All the Facets of the Six Point Hamming Cone

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ABSTRACT

A finite semimetric is L^1 − embeddable if it can be expressed as a non-negative combination of Hamming semimetrics. The cone of such semimetrics is called the Hamming cone. A finite semimetric is called hypermetric if it satisfies the $(2k+1)$ − *gonal* inequalities which naturally generalize the triangle inequality. With the aid of a computer we show that a semimetric on six points is hypermetric if and only if it is L^1 − embeddable and give a complete list of the the facets of the six point Hamming cone. It is known that there are seven point hypermetrics that are not L^1 − embeddable.

1. Introduction

For an integer n and a finite set $\mathbf{X} = \{x_1, \ldots, x_n\}$, let $(\mathbf{X}, \mathbf{d})$ be a semimetric space. In other words,

$$\mathbf{d}(x_i, x_j) = \mathbf{d}(x_j, x_i), \quad \mathbf{d}(x_i, x_i) = 0, \text{ and}$$

$$\mathbf{d}(x_i, x_j) \le \mathbf{d}(x_i, x_k) + \mathbf{d}(x_k, x_j) \quad \text{for distinct } i, j, k \in \{1, \ldots, n\}.$$

Such a semimetric is called *Hamming* if for some $t \ge 0$ and for some non-empty $S \subset \{1, \ldots, n\}$

$$\mathbf{d}(x_i, x_j) = \begin{cases} t & i \in S, j \notin S \\ 0 & \textit{otherwise.} \end{cases} \tag{1}$$

It is convenient to represent a semimetric **d** as a point d in $R^{\binom{n}{2}}$ by writing $d = (\mathbf{d}(x_1, x_2), \mathbf{d}(x_1, x_3), \ldots, \mathbf{d}(x_{n-1}, x_n))$. Using this representation, the convex hull of the set of all Hamming semimetrics is called the *Hamming cone*, H_n. For a fixed subset S, as we let t vary we get an extreme ray (generator) of H_n. Since S and its complement give the same semimetric, there are $2^{n-1} - 1$ extreme rays of H_n. In this paper, we give a list of all of the facets of the Hamming cone on six points, H_6.

An alternative framework for describing this result is in terms of embeddability of finite semimetrics. A semimetric is L^1-embeddable if it is contained in the Hamming cone, H_n. For other equivalent definitions the reader is referred to the monograph by P. Assouad and M. Deza[3]. For example if **d** is rational, $(\mathbf{X}, \mathbf{d})$ is L^1-embeddable if and only if $(\mathbf{X}, k\mathbf{d})$ embeds isometrically into a Hamming N-cube (hypercube) for some N and some $k > 0$. In this paper we give a minimal list of inequalities such that a semimetric on six points is L^1-embeddable if and only if it satisfies all of the inequalities. These inequalities are a subset of a class of inequalities introduced by Deza[8] [7] that are known as the $(2k+1)-$*gonal* inequalities:

$$\sum_{1 \le i,j \le 2k+1} \lambda_i \lambda_j \mathbf{d}(y_i, y_j) \le 0 \tag{2}$$

for $\lambda_1 = \lambda_2 = \cdots = \lambda_{k+1} = 1$, $\lambda_{k+2} = \lambda_{k+3} = \cdots \lambda_{2k+1} = -1$, and $y_1, \ldots, y_{2k+1} \in X$. A semimetric space $(\mathbf{X}, \mathbf{d})$ is $(2k+1)-$*gonal* if it satisfies all of the $(2k+1)$-gonal inequalities. Since we do not require that the y_i be distinct in (2), it is easy to verify that a semimetric space that is $(2k+1)$-gonal is also $(2k-1)$-gonal. A similar set of inequalities were discovered by J. Kelly[10] [11], who calls a semimetric *hypermetric* if it is $(2k+1)-$*gonal* for all k. The 3-gonal inequalities are just the familiar triangle inequalities, and so all semimetrics are 3-gonal. Deza[7] showed that all four point semimetrics are L^1-embeddable and that all 5-gonal five point semimetrics are L^1-embeddable. The facets of the H_4 are all of the 3-gonal (triangle) inequalities with distinct y_i. The facets of H_5 are all of the 3-gonal and 5-gonal inequalities with distinct y_i. For seven points, the first author[5] [6] found an example of a seven point hypermetric that was not L^1-embeddable. Dually, Assouad[2] found an example of a facet of H_7 that was not a

$(2k+1)$-gonal inequality. Additional facets of H_7 are given in[4]. We now settle the question for the six point Hamming cone by giving a complete list of its facets, all of which are $(2k+1)$-gonal.

Theorem. The facets of H_6 are the 3,5-gonal inequalities with distinct y_i and the 7-gonal inequalities with 6 distinct y_i.

A simple count shows that there are 210 such facets. This theorem settles affirmatively a conjecture given by Deza and the first author[1]. It is known that all L^1-embeddable semimetrics are hypermetric[8]. By the theorem, all 3,5,7-gonal semimetrics are L^1-embeddable, and hence hypermetric. Therefore we obtain the following corollary.

Corollary. For a six point semimetric space $(\mathbf{X}, \mathbf{d})$ the following are equivalent:

(a) $\mathbf{d}$ is hypermetric;

(b) $\mathbf{d}$ is L^1-embeddable;

(c) $\mathbf{d}$ is 7-gonal.

Let $f(m)$ be the smallest integer such that every m point semimetric that is $f(m)$-gonal is hypermetric. Deza showed that $f(3) = f(4) = 3$, $f(5) = 5$ and the corollary gives $f(6) = 7$. A second conjecture given by Deza and the first author is that $f(m) < \infty$ for all m. This is still unsolved for all $m \geq 7$.

2. Proof of Theorem.

In this section we assume throughout that $(\mathbf{X}, \mathbf{d})$ is a six point semimetric space and that $\mathbf{d}$ is represented by a point d in R^{15}. As we saw in the last section there are 31 extreme rays of H_6. Each facet of H_6 contains at least 14 extreme rays and so in principal the theorem can be checked by verifying which of the $\binom{31}{14} = 265,182,525$ hyperplanes defined by these rays define facets. This is computationally infeasible. We reduce this computation by introducing an equivalence relation on the facets, and then generate at least one facet from each equivalence class.

For $c,\ x \in R^{15}$ the inequality $cx \le 0$ defined by c is *valid* for H_6 if $cd \le 0$ for all $d \in H_6$. Let us index the coordinates of c lexicographically so that $c = (c_{1,2}, c_{1,3}, \ldots, c_{n-1,n})$. Let $\pi = (\pi(1), \ldots, \pi(6))$ be any permutation of $1, \ldots, 6$ with inverse permutation π^{-1}. For any $z \in R^{15}$ let $z^{\pi} \in R^{15}$ be defined by

$$z^{\pi}{}_{i,j} = \begin{cases} z_{\pi(i),\pi(j)} & \pi(i) < \pi(j) \\ z_{\pi(j),\pi(i)} & \pi(j) < \pi(i) \end{cases}.$$

For any $t > 0$, if c is valid then $t\ c^{\pi}$ defines a valid inequality over H_6 since

$$t\ c^{\pi}\ d = t\ c\ d^{\pi^{-1}} \le 0.$$

for all $d \in H_6$. We say that c is equivalent to $t\ c^{\pi}$ and this clearly defines an equivalence relation over vectors defining valid inequalities. An inequality $cx \le 0$ is $(2k+1) -$ *gonal* if

$$c_{i,j} = \mu_i \mu_j \quad 1 \le i < j \le 6$$

for integers μ_i such that

$$\sum_{i=1}^{6} |\mu_i| = 2k+1, \qquad \sum_{i=1}^{6} \mu_i = 1.$$

In comparing this with (1), the magnitude of μ_i corresponds to the number of repetitions of the point x_i in the set of $y_j,\ j = 1, \ldots, 2k+1$. Representatives of the inequalities mentioned in the theorem can be obtained by setting $\mu = (1,1,-1,0,0,0)$, $(1,1,1,-1,-1,0)$, $(2,1,1,-1,-1,-1)$ and $(1,1,1,1,-1,-2)$. We say that the corresponding facets are of types A,B,C and D respectively. It is clear that if c defines an inequality satisfying the conditions of the theorem, then so do all inequalities equivalent to c. Our task will be to generate at least one inequality from each equivalence class and verify that it satisfies the conditions of the theorem.

We split the extreme rays of H_6 given by (1) into classes 1,2 and 3 depending on whether S has cardinality 1,2 or 3 respectively. There are 6 class 1 rays, 15 class 2 rays and 10 class 3 rays. Let F be a facet of H_6 and choose arbitrarily 14 linearly independent extreme rays contained in F. Denote by S_1, S_2, S_3 the (possibly empty) subsets of class 1, 2 and 3 rays making up this set of 14 rays. Consider the class 2 extreme rays chosen. These can be represented by a graph

$G = (V, E)$ on the six points $x_1, \ldots, x_6$. The pair (x_i, x_j) is an edge in E whenever x_i, x_j is the two element set in a class 2 ray defining F. We claim that a facet equivalent to F will be contained in the list of facets generated as follows:

(i) Start with any graph G^* isomorphic to G and choose class 2 rays given by the edges as described above.

(ii) Augment the rays in (i) by the each of the $\binom{25}{14 - |S_2|}$ possible subsets of class 1 and 3 rays. Find the hyperplane through the augmented set and see if it is a facet of H_6.

To verify that F is produced by the above procedure, consider the permutation π which when applied to the vertices in G gives G^*. Apply the same permutation to the rays in S_1 and S_3. When these new rays are considered with the class 2 rays defined by G^* in step (ii) a hyperplane $\bar{c}x = 0$ will be generated. If $cx = 0$ is the hyperplane containing F, then we have $\bar{c} = c^{\pi}$ and so we have generated a hyperplane equivalent to c.

A procedure for verifying the theorem is now easy: simply perform steps (i) and (ii) for each graph on six points. Let us estimate the computational effort. There are 156 graphs on six points, but we eliminate the complete graph as it has too many edges. A simple count shows that there some 1,131,555 cases to be considered in step (ii). For each case we must first find the hyperplane through the given rays by solving a system of 14 equations in 14 unknowns. If the hyperplane is of full dimension, we determine if it is a facet by choosing points different from the origin on each of the other rays and see if all of the points lie on the same side of the computed hyperplane. The required computation was done by the second author, a SUN 3/50 workstation, in ten and a half hours.

In Figure 1 we give a graph, $K_{1,4} \cup K_1$, that generates 11 facets of H_6, 10 of type A and one of type D. The type D facets are simplicial, and so up to relabelling, this is the unique graph generating type D facets. We show a facet of each type generated, along with the additional rays of classes 1 and 3 required. These rays are given by listing the subsets S used to define them in (1). Similarly, Figure 2 gives a graph, $K_{3,3}$, that generates 12 facets of H_6, 11 of type B and one of type C. The type C facets are also simplicial.

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