrics that contain L^1. A finite semimetric is of
if it is the square of a semimetric that can be isometrically embedded in Euclidean space. It is known that a semimetric that is hypermetric is of negative type; and the distance matrix of a semimetric of negative type has one positive eigenvalue. All of the inclusions in the above hierarchy are strict. We demonstrate this latter fact by the 1-extensions of $K_4 - P_3$. Indeed, $K_6 - P_3$ is L^1 − embeddable and $K_7 - P_3$ is hypermetric but not L^1 − embeddable. Here we show that $K_8 - P_3$ is hypermetric but $K_9 - P_3$ is not; $K_{10} - P_3$ is of negative type but $K_{11} - P_3$ is not; and $K_{11} - P_3$ has one positive eigenvalue but $K_{12} - P_3$ does not. This situation can be contrasted with the collapse of the metric hierarchy for bipartite graphs.

1. Introduction

For an integer n and a finite set $\mathbf{S} = \{s_1, \ldots, s_n\}$, let $(\mathbf{S}, \mathbf{d})$ be a semimetric space. In other words,

$$\mathbf{d}(s_i, s_j) = \mathbf{d}(s_j, s_i), \quad \mathbf{d}(s_i, s_i) = 0, \quad \text{and}$$

$$\mathbf{d}(s_i, s_j) \leq \mathbf{d}(s_i, s_k) + \mathbf{d}(s_k, s_j) \quad \text{for distinct } i, j, k \in \{1, \ldots, n\}.$$

Throughout the paper, when we talk of the metric induced by a connected graph, we are referring to the metric on the vertices where the distance between two vertices is the number of edges on a shortest path between them. K_n denotes the complete graph on n vertices, $K_{m,n}$ denotes the complete bipartite graph with m vertices in one part and n vertices in the other, and P_n denotes the path on n vertices. Where H is a subgraph of G, $G - H$ denotes the graph G with the edges of H removed.

A semimetric is called *Hamming* if for some $t \geq 0$ and for some non-empty $S \subset \{1, \ldots, n\}$

$$\mathbf{d}(s_i, s_j) = \begin{cases} t & i \in S,\ j \notin S \ \ or \ \ i \notin S,\ j \in S \\ 0 & \text{otherwise.} \end{cases} \tag{1.1}$$

It is convenient to represent a semimetric $\mathbf{d}$ as a point d in $R^{\binom{n}{2}}$ by writing $d = (\mathbf{d}(s_1, s_2), \mathbf{d}(s_1, s_3), \ldots, \mathbf{d}(s_{n-1}, s_n))$. Using this representation, the convex hull of the set of all Hamming semimetrics is called the *Hamming cone*, H_n. A semimetric $\mathbf{d}$ is called L^1-embeddable if d is contained in H_n. For other equivalent definitions and properties of these cones the reader is referred to the monograph by[1] and the survey paper[5]. For example if $\mathbf{d}$ is rational, $(\mathbf{S}, \mathbf{d})$ is L^1-embeddable if and only if $(\mathbf{S}, k\mathbf{d})$ embeds isometrically into a Hamming N-cube (hypercube) for some N and some $k > 0$. A class of inequalities that are valid over the Hamming cone were introduced by Deza[7] [6] and are known as the $(2k+1)-$*gonal* inequalities:

$$\sum_{1 \leq i,j \leq n} \lambda_i \lambda_j \mathbf{d}(s_i, s_j) \leq 0 \tag{1.2}$$

where

$$\sum_{i=1}^{n} \lambda_i = 1 \qquad \sum_{i=1}^{n} |\lambda_i| = 2k+1 \qquad \lambda_i \text{ integer.}$$

A semimetric space $(\mathbf{S}, \mathbf{d})$ is $(2k+1)-$*gonal* if it satisfies all of the $(2k+1)$-gonal inequalities. If we replace $2k+1$ by $2k$ in (1.2) we get the $2k$-gonal inequalities. These are of limited interest, since Deza[6] has shown that a semimetric that is $(2k+1)$-gonal is also $(2k+2)$-gonal. A similar set of inequalities were discovered by J. Kelly[9] [10], who calls a semimetric *hypermetric* if it is $(2k+1)-$*gonal* for all k. The 3-gonal inequalities are just the familiar triangle inequalities, and so all semimetrics are 3-gonal. Deza[6] showed that all L^1-embeddable semimetrics are hypermetric and the reverse implication is true for metrics on at most 5 points. Avis and Mutt[4] showed that hypermetrics on six points are L^1-embeddable. Avis[2] [3] showed that the metric induced by $K_6 - P_3$ is L^1-embeddable but that the metric induced by $K_7 - P_3$ is hypermetric but not L^1-embeddable.

For a positive number r, an $r-$*extension* of a semimetric is formed by adding new points from which all distances are r. If the original metric has all distances one or two, then all 1-extensions and 2-extensions are also metrics. $K_n - P_3$, $n \geq 5$ is the family of 1-extensions of $K_4 - P_3$. We were led to the notion of extensions by the following question of M. Deza: Are there infinite hypermetrics that are not L^1-embeddable? In Section 2 we answer this affirmatively by showing that all 2-extensions of $K_7 - P_3$ are hypermetric. Unfortunately the class of hypermetrics with distances one and two is not closed under 2-extension: we show that the L^1-embeddable (and hence hypermetric) star $K_{1,5}$ has a 2-extension with three new points which is not hypermetric.

In section 3, we show that 1-extensions of $K_4 - P_3$ play a useful role in spanning a hierarchy based on L^1. The first three classes in the hierarchy are metrics embeddable in a hypercube, metrics embeddable in L^1 and hypermetrics. Similar in structure to the hypermetric inequalities are the negative type inequalities:

$$\sum_{1 \leq i,j \leq n} \lambda_i \lambda_j \mathbf{d}(s_i, s_j) \leq 0 \tag{1.3}$$

where

$$\sum_{i=1}^{n} \lambda_i = 0 \qquad \lambda_i \text{ real.}$$

A semimetric space $(\mathbf{S}, \mathbf{d})$ is of if it satisfies all of the inequalities (1.3) for all choices of real λ_i. It follows from I.J. Schoenberg's[13] characterisation of metrics that can be embedded in Euclidean space, that a metric is of negative type if it is the square of a metric embeddable in Euclidean space. It is quite easy to show that semimetrics that are hypermetric are of negative type. From a semimetric space $(\mathbf{S}, \mathbf{d})$ we define the distance matrix D to be the $n \times n$ symmetric matrix with zeroes on the diagonal and $D_{i,j} = \mathbf{d}(s_i, s_j)$. It is known[8] that the distance matrices of semimetrics of negative type have one positive eigenvalue. The L^1-hierarchy can be summarized:

$$d \text{ is embeddable in a hypercube } \rightarrow d \text{ is embeddable in } L^1 \rightarrow d \text{ is hypermetric } \rightarrow$$

Although it is known that inclusion in the hierarchy is strict, there is no unified set of examples. In Section 3, we will show that 1-extensions of $K_4 - P_3$ fill this role. To begin with it is easy to verify that $K_4 - P_3$ cannot be embedded isometrically into a hypercube (since it contains a triangle with all distances one), but it can be embedded into L^1 (since $2d$ is hypercube embeddable). As mentioned, $K_7 - P_3$ is hypermetric but not L^1-embeddable. Assouad and Deza observed that the metric induced by $K_5 - K_3$ is of negative type but is not hypermetric. We give another example by showing that whereas $K_8 - P_3$ is hypermetric, $K_9 - P_3$ is of negative type but is not hypermetric. P. Winkler showed that the graph $K_7 - K_5$ has distance matrix with one positive eigenvalue but is not of negative type[14]. We again illustrate this by showing that whereas $K_{11} - P_3$ is of negative type, the distance matrix of $K_{12} - P_3$ has one positive eigenvalue but $K_{12} - P_3$ is not of negative type.

In summary, we show that all of the inclusions in the L^1-hierarchy are strict by means of a single family. By way of contrast, we mention that R.L. Roth and P.M. Winkler[12] have proved that the hierarchy collapses for bipartite graphs. Specifically they showed that the metric induced by a connected bipartite graph is hypercube embeddable if and only if it has one positive eigenvalue.

2. 2-Extensions

In this section we will show that all 2-extensions of $K_7 - P_3$ are hypermetric. To begin with, we give an equivalent form of (1.2) which is more convenient for our purposes. Let $(\mathbf{S}, \mathbf{d})$ be a semimetric. For points $x, y \in \mathbf{S}$ let $\overline{xy}$ denote $\mathbf{d}(x, y)$. Then (1.2) is equivalent to

$$\sum_{1 \le i < j \le k} \overline{x_i x_j} + \sum_{1 \le i < j \le k+1} \overline{y_i y_j} \le \sum_{\substack{1 \le i \le k \\ 1 \le j \le k+1}} \overline{x_i y_j}, \tag{2.1}$$

where $x_1, \ldots, x_k, y_1, \ldots, y_{k+1}$ is a $(2k+1)$-tuple from S, where the x_i, y_i are not necessarily distinct. To see this, consider a positive λ_i in (1.2). Choose λ_i of the $y_1, \cdots, y_{k+1}$ to be s_i. If λ_i is negative, choose $-\lambda_i$ of the $x_1, \cdots, x_k$ to be s_i. Rearranging terms and changing the notation, (1.2) becomes (2.1). On the other hand, if

$$\{x_1, \ldots, x_k\} \cap \{y_1, \ldots, y_{k+1}\} = \varnothing$$

then the reverse transformation shows that (2.1) implies (1.2). If the intersection is not empty, then (2.1) implies a $(2j+1)$-gonal inequality for some $j < k$. We begin with a simple lemma.

Lemma2.1. If S is (2k-1)-gonal and

$$\sum_{j=2}^{k+1} \overline{y_1 y_j} \le \sum_{i=1}^{k} \overline{x_i y_1}. \tag{2.2}$$

then (2.1) holds for $x_1, \ldots, x_k, y_1, \ldots, y_{k+1}$.

Proof: Since S is $(2k-1)$-gonal, by the remark in the introduction, it is $2k$-gonal. Therefore the $2k$-gonal inequality is satisfied for

$$x_1, \ldots, x_k, y_2, \ldots, y_{k+1}.$$

Adding this inequality to (2.2) gives (2.1) □.

The following theorem generalizes the proof in[2] that $K_7 - P_3$ is hypermetric.

Theorem 2.1. All 2-extensions of $K_7 - P_3$ are hypermetric.

Proof: Let $(\mathbf{S},\mathbf{d})$ be the metric space corresponding to a 2-extension of $K_7 - P_3$. Suppose that the points are labelled so that $s_1, \ldots, s_7$ are the points of $K_7 - P_3$ and $\overline{s_1 s_2} = \overline{s_1 s_3} = 2$. Colour the points of S as follows: s_1 is red; s_2 and s_3 are blue; s_4, s_5, s_6, s_7 are yellow; all other points are white (see Figure 2.1). Clearly S is a metric and so it satisfies the 3-gonal (triangle) inequalities.

Assume inductively that S is $2k-1$-gonal. We will show that it is $2k+1$-gonal. Let $x_1, \ldots, x_k, y_1, \ldots, y_{k+1}$ be a $(2k+1)$-tuple from S. We may assume that

$$\{x_1, \ldots, x_k\} \cap \{y_1, \ldots, y_{k+1}\} = \varnothing$$

for otherwise (2.1) follows from the induction hypothesis.

First observe that if y_1 is white, then

$$\sum_{j=2}^{k+1} \overline{y_1 y_j} \le 2k = \sum_{i=1}^{k} \overline{x_i y_1}, \tag{2.2}$$

and (2.1) holds by the lemma.

We may therefore suppose that $Y = \{y_1, \ldots, y_{k+1}\}$ contains no white points. If y_1 is yellow, it must be distance one from all other points in Y. Therefore

$$\sum_{j=2}^{k+1} \overline{y_1 y_j} \le k \le \sum_{i=1}^{k} \overline{x_i y_1}.$$

and again (2.1) is holds by the lemma.

We may now suppose that the points of Y are coloured red or blue. Suppose that some point, say y_1 occurs more than $k/2$ times in Y. Then we have

$$\sum_{j=2}^{k+1} \overline{y_1 y_j} \le 2 \times \frac{k}{2} \le \sum_{i=1}^{k} \overline{x_i y_1}$$

which again implies (2.1). We conclude therefore that $Y = \{s_1, s_2, s_3\}$ and $x_1, \cdots, x_{k+1}$ consists of yellow and white points. Relabelling if necessary, suppose $y_1 = s_1$, the unique red point, and y_{k+1} is the blue point that occurs at least as often as the other blue point. Let r denote the number of times s_1 occurs in $y_1, \ldots, y_{k+1}$. Let $A = \sum_{j=2}^{k+1} \overline{y_1 y_j}$, $B = \sum_{i=1}^{k} \overline{y_i y_{k+1}}$, and $f = min(A, B)$. Since

$A = 2k - 2r + 2$ and $B \le (k + 1 - r)/2 + 2r$, a simple calculation shows that

$$f \le \lfloor 8(k + 1)/7 \rfloor = k + 1 + \lfloor (k + 1)/7 \rfloor.$$

In what follows, we assume that

$$\sum_{j=2}^{k+1} \overline{y_1 y_j} \le f,$$

the case where $B \le f$ is similar.

Let w be the number of times white points appear in $x_1, \ldots, x_k$. If $w \ge 1 + \lfloor (k + 1)/7 \rfloor$ then

$$\sum_{i=1}^{k} \overline{y_1 x_i} = k + w \ge k + 1 + \lfloor (k + 1)/7 \rfloor \ge f \ge \sum_{j=2}^{k+1} \overline{y_1 y_j},$$

and (2.1) follows from the lemma.

Suppose now that $w \le \lfloor (k + 1)/7 \rfloor$. Since a yellow point, say x_1 appears at least $(k - w)/4$ times,

$$\sum_{i=2}^{k} \overline{x_1 x_i} \le (k - 1) + w - \frac{k - w}{4} + 1 = \frac{3k}{4} + \frac{5w}{4},$$

and hence

$$\sum_{i=2}^{k} \overline{x_1 x_i} + \sum_{i=2}^{k+1} \overline{y_1 y_i} \le \frac{3k}{4} + \frac{5w}{4} + f.$$

On the other hand, since

$$\sum_{i=1}^{k} \overline{y_1 x_i} = k + w,$$

it follows that

$$\sum_{j=2}^{k+1} \overline{x_1 y_j} + \sum_{i=1}^{k} \overline{x_i y_1} = 2k + w.$$

Now since $w \le \lfloor (k + 1)/7 \rfloor$, we have

$$\sum_{j=2}^{k+1} \overline{x_1 y_j} + \sum_{i=1}^{k} \overline{x_i y_1} - \sum_{i=2}^{k} \overline{x_1 x_i} - \sum_{i=2}^{k+1} \overline{y_1 y_i} > 2k + w - \frac{3k}{4} - \frac{5w}{4} - f = \frac{5k}{4} - \frac{w}{4} - f$$

$$\leq \frac{5k}{4} - \frac{w}{4} - (k+1) - \lfloor \frac{k+1}{7} \rfloor \geq -1 + \frac{k}{4} - \frac{5}{4} \lfloor \frac{k+1}{7} \rfloor > -1.$$

In fact since all numbers are integers we have

$$\sum_{j=2}^{k+1} \overline{x_1 y_j} + \sum_{i=1}^{k} \overline{x_i y_1} - \sum_{i=2}^{k} \overline{x_1 x_i} - \sum_{i=2}^{k+1} \overline{y_1 y_i} \geq 0. \tag{2.3}$$

Now $x_2, \ldots, x_k, y_2, \ldots, y_{k+1}$ satisfy the $(2k-1)$-gonal inequality by the inductive hypothesis. Adding this inequality to (2.3) we obtain (2.1). $\square$.

We may now answer Deza's question about the existence of infinite metrics that are hypermetric but not L^1 – embeddable. Let S be an infinite metric in which all distances are two except that a subset of 7 points induce a $K_7 - P_3$. The resulting metric must be hypermetric, for otherwise it would not be $(2k+1)$-gonal for some k. The set of points $x_1, \ldots, x_k, y_1, \ldots, y_{k+1}$ which violate the inequality embed into a finite extension of $K_7 - P_3$, contradicting Theorem 2.1.

Unfortunately, the class of hypermetrics with distances one or two is not closed under 2-extension. Consider the metric induced by $K_{1,5}$. This metric is L^1 – embeddable, in fact it is embeddable in a hypercube, as shown in Figure 2.2(a). However the 2-extension formed by adding three new points is not hypermetric since it is not 19-gonal, as shown in Figure 2.2(b).

3. 1-Extensions

In this section we show that $K_n - P_3$ forms a family of metrics that span the L^1 hierarchy. In what follows we assume that the vertices are labelled $s_1, \ldots, s_n$ and the non-edges are $s_1 s_2$ and $s_1 s_3$. It is easy to check that $K_4 - P_3$ in not embeddable in Euclidean space. As mentioned in the introduction, $K_6 - P_3$ is L^1 – embeddable but $K_7 - P_3$ is not. Along the lines of the proof of Theorem 2.1, one can prove that $K_8 - P_3$ is hypermetric. A non-hypermetric member of the family is $K_9 - P_3$. To see this assign the weights

$$\lambda_1 = 3,\ \lambda_2 = \lambda_3 = 2,\ \lambda_4 = \cdots = \lambda_7 = -1.$$

One can readily check that the left hand side of (1.2) has the value 1. A member of the family which is not of negative type is $K_{11} - P_3$. To see this assign the weights

$$\lambda_1 = \frac{24}{7}\,,\ \lambda_2 = \lambda_3 = \frac{16}{7}\,,\ \lambda_4 = \cdots = \lambda_{11} = -1.$$

One can readily check that the left hand side of (1.3) has the value 4/7. The next theorem shows that $K_{11} - P_3$ has one positive eigenvalue, but $K_{12} - P_3$ has two.

Theorem 3.1 The distance matrix D of $K_{n+3} - P_3$ has one positive root for $n \le 8$ and two positive roots for $n \ge 9$.

Proof: We calculate the characteristic polynomial of D.

$$|\, D - xI \,| = \begin{vmatrix} -x & 2 & 2 & 1 & 1 & . & . & 1 \\ 2 & -x & 1 & 1 & 1 & . & . & 1 \\ 2 & 1 & -x & 1 & 1 & . & . & 1 \\ 1 & 1 & 1 & -x & 1 & . & . & 1 \\ . & . & . & 1 & . & . & . & . \\ . & . & . & . & . & . & . & 1 \\ 1 & 1 & 1 & 1 & 1 & . & 1 & -x \end{vmatrix}$$

$$= \begin{vmatrix} -x & x+2 & x+2 & 1 & 1 & . & . & 1 \\ x+2 & -2(x+2) & -(x+3) & 0 & 0 & . & . & 0 \\ x+2 & -(x+3) & -2(x+2) & 0 & 0 & . & . & 0 \\ 1 & 0 & 0 & -x & 1 & . & . & 1 \\ . & . & . & 1 & . & . & . & . \\ . & . & . & . & . & . & . & 1 \\ 1 & 0 & 0 & 1 & 1 & . & 1 & -x \end{vmatrix}$$

$$= \begin{vmatrix} -x & x+2 & x+2 & 1 & 1 & . & . & 1 \\ 0 & -2(x+2) & -(x+3) & 0 & 0 & . & . & 0 \\ 0 & -(x+3) & -2(x+2) & 0 & 0 & . & . & 0 \\ 0 & 0 & 0 & -x & 1 & . & . & 1 \\ . & . & . & 1 & . & . & . & . \\ . & . & . & . & . & . & . & 1 \\ 0 & 0 & 0 & 1 & 1 & . & 1 & -x \end{vmatrix} + \begin{vmatrix} 0 & x+2 & x+2 & 1 & 1 & . & . & 1 \\ x+2 & -2(x+2) & -(x+3) & 0 & 0 & . & . & 0 \\ x+2 & -(x+3) & -2(x+2) & 0 & 0 & . & . & 0 \\ 1 & 0 & 0 & -x & 1 & . & . & 1 \\ . & . & . & 1 & . & . & . & . \\ . & . & . & . & . & . & . & 1 \\ 1 & 0 & 0 & 1 & 1 & . & 1 & -x \end{vmatrix}$$

By the definition of determinant, we have

$$\begin{vmatrix} 0 & c & d \\ a & A & 0 \\ b & 0 & B \end{vmatrix} = \begin{vmatrix} 0 & c \\ a & A \end{vmatrix} |B| + |A| \begin{vmatrix} 0 & d \\ b & B \end{vmatrix}.$$

Applying this formula we have

$$|\, D - xI \,| = -\, x(x+1)(3x+7)\phi_n(x) + \begin{vmatrix} 0 & x+2 & x+2 \\ x+2 & -2(x+2) & -(x+3) \\ x+2 & -(x+3) & -2(x+2) \end{vmatrix} \phi_n(x)$$

$$+ (x+1)(3x+7)\psi_{n+1}(x),$$

where $\psi_{n+1}(x)$ is the determinant of the $n+1 \times n+1$ matrix

$$\begin{vmatrix} 0 & 1 & 1 & . & . & 1 \\ 1 & -x & 1 & . & . & 1 \\ 1 & 1 & -x & . & . & 1 \\ . & . & . & . & . & . \\ . & . & . & . & -x & 1 \\ 1 & 1 & 1 & . & 1 & -x \end{vmatrix},$$

and $\phi_n(x)$ is the determinant of the $n \times n$ matrix

$$\begin{vmatrix} -x & 1 & . & . & 1 \\ 1 & -x & . & . & . \\ . & . & . & . & . \\ . & . & . & -x & 1 \\ 1 & 1 & . & 1 & -x \end{vmatrix}.$$

Now, since

$$\phi_{n+1}(x) = -\,x\phi_n(x) + \psi_{n+1}(x),$$

we have

$$\psi_{n+1}(x) = \phi_{n+1}(x) + x\phi_n(x).$$

Hence,

$$|\,D - xI\,| = 2(x+1)(x+2)^2\phi_n(x) + (x+1)(3x+7)\phi_{n+1}(x).$$

Now $\phi_n(x) = (-1)^{n-1}(n-1-x)(x+1)^{n-1}$ and so

$$|\,D - xI\,| = (-1)^{n-1}(x+1)^n g_n(x)$$

where

$$g_n(x) = x^3 - nx^2 - (2n+9)x + (n-8).$$

Evaluation of the derivative of $g_n(x)$ shows that it has one negative and one positive root, say $\alpha < 0 < \beta$. Inspection shows that $g_n(x)$ increases to $g_n(\alpha)$, decreases to $g_n(\beta)$ and then increases again. Since $g_n(0) = n-8$, $g_n(x) = 0$ has one positive root if $n \le 8$ otherwise it has two positive roots, proving the theorem. □

Let G be a graph of order n and let $\phi(G: x)$ be its characteristic polynomial. Let

$$P(G: x) = \phi(G: -x) - (-1)^n \phi(\bar{G}: x - 1)$$

where $\bar{G}$ is the complement of G. If G has at least one edge, then it can be shown that $P(G: x)$ has at least one positive root. The following theorem of Maehara[11] gives a condition for a special Euclidean embedding of G.

Theorem 3.2 G can be embedded in Euclidean space in such a way that all edges are of unit length and all non-edges are of length $t > 1$ if and only if

$$1 - \frac{1}{t^2} \le \frac{1}{z_{\max}} ,$$

where $z_{\max}$ is the maximum root of the polynomial $P(G: x)$. □

Using this result, we can obtain the following theorem.

Theorem 3.3 $K_{n+3} - P_3$ is of negative type if and only if $n \le 7$.

Proof: We calculate $P(K_{n+3} - P_3: x)$. First we calculate the characteristic polynomial $\phi(K_{n+3} - P_3: x)$ along similar lines as the calculation of $|D - xI|$ in Theorem 3.1. We have

$$\phi(K_{n+3} - P_3: x) = \begin{vmatrix} -x & 0 & 0 & 1 & 1 & . & . & 1 \\ 0 & -x & 1 & 1 & 1 & . & . & 1 \\ 0 & 1 & -x & 1 & 1 & . & . & 1 \\ 1 & 1 & 1 & -x & 1 & . & . & 1 \\ . & . & . & 1 & . & . & . & . \\ . & . & . & . & . & . & . & 1 \\ 1 & 1 & 1 & 1 & 1 & . & 1 & -x \end{vmatrix} = (-1)^{n-1}(x+1)^n \{\ x^3 - nx^2 - (2n+1)x + n\ \}.$$

Since

$$\phi(\overline{K_{n+3} - P_3}: x) = \phi(P_3 \cup \bar{K}_n: x) = (-x^3 + 2x)(-x)^n,$$

we have

$$\begin{aligned} P(K_{n+3} - P_3: x) &= (-1)^{n-1}(1-x)^n \{ -x^3 - nx^2 + (2n+1)x + n\ \} \\ &\quad - (-1)^{n-1}(1-x)^n \{ -(x-1)^3 + 2(x-1) \} \\ &= (-1)^{n-1}(1-x)^n\ f_n(x) \end{aligned} \tag{3.1}$$

where

$$f_n(x) = -(n+3)x^2 + (2n+2)x + (n+1).$$

The metric induced by $K_{n+3} - P_3$ is of negative type if and only if it is the square of a metric that is embeddable in Euclidean space. But this is the case if and only if there is a Euclidean embedding of $K_{n+3} - P_3$ such that all edges are of length 1 and all non edges are of length $t = \sqrt{2}$. By Theorem 3.2 this can be achieved if and only if $z_{\max} \le 2$. From (3.1) this is the case if and only if the maximum root of $f_n(x)$ is at most 2. Since $f_n(x) = 0$ has one positive root, it suffices to check whether $f_n(2) \le 0$. But $f_n(2) = n - 7$ and so $z_{\max} \le 2$ if and only if $n \le 7$ as required. □

Table 3.1 summarizes the results for the metric properties of $K_n - P_3$. A "*" in the table indicates that the metric is in the given class, a "-" indicates that it is not.

Metric Type	Embeddability of $K_n - P_3$, for $n =$						
	6	7	8	9	10	11	12
Hypercube Embeddable	-	-	-	-	-	-	-
L^1-Embeddable	*	-	-	-	-	-	-
Hypermetric	*	*	*	-	-	-	-
Negative Type	*	*	*	*	*	-	-
One Positive Eigenvalue	*	*	*	*	*	*	-

Table 3.1

References

1. P. Assouad and M. Deza, "Metric Subspaces of L^1," *Publications Mathématiques d'Orsay*, vol. 3, 1982. 47 pages

2. D. Avis, "On the Hamming Cone," Technical Report 77-5, Dept. of Operations Research, Stanford University, 1977.

3. D. Avis, "Hypermetric Spaces and the Hamming Cone," *Canadian Journal of Mathematics*, vol. 33, pp. 795-802, 1981.

4. D. Avis and Mutt, "All the Facets of The Six Point Hamming Cone," *European Journal of Combinatorics*, vol. 10, pp. 309-312, 1989.

5. D. Avis and M. Deza, "L1-Embedability, Complexity and Multicommodity Flows," *Networks*, vol. 21, pp. 595-617, 1991.

6. M. Deza, "Matrices de formes quadratiques non negatives pour des arguments binaires," *C.R. Acad. Sc. Paris*, vol. 277, pp. 873-875, 1973.

7. M. Deza(Tylkin), "On Hamming Geometry of Unit Cubes," *Doklady Akad. Nauk. SSR.*, vol. 134, pp. 1037-1040, 1960. English translation in Soviet Physics Dokl. 5 (1961), 940-943.

8. R.L. Graham and P. M. Winkler, "On Isometric Embeddings of Graphs," *Trans. A.M.S.*, vol. 288, pp. 527-536, 1985.

9. J.B. Kelly, "Metric Inequalities and Symmetric Differences," in *Inequalities II*, ed. Oved Shisha, pp. 193-212, Academic Press, New York, 1970.

10. J. B. Kelly, "Hypermetric Spaces," in *Lecture Notes in Math.*, vol. 490, pp. 17-31, Springer-Verlag, Berlin, 1975.

11. H. Maehara, "Regular Embeddings of a Graph," *Pacific J. Math.*, vol. 107, pp. 393-402, 1983.

12. R.L. Roth and P. M. Winkler, "Collapse of the Metric Hierarchy for Bipartite Graphs," *European Journal of Combinatorics*, vol. 7, pp. 371-375, 1986.

13. I.J. Schoenberg, "Remarks to Maurice Frechet's article," *Annals of Math.*, vol. 36, pp. 724-732, 1935.

14. P.M. Winkler, "On Graphs which are Metric Spaces of Negative Type," in *Graph Theory with Applications to Algorithms and Computer Science*, ed. Y. Alavi et al., pp. 801-810, Wiley, 1985.