On the Directed Cut Cone and Polytope

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Abstract

In this paper we study the directed cut cone and polytope which are the positive hull and convex hull of all directed cut vectors of a complete directed graph, respectively. We present results on the polyhedral structure of these polyhedra. A relation between directed cut polyhedra and undirected cut polyhedra is established that provides families of facet defining inequalities for directed cut polyhedra from those known for undirected cut polyhedra.

1 Introduction

Let $\vec{K}_n = (V_n, A_n)$ denote the complete directed graph with n nodes, V_n, and $n(n-1)$ *arcs*, A_n, one for each distinct ordered pair of vertices. The mapping $\delta^+ \colon S \to \{0,1\}^{|A_n|}, S \subseteq V_n$, defines the set of *directed cut vectors*, which we will usually refer to as *dicut vectors*, where $\delta^+(S)_{ij} = 1$, if $i \in S$ and $j \notin S$, and is zero otherwise. A dicut vector is therefore the arc incidence vector of all arcs ij with $i \in S$ and $j \notin S$. Note that the dicut vectors for S and its complement, $V_n \setminus S$, are different, except when $S = \emptyset$ or $S = V_n$. Hence there are $2^n - 1$ distinct dicut vectors for $\vec{K}_n$. The *directed cut cone*, $DCUT_n$, is the positive hull of the set of all directed cut vectors, and the *directed cut polytope*, $DCUT_n^\square$, is their convex hull.

We contrast the above definitions with corresponding definitions for the undirected complete graph $K_n = (V_n, E_n)$, where V_n is a set of n nodes and E_n the set of all $n(n-1)/2$ undirected *edges*. The mapping $\delta \colon S \to \{0,1\}^{|E_n|}, S \subseteq V_n$, defines the set of *cut vectors*, where $\delta(S)_{i,j} = 1$, if precisely one of i and j is in S, and is zero otherwise. A cut vector is therefore the edge incidence vector of all edges with precisely one endpoint in S. Note that the cut vectors for S and its complement, $V_n \setminus S$, are identical, and that there are 2^{n-1} distinct cut vectors for K_n. The *cut cone*, CUT_n, is the positive hull of the set of all cut vectors, and the *cut polytope*, $CUT_n^\square$, is their convex hull.

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The cut cone and polytope arise in many fields and have been intensively studied. The book by Deza and Laurent [8] provides an encyclopedic reference. Directed cut polyhedra have been much less studied. This is somewhat surprising, as the celebrated Ford-Fulkerson algorithm computes a minimum weight dicut in a network with non-negative edge weights which separates two given vertices. However, Karp [14] showed that the undirected maximum cut problem with non-negative weights is NP-hard. By replacing each weighted undirected edge by two oppositely directed arcs, each with the original weight, this result applies to the directed case also. Hence for arbitrary edge weights both the minimum and maximum dicut problems are NP-hard. This has prompted attempts to get good approximation algorithms. For the maximum cut problem with non-negative edge weights good worst case bounds are known in both the directed and undirected cases. These methods are usually based on semi-definite programming, see [10] and [11] for example.

In mining optimization, the original Ford-Fulkerson algorithm has long been used to compute what is called the 'optimum pit' for open pit mining with slope constraints. However the addition of other realistic constraints, such as the maximum amount of material to be removed, renders the problem NP-hard [12]. The corresponding optimization problem can be formulated as optimizing over the directed cut polytope with an additional knapsack constraint. Commercial software has had success applying techniques like Lagrangian relaxation [4] and branch and cut [5]. However, in order to effectively incorporate branch and cut type procedures for these hard problems, a study of the structure of directed cut polyhedra is essential.

As far as we know, this paper presents the first systematic study of directed cut polyhedra. The only related work we are aware of is the study of the directed cut polyhedra restricted to the series parallel directed graphs [7] rather than the complete directed graph. The recent study of the dominant of the s-t-cut polytope, for which the complete facet structure is known [13], is also a related polyhedra.

The paper is organized as follows. In the next section we provide basic background on the cut polyhedra and their LP-relaxations, the semimetric and rooted semimetric polyhedra, and give analogues for directed cuts. In Section 3 we give the dimension of the directed cut polyhedra and prove the validity of the LP-relaxations given in Section 2. In Section 4 we link the directed cut and cut polyhedra in a fundamental way. This allows us to generate facets of the dicut polyhedra from facets of the cut polyhedra, as is demonstrated in Section 5. In Section 6 we show that the well known technique of zero-lifting a facet to higher dimensions applies to directed cut polyhedra. Finally in Section 7 we completely analyze the facets of directed cut polytopes on three and four vertices.

2 Distances, semimetrics, and LP-relaxations

In the sequel we will use the term *arc* for a directed edge and often simply call an undirected edge an edge. Since we will be defining variables on both arcs and edges, we will use the notation x_{ij} to denote a variable defined on the arc ij, and the notation $x_{i,j}$ to denote a variable defined on the undirected edge (i, j). In this way x_{ij} and x_{ji} are distinct variables, whereas $x_{i,j}$ is identical to $x_{j,i}$. The edge and arc variables will be referred to as edge and arc *weights*.

If $C \subseteq E(G)$ is a cycle of undirected graph G and $|F|$ is odd for $F \subseteq C$ then the inequality:

$$\sum_{e \in F} x_e - \sum_{e \in C \setminus F} x_e \leq |F| - 1$$

is defined as a *cycle inequality*. For the special case where $|C|$ is 3 this cycle inequality is referred to as a *triangle inequality*.

Following Deza and Panteleeva [9] a *quasi-semimetric*[1] d on a set S is a function $d\colon S^2 \to \mathbb{R}$ such that for all distinct i, j, k in S, $d(i, j) \geq 0$, $d(i, i) = 0$ and the triangle inequality is satisfied:

$$d(i, k) \leq d(i, j) + d(j, k). \tag{1}$$

If in addition d is symmetric, that is for all $i, j \in S$ $d(i, j) = d(j, i)$, then it is called a *semimetric*.

Semimetric spaces have a fundamental polyhedral connection to cut polyhedra. As before, let $K_n = (V_n, E_n)$ be the undirected complete graph. For the following facts, the reader is referred to [8]. We define polyhedra

$$MET_n = \{x \in \mathbb{R}^{|E_n|} : x_{i,k} \leq x_{i,j} + x_{j,k}, 1 \leq i, j, k \leq n, \text{ distinct}\}$$

and

$$MET_n^\square = \{x \in \mathbb{R}^{|E_n|} : x_{i,k} \leq x_{i,j} + x_{j,k},\ x_{i,j} + x_{j,k} + x_{i,k} \leq 2,\ 1 \leq i, j, k \leq n, \text{ distinct}\},$$

where in the above definitions we formally identify $x_{i,j} = x_{j,i}$ for all indices i, j. Let $S = V_n$. Given a semimetric d, we may further identify $x_{i,j} = d(i, j)$ and see that x is contained in MET_n. Here we note that the non-negativity constraints are implied by the triangle inequalities. Similarly any $x \in MET_n$ corresponds to a semimetric d, which justifies calling MET_n the *semimetric cone*. It is well known that each non-zero cut vector generates an extreme ray of MET_n. Similarly, the set of cut vectors form vertices of the *semimetric polytope* $MET_n^\square$. In fact they are the only integral vertices. Therefore $MET_n^\square$ is a natural LP-relaxation for $CUT_n^\square$.

We can get a weaker LP-relaxation by considering the *rooted semimetric polytope* $RMET_n^\square$, which is defined by the subset of triangle inequalities obtained by fixing one of the indices, say by setting $k = 1$. The *rooted semimetric cone*, $RMET_n$, is defined similarly. From the above discussion it is clear that

$$CUT_n \subseteq MET_n \subseteq RMET_n \textit{ and } CUT_n^\square \subseteq MET_n^\square \subseteq RMET_n^\square. \tag{2}$$

[1] In [6], Charikar et al. call this a *directed semimetric*.

We will develop analogous polyhedra for dicuts. As before, $\vec{K}_n = (V_n, A_n)$ denotes the complete directed graph. Firstly we note that dicut vectors are not symmetric and in fact they exhibit a partial asymmetry. Nevertheless, they do possess a weaker *three point symmetry*:

$$\delta^+(S)_{ij} + \delta^+(S)_{jk} + \delta^+(S)_{ki} \quad = \quad \delta^+(S)_{ji} + \delta^+(S)_{kj} + \delta^+(S)_{ik}. \tag{3}$$

This is evident as both sides of equation (3) are equal to 0 if $i, j, k \in S$ or $i, j, k \notin S$. If S contains only one of i, j, k say i and $j, k \notin S$ then $\delta^+(S)_{ij} = 1$ and the other two terms on the left hand side of (3) are equal to 0 while the right hand side has $\delta^+(S)_{ik} = 1$ and the other two terms are equal to zero. Finally, if S contains two nodes, say i, k and $j \notin S$ then $\delta^+(S)_{ij} = 1$ and $\delta^+(S)_{kj} = 1$ and all other terms are 0, so (3) is satisfied. By relabeling nodes if needed it is easy to see that (3) holds for all possible sets S. An interesting consequence of (3) is that it implies that, unlike $CUT_n^\square$, $DCUT_n^\square$ is not full dimensional. We return to this issue in the next section.

We define a *3-semimetric* as a quasi-metric d that satisfies the three point symmetry, that is

$$d(i,j) + d(j,k) + d(k,i) = d(j,i) + d(k,j) + d(i,k) \tag{4}$$

for all distinct $i, j, k \in S$. Note that this equation trivially holds for semimetrics. We call (S, d) a *3-semimetric* space. One can easily check that with $S = V_n$, (S, δ^+) is a 3-semimetric space. We define the *3-semimetric cone*

$$\begin{aligned} 3MET_n = \{x \in \mathbb{R}^{|A_n|} : x_{ij} + x_{jk} + x_{ki} \quad &= \quad x_{ji} + x_{kj} + x_{ik} \\ x_{ik} - x_{ij} - x_{jk} \quad &\leq \quad 0, \quad x_{ij} \geq 0, \qquad 1 \leq i,j,k \leq n, \text{ distinct}\}. \end{aligned}$$

To define the *3-semimetric polytope* we add perimeter inequalities which say that the sum of arc variables around each directed triangle is at most one:

$$\begin{aligned} 3MET_n^\square = \{x \in \mathbb{R}^{|A_n|} : x_{ij} + x_{jk} + x_{ki} \quad &= \quad x_{ji} + x_{kj} + x_{ik} \leq 1 \\ x_{ik} - x_{ij} - x_{jk} \quad &\leq \quad 0, \quad x_{ij} \geq 0, \qquad 1 \leq i,j,k \leq n, \text{ distinct}\}. \end{aligned}$$

In these definitions, note that the order of the indices is crucial and that the non-negativity inequalities are now included. We will show various analogies with the undirected case. As an initial observation, it is easy to verify that the dicut vectors are members of these polyhedra.

If we only consider the triangle inequalities and cycle equalities that involve a given node $1 \in V_n$ we can define rooted relaxations of the 3-semimetric cone and polytope. We define the rooted 3-semimetric cone, $\mathrm{R3MET}_n$, as the cone defined by:

$$\begin{aligned} x_{1i} + x_{ij} + x_{j1} &= x_{1j} + x_{ji} + x_{i1} && (5) \\ x_{1i} - x_{1j} - x_{ji} &\leq 0 && (6) \\ x_{i1} - x_{ij} - x_{j1} &\leq 0 && (7) \\ x_{ij} - x_{i1} - x_{1j} &\leq 0 && (8) \\ x_{ij} &\geq 0 \qquad \text{for each } ij \in A_n. && (9) \end{aligned}$$

The rooted 3-semimetric polytope, $\mathrm{R3MET}_n^\square$ is the polytope defined by the inequalities above and the perimeter inequalities:

$$x_{1i} + x_{ij} + x_{j1} \leq 1 \qquad (10)$$

where we take all $ij \in A_n$. From the definitions and discussion above we have the following relationships:

$$DCUT_n \subseteq 3MET_n \subseteq R3MET_n \text{ and } DCUT_n^\square \subseteq 3MET_n^\square \subseteq R3MET_n^\square. \qquad (11)$$

Observe that these relationships are analogous to those given above for the undirected case. We strengthen the analogy in the following sections. In Section 3 we compute the dimension of these polyhedra, and prove that dicut vectors are vertices of $\mathrm{3MET}_n^\square$ and $\mathrm{R3MET}_n^\square$. In Section 4 we give a fundamental relationship between $\mathrm{DCUT}_n^\square$ and $\mathrm{CUT}_n^\square$. This is used in Section 5 to prove that the inequalities used to define $\mathrm{3MET}_n^\square$ are facets of $\mathrm{DCUT}_n^\square$, and to give additional classes of facets for this polytope.

3 Basic properties of $\mathbf{DCUT}_n^\square$, $\mathbf{3MET}_n^\square$ and $\mathbf{R3MET}_n^\square$

Unlike the polyhedra related to cuts in an undirected graph, the polyhedra related to dicuts defined in the previous section are not full dimensional. This is evident from the linearities in the definitions. Their dimension is given by the next result.

Proposition 1 *All polyhedra listed in (11) have dimension* $\binom{n}{2} + n - 1$.

Proof. We first show an upper bound on the dimension of $R3MET_n$. By repeated use of equation (5) for each arc $ji \in A_n$, one can replace x_{ji} with $x_{1i} + x_{ij} + x_{j1} - x_{1j} - x_{i1}$ where $j > i \geq 2$. This eliminates $\binom{n}{2} - n + 1$ variables, since each variable x_{ji} only occurs in one of the set of equations chosen. This leaves $\binom{n}{2} + n - 1$ variables and proves the upper bound.

Next we give an identical lower bound on the dimension of $\mathrm{DCUT}_n^\square$. For this, a set of $\binom{n}{2} + n - 1$ linearly independent dicut vectors is constructed. Let $S_{i,j} = \{k \in V_n : k \leq i\} \cup \{k \in V_n : j < k \leq n\}$ for $1 \leq i < j \leq n$, and let $T_i = \{k : 2 \leq k \leq i\}$ for $2 \leq i \leq n$. We claim that the dicut vectors $C = \{\delta^+(S_{i,j}) : 1 \leq i < j \leq n\} \cup \{\delta^+(T_i) : 2 \leq i \leq n\}$ are linearly independent. By construction, the matrix formed by using these dicut vectors as the rows where the columns are ordered lexicographically by ij for $1 \le i < j \le n$, followed by the columns $j1$, $2 \le j \le n$, is lower triangular with all 1's on the diagonal. These $\binom{n}{2} + n - 1$ vectors along with the dicut vector corresponding to the empty set, $S = \emptyset$, form $\binom{n}{2} + n$ affinely independent vectors.

The proposition now follows by using the inclusions in (11) and the trivial inclusions $DCUT_n^\square \subseteq DCUT_n$ and $R3MET_n^\square \subseteq R3MET_n$. ■

It is useful to project the dicut polyhedra onto a lower dimensional space where they are full dimensional. Consider the directed cut polytope and cone on the graph $\vec{J}_n$ with n nodes and arc set $A(\vec{J}_n) = \{ij : 1 \le i < j \le n\} \cup \{i1 : 2 \le i \le n\}$ as opposed to the complete directed graph $\vec{K}_n$. The polyhedra DCUT_n, $\mathrm{DCUT}_n^\square$, $\mathrm{3MET}_n$, $\mathrm{3MET}_n^\square$, $\mathrm{R3MET}_n$ and $\mathrm{R3MET}_n^\square$ become full-dimensional when restricted to the arc set $A(\vec{J}_n)$. Depending on the situation, we will use either one or the other of these two representations.

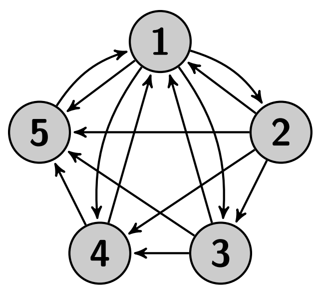

Figure 1: The directed graph $\vec{J}_5$.

The sets of inequalities that define $\mathrm{3MET}_n$, $\mathrm{3MET}_n^\square$, $\mathrm{R3MET}_n$ and $\mathrm{R3MET}_n^\square$ will be rewritten slightly as some of the arcs used to index the inequalities do not exist in $A(\vec{J}_n)$. As mentioned in the proof of Theorem 1 we can substitute

$$x_{ji} = x_{ij} + x_{1i} - x_{i1} + x_{j1} - x_{1j} \qquad \text{whenever } j > i \ge 2. \tag{12}$$

This yields the following set of inequalities that define the cone $\mathrm{3MET}_n$.

For $2 \le i \le n$,

$$x_{i1} \ge 0, \quad x_{1i} \ge 0. \tag{13}$$

For $2 \le i < j \le n$,

$$x_{ij} \ge 0 \tag{14}$$
$$-x_{ij} + x_{i1} - x_{1i} + x_{1j} - x_{j1} \le 0 \tag{15}$$
$$-x_{ij} + x_{i1} - x_{j1} \le 0 \tag{16}$$
$$-x_{ij} - x_{1i} + x_{1j} \le 0 \tag{17}$$
$$x_{ij} - x_{i1} - x_{1j} \le 0. \tag{18}$$

For $2 \le i < j < k \le n$,

$$\begin{aligned} -x_{ij} + x_{ik} - x_{jk} &\le 0 & (19)\\ x_{ij} - x_{ik} - x_{jk} + x_{1k} - x_{k1} + x_{j1} - x_{1j} &\le 0 & (20)\\ -x_{ij} - x_{ik} + x_{jk} + x_{i1} - x_{1i} + x_{1j} - x_{j1} &\le 0. & (21)\end{aligned}$$

The inequalities (13)-(15) above are the non-negativity constraints, and the remaining inequalities are the triangle inequalities. All of the above inequalities along with the following *perimeter inequalities* define $3\mathrm{MET}_n^\square$.

For $2 \le i < j \le n$,

$$x_{1i} + x_{ij} + x_{j1} \le 1. \quad (22)$$

For $2 \le i < j < k \le n$,

$$x_{ij} + x_{jk} + x_{ik} + x_{1i} - x_{i1} + x_{k1} - x_{1k} \le 1. \quad (23)$$

Using a vertex enumeration software, such as *lrs* [1][2], we can easily check that the vertices of $3\mathrm{MET}_3^\square$ are precisely the seven dicut vectors, so the 10 inequalities (13)-(18),(22) obtained when $n = 3$ are the complete set of facets of $\mathrm{DCUT}_3^\square$. See Section 7 for further information. The situation mirrors that of $\mathrm{MET}_3^\square$ and $\mathrm{CUT}_3^\square$. However there is an important difference. The vertices of $\mathrm{CUT}_n^\square$, $n \ge 3$, are all equivalent under an automorphism called the switching operation, that will be reviewed in Section 5. However this is not the case with $\mathrm{DCUT}_3^\square$. The dicut vector $\delta^+(S)$ with $S = \{\emptyset\}$ is contained on 9 facets - all except the perimeter inequality (22). However all other dicut vectors lie on 7 facets. For example, the dicut vector $\delta^+(S)$ with $S = \{2\}$ does not lie on the facets $x_{21} \ge 0, x_{23} \ge 0, -x_{12} + x_{13} - x_{23} \le 0$. This fundamental difference greatly complicates the study of the dicut polyhedra.

The following result shows that $\mathrm{R3MET}_n^\square$ and $3\mathrm{MET}^\square$ are LP-relaxations of the directed cut polytope.

Theorem 2 *The only integral vectors that are contained in $R3MET_n^\square$ are the dicut vectors $\delta^+(S)$ for $S \subseteq V_n$ and every dicut vector is a vertex of $R3MET_n^\square$.*

Proof. In the first part of the proof we will use the original definition of $\mathrm{R3MET}_n^\square$ given in terms of all $n(n-1)$ variables to show that the only integral vectors of $\mathrm{R3MET}_n^\square$ are dicut vectors. In the second part of the proof we show that every dicut vector is a vertex of $\mathrm{R3MET}_n^\square$.

The non-negativity and the perimeter inequalities imply that the only integral vectors in $\mathrm{R3MET}_n^\square$ are 0/1 valued. Let $x \in R3MET_n^\square \cap \{0,1\}^{|A_n|}$. Let $I = \{i : x_{i1} = 1\}$ and $J = \{i : x_{1i} = 1\}$. We will first show that one of $|I| = 0$ or $|J| = 0$ holds. Indeed, if $i \in I$

and $j \in J$, $i \neq j$, the rooted perimeter inequality $x_{i1}+x_{1j}+x_{ji} \leq 1$ would be violated by x. Otherwise, if there exists $i \in I \cap J$ then $x_{1i} + x_{i1} = 2$ but summing R3MET$_n^\square$ inequalities:

$$\begin{aligned} x_{1i} + x_{ij} + x_{j1} &\leq 1 \qquad (24) \\ x_{i1} - x_{ij} - x_{j1} &\leq 0 \qquad (25) \end{aligned}$$

together yields

$$x_{1i} + x_{i1} \leq 1. \qquad (26)$$

If both I and J are empty then x corresponds to the cut $\delta^+(V_n)$ or $\delta^+(\emptyset)$ since $x_{ij} = 1$ implies that at least one of $x_{i1} = 1$ or $x_{1j} = 1$ by (8).

Assume $I \neq \emptyset$, consider an index $i \in I$. For any $j \in I, i \neq j$, the perimeter inequalities (10) for arcs ij and ji prove that $x_{ij} = x_{ji} = 0$. Therefore all arcs ij with both endpoints in I have $x_{ij} = 0$.

Now consider any $j \notin I$. The perimeter inequality (10) for ji implies that $x_{ji} = x_{1j} = 0$. As this inequality is satisfied as an equation, by the linearity (5) we have that

$$x_{1i} + x_{ij} + x_{j1} = 1.$$

However, $x_{j1} = 0$ since $j \notin I$ and $x_{1i} = 0$ by (26), so $x_{ij} = 1$. Lastly, if $j, k \notin I$ then $x_{jk} = 0$ follows from the fact that $x_{jk} \leq x_{j1} + x_{1k}$, $x_{j1} = 0$ and $x_{1k} = 0$ as J is empty. We have shown that $x = \delta^+(I)$.

Assume $J \neq \emptyset$, then by the inequalities given by $R3MET_n^\square$ we can show as above that $x_{ij} = 0$ if $i, j \in J$, $i, j \notin J$ or $j \in J$ and $i \notin J$ and $x_{ij} = 1$ if $i \in J$ and $j \notin J$. This proves that $x = \delta^+(J)$.

To show that every dicut vector is a vertex of R3MET$_n^\square$ we use induction on n. In this part of the proof we use the full dimensional definition of R3MET$_n^\square$ formed by eliminating variables x_{ji} for $j > i \geq 2$ using the linearities. This makes verification of linear independence simpler. For the base case $n = 3$, one can easily check that R3MET$_3^\square$ =DCUT$_3^\square$. See Section 7 for a complete description of $DCUT_3^\square$.

By a *root* of an inequality we mean a vector x for which it is satisfied as an equality. For $n \geq 4$ we assume inductively that a dicut vector x that corresponds to a directed cut $\delta^+(S)$ in $\vec{K}_{n-1}$ and is a root of $\binom{n-1}{2} + (n-1) - 1$ linearly independent inequalities. Call this set of inequalities T. We will extend T to a set of $\binom{n}{2} + n - 1$ linearly independent inequalities from R3MET$_n^\square$ that are satisfied with equality. Doing so involves considering the possible cases of whether or not nodes 1 and n are in S.

Case 1: $1 \in S$ and $n \in S$.
The vector corresponding to $\delta^+(S)$ is a root of the inequalities $x_{in} \geq 0$ for $i = 1, ..., n-1$ and $x_{n1} \geq 0$. These n inequalities along with the inequalities in T are linearly independent.

Case 2: $1 \notin S$ and $n \notin S$.
The vector $\delta^+(S)$ is a root of the inequalities $x_{n1} \geq 0$, $x_{in} \geq 0$ for $i \notin S$, $i \neq n$, and

$x_{in} + x_{n1} + x_{1i} \leq 1$ for $i \in S$. These n inequalities along with the inequalities in T are all linearly independent.

Case 3: $1 \in S$ and $n \notin S$.

Firstly suppose $|S| = n-1$. Then $\delta^+(S)$ is a root of the n inequalities $x_{in} + x_{n1} + x_{1i} \leq 1$, $2 \leq i \leq n-1$, $x_{n1} \geq 0$, and $-x_{1n} + x_{2n} + x_{12} - x_{21} + x_{n1} \geq 0$ (corresponding to $x_{n2} \geq 0$). Together with the inequalities in T they are also linearly independent. Otherwise, $|S| \leq n-2$. Then $\delta^+(S)$ is a root of the n inequalities $x_{in} \geq 0$ for $i \notin S$, $i \neq n$, $x_{n1} \geq 0$, $x_{1i} + x_{in} + x_{n1} \leq 1$ for all $i \in S, i \neq 1$, and $-x_{1n} + x_{jn} + x_{1j} - x_{j1} + x_{n1} \geq 0$ for some $j \notin S$, $j \neq n$. Note that the last inequality corresponds to $x_{nj} \geq 0$ and the index j exists by the assumption on the cardinality of S. These inequalities along with the inequalities in T are all linearly independent.

Case 4: $1 \notin S$ and $n \in S$.

The vector $\delta^+(S)$ is a root of the n inequalities $x_{in} \geq 0$ for $i = 1, ..., n-1$ and $x_{n1}+x_{1i}+x_{in} \leq 1$ for some $i \notin \{1, n\}$. Together with the inequalities in T they are all linearly independent. ■

The following corollary is evident as R3MET$_n^\square$ contains a subset of the inequalities that define 3MET$_n^\square$ and no directed cuts violate inequalities of 3MET$_n^\square$.

Corollary 3 *The only integral vectors of 3MET$_n^\square$ are the directed cut vectors $\delta^+(S)$ for $S \subseteq V_n$ and every dicut vector is a vertex of 3MET$_n^\square$.*

4 A fundamental relationship between DCUT$_n^\square$ and CUT$_n^\square$

Our most powerful tool for obtaining results about directed cut polyhedra is to show a fundamental relation to cut polyhedra. Then we will be able to use many of the known results on cut polyhedra, such as classes of facets for example, to obtain results for directed cut polyhedra. To begin with we extend two basic operations for cut polyhedra, permuting and collapsing, to our setting. We will leave the operation of zero-lifting, until later.

For a permutation σ of the nodes $\{1, ..., n\}$ and a vector $v \in \mathbb{R}^{|A_n|}$ we define $\sigma(v) \in \mathbb{R}^{|A_n|}$ as $\sigma(v)_{ij} = v_{\sigma(i)\sigma(j)}$. The following lemma trivially holds as the nodes in $\vec{K}_n$ can be relabelled.

Lemma 4 *Given $v \in \mathbb{R}^{|A_n|}$, $v_0 \in \mathbb{R}$ and σ a permutation of $\{1, ..., n\}$, the following statements are equivalent:*

- *The inequality $v^T x \leq v_0$ is valid (resp. facet inducing) for DCUT$_n^\square$.*
- *The inequality $\sigma(v)^T x \leq v_0$ is valid (resp. facet inducing) for DCUT$_n^\square$.*

We can define a similar type of collapsing operation that constructs a valid inequality for DCUT$_m^\square$ from a valid inequality for DCUT$_n^\square$, where $m < n$. Let $\pi = (M_1, ..., M_m)$ be a partition of V_n into m non-empty sets. If $v \in \mathbb{R}^{|A_n|}$, the collapse of v according to π is:

$$v^{\pi}_{ij} = \sum_{s \in M_i, t \in M_j} v_{st}. \tag{27}$$

The directed collapsing operation has many similar properties as the undirected collapsing operation. For instance, if S^π is defined to be $\bigcup_{k \in S} M_k$ for $S \subseteq \{1, ..., m\}$ then $v^{\pi T}\delta^+(S) = v^T\delta^+(S^\pi)$. This gives the following lemma which has the undirected equivalent stated as Lemma 26.4.1 in Deza and Laurent's book [8], and whose proof is straightforward.

Lemma 5 *Let $v \in \mathbb{R}^{|A_n|}$, $v_0 \in \mathbb{R}$ and $\pi = (M_1, ..., M_m)$ be a partition of the vertices of V_n into m non-empty sets. The following are true:*

1. *If $v^T x \leq v_0$ is a valid inequality for $DCUT^{\square}_n$ then $v^{\pi T} x \leq v_0$ is a valid inequality for $DCUT^{\square}_m$.*
2. *If $\delta^+(S)$, for some $S \subseteq \{1, ..., m\}$, is a root of inequality $v^{\pi T} x \leq v_0$, then $\delta^+(S^\pi)$ is a root of $v^T x \leq v_0$.*

We now move to the main topic of this section where we will be working with the full dimensional representations of dicut polyhedra described in the previous section. We begin by considering a partition of the set of all subsets of nodes of $\vec{J}_n$, $V(\vec{J}_n) = \{1, ..., n\}$, into two families, $\mathcal{S}_1$ and $\mathcal{S}_2$, where $\mathcal{S}_1$ contains all subsets S such that $1 \in S$ and $\mathcal{S}_2$ contains all subsets S with $1 \notin S$.

Define the polytope $\mathcal{P}_{n,1}$ to be the convex hull of dicut vectors associated with the subsets of $\mathcal{S}_1$, $\mathcal{P}_{n,1} = conv\{\delta^+(S) : S \in \mathcal{S}_1\}$. Similarly, define $\mathcal{P}_{n,2} = conv\{\delta^+(S) : S \in \mathcal{S}_2\}$. Clearly, the directed cut polytope is the convex hull of the two polytopes $\mathcal{P}_{n,1}$ and $\mathcal{P}_{n,2}$ with the two polytopes only intersecting in a single point, the cut vector $\delta^+(V(\vec{J}_n)) = \delta^+(\emptyset)$.

The benefit of defining polytopes $\mathcal{P}_{n,1}$ and $\mathcal{P}_{n,2}$ is the fact that both are bijections of the undirected cut polytope. We define the mappings

$$\xi_1 \colon \mathbb{R}^{\binom{n}{2}} \to (\mathbb{R}^{\binom{n}{2}}, \{0\}^{n-1}) \text{ and } \xi_2 \colon \mathbb{R}^{\binom{n}{2}} \to (\mathbb{R}^{\binom{n}{2}}, \{0\}^{n-1})$$

between the cut polytope CUT_n and $\mathcal{P}_{n,1}$ and $\mathcal{P}_{n,2}$ respectively. The co-ordinates of the range space, $(\mathbb{R}^{\binom{n}{2}}, \{0\}^{n-1})$ are labelled $(x_{ij} : 1 \leq i < j \leq n, x_{i1} : 2 \leq i \leq n)$. In Section 5, these mappings are used to obtain valid inequalities and facets of the directed cut polytope from valid inequalities and facets of the cut polytope.

The mapping ξ_1 is defined by

$$\begin{cases} x_{i1} = 0 & \text{for } 2 \leq i \leq n, \\ x_{1i} = x_{1,i} & \text{for } 2 \leq i \leq n, \\ x_{ij} = \frac{1}{2}(x_{i,j} + x_{1,j} - x_{1,i}) & \text{for } 2 \leq i < j \leq n, \end{cases}$$

equivalently ξ_1^{-1} is defined by

$$\begin{cases} x_{1,i} = x_{1i} & \text{for } 2 \leq i \leq n, \\ x_{i,j} = x_{ij} + x_{ji} = x_{1i} - x_{1j} + 2x_{ij} & \text{for } 2 \leq i < j \leq n. \end{cases}$$

The mapping ξ_2 is defined by

$$\begin{cases} x_{i1} = x_{1,i} & \text{for } 2 \leq i \leq n, \\ x_{1i} = 0 & \text{for } 2 \leq i \leq n, \\ x_{ij} = \frac{1}{2}(x_{i,j} + x_{1,i} - x_{1,j}) & \text{for } 2 \leq i < j \leq n, \end{cases}$$

equivalently ξ_2^{-1} is defined by

$$\begin{cases} x_{1,i} = x_{i1} & \text{for } 2 \leq i \leq n, \\ x_{i,j} = x_{ij} + x_{ji} = x_{j1} - x_{i1} + 2x_{ij} & \text{for } 2 \leq i < j \leq n. \end{cases}$$

For any $S \subseteq \mathcal{S}_1$, ξ_1 has the property that,

$$\xi_1(\delta(S)) = \delta^+(S). \tag{28}$$

Figure 2 is an example of these mappings for $n = 4, S = \{1, 4\}$. Referring to the figure:

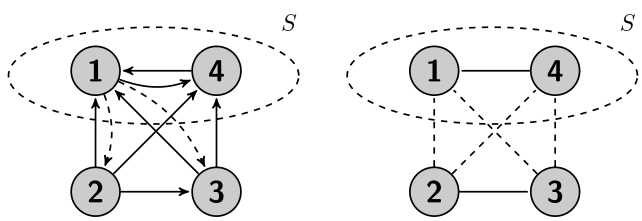

Figure 2: The mappings ξ_1 and ξ_1^{-1}: solid edges weight zero, dashed weight one

$$\begin{aligned} \xi_1^{-1}(\delta^+_{\vec{J}_4}(S)) &= \xi_1^{-1}(x_{12}, x_{13}, x_{14}, x_{23}, x_{24}, x_{34}, x_{21}, x_{31}, x_{41}) \\ &= \xi_1^{-1}((1,1,0,0,0,0,0,0,0)) \\ &= (1,1,0,0,1,1) \\ &= (x_{1,2}, x_{1,3}, x_{1,4}, x_{2,3}, x_{2,4}, x_{3,4}) = \delta_{K_4}(S) \end{aligned}$$

Similarly, for any subset S of $\mathcal{S}_2$,

$$\xi_2(\delta(S)) = \delta^+(S). \tag{29}$$

Figure 3 is an example of these mappings for $n = 4, S = \{2, 3\}$. Referring to the figure:

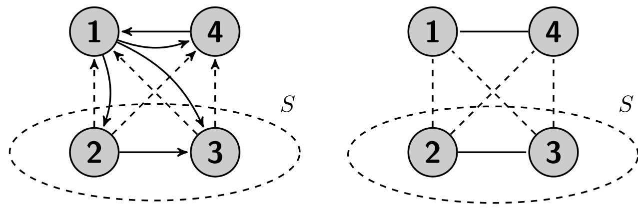

Figure 3: The mappings ξ_2 and ξ_2^{-1}: solid edges weight zero, dashed weight one

$$\begin{aligned} \xi_2^{-1}(\delta^+_{\vec{J}_4}(S)) &= \xi_2^{-1}(x_{12}, x_{13}, x_{14}, x_{23}, x_{24}, x_{34}, x_{21}, x_{31}, x_{41}) \\ &= \xi_2^{-1}((0,0,0,0,1,1,1,1,0)) \\ &= (1,1,0,0,1,1) \\ &= (x_{1,2}, x_{1,3}, x_{1,4}, x_{2,3}, x_{2,4}, x_{3,4}) = \delta_{K_4}(S) \end{aligned}$$

It follows from (28) and (29) that $\xi_1(CUT_n) = \mathcal{P}_{n,1}$ and $\xi_2(CUT_n) = \mathcal{P}_{n,2}$. This observation yields the following proposition:

Proposition 6 *The directed cut polytope is the convex hull of the linear transformation of two cut polytopes (undirected) that only intersect in a single point, the directed cut* $\delta^+(V_n) = \delta^+(\emptyset) = (0, 0, \cdots, 0)$.

Balas's [3] work on the union of polyhedra states:

Theorem 7 *[3] Given polyhedra* $P_i = \{x \in \mathbb{R}^n : A^i x \geq b^i\} \neq \emptyset, i \in Q$, *$Q$ is an index set, the closed convex hull of* $\cup_{i \in Q} P_i$ *is the set of those* $x \in \mathbb{R}^n$ *for which there exist vectors* $(y^i, y_0^i) \in \mathbb{R}^{n+1}, i \in Q$, *satisfying*

$$\begin{aligned} x - \sum(y^i : i \in Q) &= 0 \\ A^i y^i - b^i y_0^i &\geq 0 \\ y_0^i &\geq 0 \\ \sum(y_0^i : i \in Q) &= 1. \end{aligned}$$

This theorem provides a framework for optimizing over the directed cut polytope through the knowledge of the structure of polytopes $\mathcal{P}_{n,1}$ and $\mathcal{P}_{n,2}$ (or the undirected cut polytope). With this strong relationship to the well studied undirected polytope, a natural question arise as to why try and study the directed cut polyhedra structures? Our main motivation for better understanding this natural structure is that many industrial problems can be formulated as a directed cut problem with side constraints. These side constraints often have a well studied structure. For instance, in the case of open pit mining a common type of side constraint takes the form of a classic knapsack type inequality. As these types of constraints are well studied and cutting planes for them are implemented efficiently in commercial mixed integer program solvers, we would like to preserve their structure. The linear transformation defined by the mappings ξ_1 and ξ_2 can alter the knapsack type constraints so that some variables contain negative coefficients, eliminating their knapsack structure.

The relation between $\mathrm{CUT}_n^\square$, $\mathcal{P}_{n,1}$ and $\mathcal{P}_{n,2}$ gives rise to the following proposition which has a similar flavour as Proposition 5.2.7 of [8]. A sum where the range is empty is taken to have value zero.

Proposition 8 *The inequality,*

$$\sum_{1\leq i<j\leq n} a_{i,j}x_{i,j} \leq \alpha \tag{30}$$

is valid (resp. facet defining) for the cut polytope if and only if the inequality

$$\sum_{i=2}^{n}\big(a_{1,i} - \sum_{j=2}^{i-1} a_{j,i} + \sum_{j=i+1}^{n} a_{i,j}\big)x_{1i} + 2\sum_{2\leq i<j\leq n} a_{i,j}x_{ij} \leq \alpha \tag{31}$$

is valid (resp. facet defining) for the polytope $\mathcal{P}_{n,1}$ which is in turn valid (resp. facet defining) if and only if the inequality

$$\sum_{i=2}^{n}\big(a_{1,i} + \sum_{j=2}^{i-1} a_{j,i} - \sum_{j=i+1}^{n} a_{i,j}\big)x_{i1} + 2\sum_{2\leq i<j\leq n} a_{i,j}x_{ij} \leq \alpha \tag{32}$$

is valid (resp. facet defining) for the polytope $\mathcal{P}_{n,2}$.

Proof. We first observe that substituting $x_{i,j}, 1 \leq i < j \leq n$, in (30) using the mapping ξ_1^{-1} we get (31), and using the mapping ξ_2^{-1} we get (32). Since these mappings are bijections they preserve validity and roots, so the proposition follows. ■

We obtain the following table of relations between facets of $\mathrm{CUT}_n^\square$, $\mathcal{P}_{n,1}$ and $\mathcal{P}_{n,2}$.

$CUT_n^\square$	$\mathcal{P}_{n,1}$	$\mathcal{P}_{n,2}$
$x_{1,j} - x_{1,i} - x_{i,j} \leq 0$	$x_{1j} - x_{1i} - x_{ij} \leq 0$	$x_{ij} \geq 0$
$x_{1,i} - x_{1,j} - x_{i,j} \leq 0$	x_{ij}	$x_{i1} - x_{ij} - x_{j1} \leq 0 \geq 0$
$x_{i,j} - x_{1,i} - x_{1,j} \leq 0$	$x_{ij} \leq x_{1j}$	$x_{ij} \leq x_{i1}$
$x_{1,i} + x_{i,j} + x_{1,j} \leq 2$	$x_{1i} + x_{ij} \leq 1$	$x_{j1} + x_{ij} \leq 1$

Table 1: $2 \leq i < j \leq n$

$CUT_n^\square$	$\mathcal{P}_{n,1}$	$\mathcal{P}_{n,2}$
$x_{i,k} - x_{i,j} - x_{j,k} \le 0$	$x_{ik} - x_{ij} - x_{jk} \le 0$	$x_{ik} - x_{ij} - x_{jk} \le 0$
$x_{i,j} - x_{i,k} - x_{j,k} \le 0$	$x_{ij} - x_{ik} - x_{jk} - x_{1j} + x_{1k} \le 0$	$x_{ij} - x_{ik} - x_{jk} + x_{j1} - x_{k1} \le 0$
$x_{j,k} - x_{i,j} - x_{i,k} \le 0$	$x_{jk} - x_{ik} - x_{ij} + x_{1j} - x_{1i} \le 0$	$x_{jk} - x_{ik} - x_{ij} + x_{i1} - x_{j1} \le 0$
$x_{i,j} + x_{j,k} + x_{i,k} \le 2$	$x_{ij} + x_{ik} + x_{jk} + x_{1i} - x_{1k} \le 1$	$x_{ij} + x_{ik} + x_{jk} - x_{i1} + x_{k1} \le 1$

Table 2: $2 \le i < j < k \le n$

5 Facets of the directed cut polytope and cone

In this section we will use the relation between the directed and undirected cuts established by the mappings ξ_1 and ξ_2 to extend previously known structural properties of the cut polyhedra to the directed cut polyhedra. Theorem 9 below allows us to characterize many different facets of the directed cut polytope from knowledge of past work on the cut polytope.

To begin, we will need to define some terms and notation. For a graph $G = (V, E)$ the *support graph*, $G(a) = (V(a), E(a))$, of a vector $a \in \mathbb{R}^{|E|}$ is the graph with edges $E(a) = \{e : a_e \neq 0,\ e \in E\}$ and nodes $V(a)$ such that every node in $V(a)$ is an endpoint of at least one edge in $E(a)$. For $a \in \mathbb{R}^{|E|}$ and $\alpha \in \mathbb{R}$ inequality $a^T x \le \alpha$ is said to be completely supported by $F \subset E$ when $E(a) \subseteq F$. As before, sums with empty range are taken to be zero.

Theorem 9 *If $a^T x \le \alpha$ is a facet of $CUT_n^\square$ then*

$$\begin{aligned} 2\sum_{2\le i<j\le n} a_{i,j}x_{ij} + \sum_{i=2}^{n}\left(a_{1,i} - \sum_{j=2}^{i-1} a_{j,i} + \sum_{j=i+1}^{n} a_{i,j}\right) x_{1i} & \\ + \sum_{i=2}^{n}\left(a_{1,i} + \sum_{j=2}^{i-1} a_{j,i} - \sum_{j=i+1}^{n} a_{i,j}\right) x_{i1} &\le \alpha \qquad (33) \end{aligned}$$

is a facet of $DCUT_n^\square$.

To prove Theorem 9 we need the following lemma, which is the non-homogeneous analogue to Lemma 26.5.2(ii) of [8]. Before stating the lemma, we introduce some terminology and notation from [8] (p. 404, 414). Let $\delta(S)$ be a cut vector for K_n. For a vector $a \in \mathbb{R}^{|E_n|}$ let $a^{\delta(S)} \in \mathbb{R}^{|E_n|}$ be defined by

$$a_{i,j}^{\delta(S)} := \begin{cases} -a_{i,j}, & \text{if } \delta(S)_{i,j} = 1, \\ a_{i,j}, & \text{if } \delta(S)_{i,j} = 0. \end{cases}$$

We say that the inequality

$$(a^{\delta(S)})^T x \le \alpha - a^T \delta(S) \qquad (34)$$

is obtained from the inequality $a^T x \le \alpha$ by *switching on the cut vector* $\delta(S)$. Let Δ denote the set symmetric difference operator. We observe that $\{T : T \subseteq \{1,...,n\}\} = \{T\Delta S : T \subseteq \{1,...,n\}\}$ for any $S \subseteq \{1,...,n\}$ and $\delta(S\Delta S)$ is the origin. So in view of (34), switching on the cut vector $\delta(S)$ takes an inequality with root $\delta(S)$ to one with the origin as a root. The corresponding affine bijection, called the *switching mapping*, is given by

$$r^{\delta(S)}(x)_{i,j} := \begin{cases} 1 - x_{i,j}, & i,j \in \delta(S), \\ x_{i,j}, & i,j \notin \delta(S). \end{cases}$$

It follows that the cut polytope looks the same as the cut cone at any vertex. This property is unfortunately not inherited by the dicut polytope, however it will be of use in generating some of its facets.

Given a subset F of E_n, set $\overline{F} := E_n \setminus F$. If $x \in \mathbb{R}^{|E_n|}$, let

$$x_F := (x_e)_{e \in F}.$$

If X is a subset of $\mathbb{R}^{|E_n|}$, define

$$X_F := \{x_F : x \in X\}, X^F := \{x \in X : x_F = 0\}.$$

Lemma 10 *Let $a^T x \le \alpha$ be a valid inequality for $CUT_n^\square$ and let $R(a,\alpha)$ denote its set of roots. Let F be a subset of E_n. If the inequality $a^T x \le \alpha$ is facet inducing and $a_{\bar{F}} \ne 0$ (resp. $a_{\bar{F}} = 0$) then $rank(R(a,\alpha)_F) = |F|$ (resp. $rank(R(a,\alpha)_F) = |F| - 1$).*

Proof. The proof makes use of various results contained in [8]. Since $a^T x \le \alpha$ is facet inducing it has many roots which are cut vectors. Let $\delta(S)$ be one of them. Then switching on the edge-cut $(S, \overline{S})$ we get an inequality

$$(a^{\delta(S)})^T x \le 0. \tag{35}$$

By Corollary 26.3.7 the inequality (35) is also facet inducing for $\text{CUT}_n^\square$. Let $R(a^{\delta(S)}, 0)$ denote its set of roots. By Lemma 26.5.2(ii) $R(a^{\delta(S)}, 0)$ has the properties that if $a_{\bar{F}}^{\delta(S)} \ne 0$ (resp. $a_{\bar{F}}^{\delta(S)} = 0$) then $rank(R(a^{\delta(S)}, 0))_F = |F|$ (resp. $rank(R(a^{\delta(S)}, 0)_F = |F| - 1)$. However, if $\delta(A)$ is a root of (35) then it is easy to verify that $\delta(A\Delta S)$ is a root of $a^T x \le \alpha$, where symmetric difference is taken with respect to V_n. Therefore the roots in $R(a, \alpha)$ can be obtained from those in $R(a^{\delta(S)}, 0)$ by this operation, which by Lemma 26.3.3 (iii) preserves linear independence. Then lemma follows. ■

With Lemma 10 in hand we can proceed with the proof of Theorem 9. A graph is called a *star* if it has a node that is incident to every edge.

Proof of Theorem 9. First we show that (33) is valid for $\text{DCUT}_n^\square$. Consider a dicut vector $\delta^+(S)$ with $S \in \mathcal{S}_1$, i.e., $1 \in S$, then $x_{i1} = 0, i = 2,...,n$, and (33) reduces to (31) and hence is valid. Similarly if $S \in \mathcal{S}_2$, i.e., $1 \notin S$. Then $x_{1i} = 0, i = 2,...,n$, and (33) reduces to (32) and hence is valid. Since $\mathrm{DCUT}_n^\square$ is the convex hull of $\mathcal{S}_1 \cup \mathcal{S}_2$ validity follows.

To show that (33) defines a facet, we begin by considering the case where $G(a)$ is not a star in which all edges contain vertex 1. Since $a^T x \le \alpha$ is a facet of the cut polytope, it follows that we can find $\binom{n}{2}$ affinely independent roots $\delta(S_i)$ $(1 \le i \le \binom{n}{2})$ of $a^T x \le \alpha$. Since for any cut vector, $\delta(A) = \delta(\overline{A})$, we may assume that $1 \in S_i$ for all i. Choose $F = \{(1,2),(1,3),\cdots,(1,n)\}$, applying Lemma 10 we get that $rank(R(a)_F) = |F| = n-1$ as $G(a)$ is not a star where all edges contain vertex 1. Let $\delta(T_i)$ $(1 \le i \le n-1)$ be $n-1$ roots of $a^T x \le \alpha$ whose projections on F are linearly independent. We can assume that $1 \notin T_i$, by replacing T_i by $\overline{T_i}$ if necessary, and so $\delta^+(T_i) = \xi_2(\delta(T_i))$.

We claim that the dicut vectors in $C = \{\delta^+(S_i) : 1 \le i \le \binom{n}{2}\} \cup \{\delta^+(T_i) : 1 \le i \le n-1\}$ are $\binom{n}{2} + n - 1$ affinely independent roots of the inequality (33). By construction, every cut in C is a root, so we simply need to show that they are affinely independent.

Consider the square matrix M whose rows are first the $\binom{n}{2}$ dicut vectors $\delta^+(S_i)$ followed by the $n-1$ vectors $\delta^+(T_i)$, index the columns of M by the sets $I \cup J$ where $I = \{ij : 1 \le i < j \le n\}$ and $J = \{i1 : 2 \le i \le n\}$. M has the form:

$$M = \begin{pmatrix} X & 0 \\ Z & Y \end{pmatrix}.$$

The matrix X is affinely independent as the vectors $\delta^+(S_i)$ are affinely independent. The matrix Y has full row rank, since its rows $\delta^+(T_i)_J = \delta(T_i)_F$ are linearly independent.

To complete the proof, we will show that the support graph $G(a)$ cannot be a star with all edges containing vertex 1. For suppose it was, then the inequality $a^T x \le \alpha$ becomes

$$\sum_{1j \in E(G(a))} a_{1,j} x_{1,j} \le \alpha. \tag{36}$$

Let the cut vector $\delta(S)$ be a root of (36), so that $a^T \delta(S) = \alpha$. We may assume that $1 \in S$. Suppose first that for each $j \in \overline{S}$, $a_{1,j} > 0$. If $S = \{1\}$, (36) does not define a facet, since it is a non-negative combination of valid inequalities of the form $a_{1,j} x_{1,j} \le 1$. Otherwise let k be any other element of S. If $a_{1,k} > 0$ then we have a contradiction, since $a^T \delta(S \setminus \{k\}) > \alpha$. So $a_{1,k} < 0$ for all $k \in S$ and it follows that (36) does not have any roots besides $\delta(S)$, a contradiction. Therefore there must be some $j \in \overline{S}$ for which $a_{1,j} < 0$. We again have a contradiction because $a^T \delta(S \cup \{j\}) > \alpha$. ■

5.1 The triangle inequalities

Using Theorem 9 one can obtain sets of facets for DCUT_n and $\mathrm{DCUT}_n^\square$ from the triangle inequalities of the cut cone and polytope. Recall that the triangle inequalities for the cut cone and polytope are:

$$\begin{aligned} x_{i,k} - x_{i,j} - x_{j,k} &\leq 0 \\ x_{i,j} - x_{i,k} - x_{j,k} &\leq 0 \\ x_{j,k} - x_{i,j} - x_{i,k} &\leq 0 \end{aligned} \tag{37}$$

for $1 \leq i < j < k < n$. The additional triangle inequalities, known as the perimeter inequalities, for the cut polytope are:

$$x_{i,j} + x_{i,k} + x_{j,k} \leq 2 \tag{38}$$

for $1 \leq i < j < k < n$. These inequalities appear in the first column of Tables 1 and 2.

We apply Theorem 9 to the first three rows of Table 1 and Table 2 to obtain facets for DCUT_n.

Corollary 11 *The following inequalities:*

$$\begin{aligned} x_{1j} - x_{1i} - x_{ij} &\leq 0 \\ x_{i1} - x_{j1} - x_{ij} &\leq 0 \\ -x_{i1} - x_{1j} + x_{ij} &\leq 0 \\ -x_{ij} + x_{ik} - x_{jk} &\leq 0 \\ x_{ij} - x_{ik} - x_{jk} + x_{1k} - x_{k1} + x_{j1} - x_{1j} &\leq 0 \\ -x_{ij} - x_{ik} + x_{jk} + x_{i1} - x_{1i} + x_{1j} - x_{j1} &\leq 0 \end{aligned}$$

for $2 \leq i < j < k \leq n$ are facet defining inequalities of $DCUT_n$.

We can observe that the last three inequalities are the same as (19)-(21) used to define $3\mathrm{MET}_n$.

Similarly, using the fact that the perimeter inequalities (38) are facet inducing inequalities of $\mathrm{CUT}_n^{\square}$ and applying Theorem 9 to the last row of each of Table 1 and 2 yields the following.

Corollary 12 *The inequalities:*

$$\begin{aligned} x_{1i} + x_{ij} + x_{j1} &\leq 1 \\ x_{ij} + x_{jk} + x_{ik} + x_{1i} - x_{i1} + x_{k1} - x_{1k} &\leq 1 \end{aligned}$$

for $2 \leq i < j < k \leq n$ are facet inducing inequalities of $DCUT_n^{\square}$.

We observe that these are the inequalities (22) and (23) used to define $3\mathrm{MET}_n^{\square}$.

5.2 Pentagonal inequalities

The first class of facets for $\mathrm{CUT}_n^\square$ that are not triangle inequalities are the *pentagonal inequalities* that appear when $n \geq 5$. They have the three general forms, where we assume that the indices $1 \leq i, j, k, l, m \leq n$ are distinct:

$$p_1 := x_{i,j} + x_{j,k} + x_{i,k} + x_{l,m} - (x_{i,l} + x_{j,l} + x_{k,l} + x_{i,m} + x_{j,m} + x_{k,m}) \leq 0. \quad (39)$$

$$p_2 := x_{i,j} + x_{i,k} + x_{i,l} + x_{j,k} + x_{j,l} + x_{k,l} - (x_{i,m} + x_{j,m} + x_{k,m} + x_{l,m}) \leq 2. \quad (40)$$

$$p_3 := x_{i,j} + x_{i,k} + x_{i,l} + x_{i,m} + x_{j,k} + x_{j,l} + x_{j,m} + x_{k,l} + x_{k,m} + x_{l,m} \leq 6. \quad (41)$$

The first form is illustrated in Figure 4.

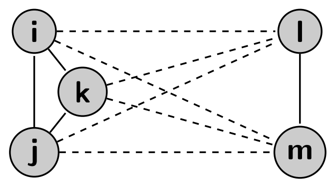

Figure 4: The pentagonal inequality (40) implies that the sum of weights on the dashed edges must be at least as large as the sum of weights on the solid edges

Applying Theorem 9 we obtain facets of the dicut polyhedra. As with the triangle inequalities, the pentagonal inequalities take different forms when expressed in terms of the variables for the full dimensional dicut polytope. For example, with $2 \leq i < j < k < l < m \leq n$ we obtain the following facets of $\mathrm{DCUT}_n^\square$:

$$x_{j1} - x_{1j} + 2(x_{k1} - x_{1k} + x_{1l} - x_{l1}) + x_{1m} - x_{m1} + p_1 \leq 0$$

$$x_{1i} - x_{i1} + x_{k1} - x_{1k} + 2(x_{l1} - x_{1l} + x_{1m} - x_{m1}) + p_2 \leq 1$$

$$2x_{1i} - 2x_{i1} + x_{1j} - x_{j1} + x_{l1} - x_{1l} + 2x_{m1} - 2x_{1m} + p_3 \leq 3.$$

Other forms are more elegant. With $i = 1, j = 3, k = 5, l = 2, m = 4$ we obtain the facet

$$x_{13} + x_{15} + x_{35} + x_{24} \leq x_{12} + x_{14} + x_{23} + x_{34} + x_{25} + x_{45} \quad (42)$$

for DCUT_n and $\mathrm{DCUT}_n^\square$ when $n \geq 5$. In the next section we describe how to generate systematically all pentagonal inequalities by relating them to the more general hypermetric inequalities.

5.3 Hypermetric inequalities

The hypermetric inequalities are valid inequalities for $\mathrm{CUT}_n^\square$ and CUT_n which generalize the triangle and pentagonal inequalities (see [8]). Let $b = (b_1, \cdots, b_n)$ be an integral vector such that $\sum_{i=1}^n b_i = 2k+1$ is odd, and there is some $S \subseteq \{1, 2, ..., n\}$ for which $\sum_{i \in S} b_i = 1 + \sum_{i \notin S} b_i$. The inequality

$$\sum_{1 \le i < j \le n} b_i b_j x_{i,j} \quad \le \quad 0 \tag{43}$$

is called a *hypermetric inequality.* Every hypermetric inequality is known to be valid for CUT_n. We observe that we get the triangle inequalities (37) by setting $b = (1, -1, 1)$,$(1, 1, -1)$ and $(-1, 1, 1)$ respectively. The triangle inequality (38) corresponds to $b = (1, 1, 1)$ and the three classes of pentagonal inequalities (40)-(41) correspond to $b = (1, 1, 1, -1, -1)$,$(1, 1, 1, 1, -1)$ and $(1, 1, 1, 1, 1)$ respectively. It is known that when n is odd and $b_i = \pm 1$ the corresponding hypermetric inequality generates a facet of CUT_n. These are known as *pure hypermetric* inequalities.

Theorem 9 has the following corollary.

Corollary 13 *The inequality*

$$\sum_{i=2}^{n} (b_1 - \sum_{k=2}^{i-1} b_k + \sum_{j=i+1}^{n} b_j) b_i x_{1i} + \sum_{i=2}^{n} (b_1 + \sum_{k=2}^{i-1} b_k - \sum_{j=i+1}^{n} b_j) b_i x_{i1} + 2 \sum_{2 \le i < j \le n} b_i b_j x_{ij} \quad \le \quad 0 \tag{44}$$

corresponding to the hypermetric inequality (43) is valid for $DCUT_n$*, and is a facet for* $DCUT_n$ *whenever (43) is a facet of* CUT_n*.*

We note that (42) corresponds to $b = (1, -1, 1, -1, 1)$. We may generalize this to a vector $b = (1, -1, 1, ..., -1, 1)$ of length n to obtain the following facet of DCUT_n and $\mathrm{DCUT}_n^\square$:

$$\sum_{1 \le i < j \le n} b_i b_j x_{ij} \quad \le \quad 0. \tag{45}$$

6 Zero-lifting the directed cut polytope

Zero-lifting is a standard way of deriving facets of higher dimensional polytopes from lower dimensional ones. For $v \in \mathbb{R}^{|A_n|}$ we define the *zero-lifting* $v' \in \mathbb{R}^{|A_{n+1}|}$ by

$$v'_{ij} = v_{ij} \qquad ij \in A_n, \qquad\qquad v'_{i\ n+1} = 0 \qquad 1 \le i \le n.$$

We say that inequality $(v')^T x \leq 0$ is obtained by zero-lifting the inequality $v^T x \leq 0$.

In this section we prove a zero-lifting theorem for the directed cut cone. We will work with the full dimensional projection of the directed cut cone, with coordinates indexed by the arc set $A(\vec{J}_n) = \{ij : 1 \leq i < j \leq n\} \cup \{i1 : 2 \leq i \leq n\}$ of the graph $\vec{J}_n$ with vertex set V_n, as discussed in Section 3. We begin by showing that Lemma 14 of [8], which we discussed earlier in Section 4 in terms of the cut polytope, can be extended to the directed cut cone.

Lemma 14 *Let $v^T x \leq 0$ be a valid inequality for $DCUT_n$ and let $R(v)$ denote its set of roots. Let F be a subset of $A(\vec{J}_n)$.*
(i) If $rank(R(v)_F) = |F|$ and $rank(R(v)^F) = |\bar{F}| - 1$, then the inequality $v^T x \leq 0$ is facet inducing.
(ii) If the inequality $v^T x \leq 0$ is facet inducing and $v_F \neq 0$ (resp. $v_{\bar{F}} = 0$), then $rank(R(v)_F) = |F|$ (resp. $rank(R(v)_F) = |F| - 1$).

The proof of Lemma 14 follows the same argument as the proof of Lemma 26.5.2 in [8]. We include it here for completeness but it required no substantial alterations.

Proof. (i) By the assumptions, a set A of $|F|$ linearly independent roots can be found whose projections on the arcs F are linearly independent. Likewise, a set B of roots of $v^T x \leq 0$ can be found whose projections on F are the zero vector where the vectors of B are linearly independent and $|B| = |\bar{F}| - 1$. It is easy to see that the vectors $A \cup B$ are a set of $\binom{n}{2} + n - 2$ linearly independent roots of $v^T x \leq 0$ which along with the empty set dicut vector form $\binom{n}{2} + n - 1$ affinely independent roots. Therefore, $v^T x \leq 0$ is a facet of DCUT_n.

(ii) If $v^T x \leq 0$ is a facet of DCUT_n, we can find a set A of $\binom{n}{2} + n - 2$ linearly independent roots of $v^T x \leq 0$. If we construct a matrix M by using the vectors A as the rows, we have a $(\binom{n}{2} + n - 2) \times (\binom{n}{2} + n - 1)$ matrix with linearly independent rows. This means that all but one column of M are linearly independent.

If the columns corresponding to arcs of F are linearly dependent then $rank(A_F) = |F| - 1$. Let $T_1 \subseteq A$ be $|F| - 1$ vectors whose projection on F are linearly independent, let $T_2 \subseteq A$ be the vectors of A whose projection on F are the zero vector and let $T_3 = A \setminus (T_1 \cup T_2)$.

For $x \in T_3$ we can express x_F (the projection of x onto the arcs of F) as a convex combination of vectors of T_1, i.e. $x_F = \sum_{y_i \in T_1} \lambda_i (y_i)_F$. A new set T_3' can be constructed where for each x in T_3, we add $x' = x - \sum_{y_i \in T_1} \lambda_i y_i$ to T_3'. The vectors in the set $T_2 \cup T_3'$ are linearly independent. It follows that $v_{\bar{F}} = 0$ as we have $|T_2 \cup T_3'| = |\bar{F}|$ linearly independent vectors satisfying $v^T x = 0$ with $x_F = 0$.

If the columns corresponding to the arcs of F are linearly independent then $rank(A_{\bar{F}}) = |\bar{F}| - 1$ and a similar argument as above can be applied to show that $v_{\bar{F}} \neq 0$. ■

We can now state and prove our zero-lifting theorem for the directed cut cone. In the proof we use a similar approach to the proof of Theorem 26.5.1 on page 415 of [8].

Theorem 15 *Given $v \in \mathbb{R}^{A(\vec{J}_n)}$ and zero-lifted $v' \in \mathbb{R}^{A(\vec{J}_{n+1})}$ the following are equivalent.*

- $v^T x \leq 0$ *is facet inducing for* $DCUT_n$.
- $v'^T x \leq 0$ *is facet inducing for* $DCUT_{n+1}$.

Proof. Assume that $v'^T x \leq 0$ is facet inducing for DCUT_{n+1} and let $R(v')$ denote its roots. Let $F = A(\vec{J}_n)$ and let $\bar{F} = \{(1, n+1), ..., (n, n+1)\} \cup (n+1, 1)$. Using Lemma 14(ii) and the fact that $v'_{\bar{F}} = 0$ we know that the rank of $R(v')_F$ is equal to $\binom{n}{2} + n - 2$, we can use $\binom{n}{2} + n - 2$ linearly independent roots from $R(V')_F$ and with the dicut vector corresponding to the empty set to form $\binom{n}{2} + n - 1$ affinely independent roots. This implies that $v^T x \leq 0$ is a facet of DCUT_n.

Assume that $v^T x \leq 0$ is facet inducing for DCUT_n and let $F = \{1n, 2n, ..., (n-1, n)\}$. As $v \neq 0$ we can assume that $v_{\bar{F}} \neq 0$ by using the permutation operation described in Lemma 4. By Lemma 14, $rank(R(v)_F) = |F|$. Let $T_j \subseteq V_n$, $j = 1, ..., n-1$, be $|F|$ subsets such that the projections of $\delta^+(T_j)$ $j = 1, ..., n-1$, onto the arc set F are linearly independent. Note that this implies that vertex n is not in any subset T_j, since if it were, the projection onto F would be the zero vector. Let S_k, $k = 1, ..., \binom{n}{2} + n - 2$, be subsets of V_n such that $\delta^+(S_k)$ are linearly independent roots of $v^T x \leq 0$.

Let $S'_k = S_k \cup \{n+1\}$ for $k = 1, ..., \binom{n}{2} + n - 2$. We claim the vectors $\delta^+(S'_k) \cup \delta^+(T_j) \cup \delta^+(\{1, ..., n\}) \cup \delta^+(\{n+1\})$ for $j = 1, ..., n-1$ and $k = 1, ..., \binom{n}{2} + n - 2$ form $n - 1 + \binom{n}{2} + n - 2 + 2 = \binom{n+1}{2} - (n+1) - 2 = Dim(DCUT_{n+1}) - 1$ linearly independent roots of $v'^T x \leq 0$. These $Dim(DCUT_{n+1}) - 1$ dicuts vectors along with the dicut vector corresponding to the empty set form $Dim(DCUT_{n+1})$ affinely independent root vectors. This proves that $v'^T x \leq 0$ is facet inducing for DCUT_{n+1}.

To see that this claim is true consider the matrix M consisting of the vectors $\delta^+(S'_k)$, $\delta^+(T_j)$, $\delta^+(\{1, ..., n\})$, and $\delta^+(\{n+1\})$ as rows for $j = 1, ..., n-1$ and $k = 1, ..., \binom{n}{2} + n - 2$. Let the columns of M be indexed by the arcs: $A(\vec{J}_n)$ followed by $(1, n+1), (2, n+1), ..., (n, n+1)$ and finally $(n+1, 1)$. The matrix M has the form:

$$M = \begin{pmatrix} X & 0 & 0 & d \\ Z & Y & 0 & 0 \\ 0 & e & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Matrix X has linearly independent rows as it corresponds to the $\binom{n}{2} + n - 2$ linearly independent vectors $\delta^+(S'_k)$. The entry d is a column vector corresponding to the $(n+1, 1)$ entry of the $\delta^+(S'_k)$ vectors. e is a row vector of all ones. Submatrix Y's columns correspond to the arcs $(1, n+1), (2, n+1) ..., (n-1, n+1)$. Recall that $n \notin T_j, j = 1, ..., n-1$. Therefore the entries of $(i, n+1)$ are identical to (i, n) in the $\delta^+(T_j)$ vectors, it follows that Y has linearly independent rows. The final two rows independence follow from their definitions. The last row can be used to eliminate the column vector d. The resulting matrix is then easily seen to have linearly independent rows. ∎

Corollary 16 *The non-negativity inequalities $x_{ij} \geq 0$ define facets of $DCUT_n$ for all $n \geq 3$.*

Proof. We will prove that $x_{12} \geq 0$ defines a facet for all $n \geq 3$ and then use the permutation operation. For $n = 3$ it is straightforward to verify that the four dicut vectors defined by $S = \{1, 2\}, S = \{2\}, S = \{2, 3\}$ and $S = \{3\}$ are a set of roots that are linearly independent. For $n \geq 4$ the result follows from Theorem 15. ■

As remarked earlier, the non-negativity inequalities do not form facets of CUT_n. Therefore this corollary could not be proved by use of Theorem 9. In the next section we will similarly use Theorem 15 to show that a further family of inequalities are facets of DCUT_n.

In this section we presented a zero-lifting lemma for DCUT_n, which as we observed earlier, is similar to a similar result for the cone CUT_n. In the latter case, by switching on cuts, the result immediately holds for the polytope $CUT_n^\square$. However, there does not seem to be such an easy way to prove a zero-lifting lemma for $DCUT_n^\square$. Such a result, subject to a few technical conditions, is contained in Meagher [12].

7 Facets of $\mathbf{DCUT}_3^\square$ and $\mathbf{DCUT}_4^\square$

In this section we give a complete characterization of the facets of $\mathrm{DCUT}_3^\square$ and $\mathrm{DCUT}_4^\square$, using results from the paper and the output of *lrs* computer runs. Except where noted, all results in this section will be in terms of the full dimensional dicut polytopes defined on the variables $x_{ij}, 1 \leq i < j \leq n$, and $x_{i1}, 2 \leq i \leq n$, so that the facets have a unique representation.

$\mathrm{DCUT}_3^\square$ is defined by a set of 15 dicut vectors in 5 dimensions and has 10 facets as follows:

- Six non-negativity constraints:

$$x_{12}, x_{13}, x_{23}, x_{21}, x_{31} \geq 0, \quad x_{23} + x_{12} - x_{21} + x_{31} - x_{13} \geq 0.$$

- Three homogeneous triangle inequalities (first three inequalities of Corollary 11):

$$x_{13} \leq x_{12} + x_{23}, \qquad x_{21} \leq x_{23} + x_{31}, \qquad x_{23} \leq x_{21} + x_{13}.$$

- One perimeter triangle inequality (first inequality of Corollary 12):

$$x_{12} + x_{23} + x_{31} \leq 1.$$

$\mathrm{DCUT}_4^\square$ is defined by a set of 31 dicut vectors in 9 dimensions and has 40 facets as follows:

- Twelve non-negativity constraints:

$$x_{ij} \geq 0, \quad 1 \leq i < j \leq 4, \qquad x_{i1} \geq 0, \quad 2 \leq i \leq 4,$$

$$x_{ij} + x_{1i} - x_{i1} + x_{j1} - x_{1j} \geq 0, \qquad 2 \leq i < j \leq 4.$$

- Twelve homogeneous triangle inequalities (Corollary 11):

$$x_{ij} \leq x_{i1} + x_{1j}, \quad x_{i1} \leq x_{ij} + x_{j1}, \quad x_{1j} \leq x_{1i} + x_{ij}, \qquad 2 \leq i < j \leq 4$$

$$-x_{23} + x_{24} - x_{34} \leq 0, \qquad x_{23} - x_{24} - x_{34} + x_{14} - x_{41} + x_{31} - x_{13}, \leq 0$$

$$-x_{23} - x_{24} + x_{34} + x_{21} - x_{12} + x_{13} - x_{31} \leq 0.$$

- Four perimeter triangle inequalities (Corollary 12):

$$x_{12} + x_{23} + x_{31} \leq 1, \qquad x_{12} + x_{24} + x_{41} \leq 1, \qquad x_{13} + x_{34} + x_{41} \leq 1,$$

$$x_{23} + x_{34} + x_{24} + x_{12} - x_{21} + x_{41} - x_{14} \leq 1.$$

- Six new homogeneous inequalities (see Figure 5):

$$x_{ik} + x_{jl} \leq x_{ij} + x_{kl} + x_{il} + x_{jk}$$

  where $\{i, j, k, l\} = \{1, 2, 3, 4\}$, $i < j$, and we have used the full coordinate system for simplicity.

- Six new non-homogeneous inequalities (see Figure 6):

$$x_{k1} + x_{1i} + x_{ij} \leq 1 + x_{i1} \quad \textit{or} \quad x_{1k} + x_{i1} + x_{ji} \leq 1 + x_{1i}$$

  where $\{i, j, k\} = \{2, 3, 4\}$ (inequality depends on whether $i < j$ or $j < i$).

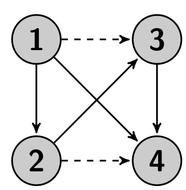

Figure 5: Example of new homogeneous facet: $x_{13} + x_{24} \leq x_{12} + x_{34} + x_{14} + x_{23}$

We observe that two new classes of facets were needed to complete the description of $\mathrm{DCUT}_4^\square$. Again we find a significant difference between the family of cut and dicut polytopes, since $\mathrm{CUT}_4^\square$ is described completely by the set of triangle inequalities. As there are no new facets for $\mathrm{CUT}_4^\square$, we cannot obtain the new facets of $\mathrm{DCUT}_4^\square$ directly from Theorem 9. For the homogeneous facets, we can use the lifting theorem to show that they are also facets of $\mathrm{DCUT}_n^\square$ for all $n \geq 4$. Due to the permutation operation described in Lemma 4 we need to consider only one member of the family.

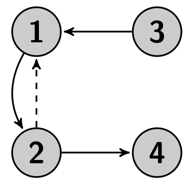

Figure 6: Example of new non-homogeneous facet: $x_{31} + x_{12} + x_{24} \leq 1 + x_{21}$

Theorem 17 *The inequality:*

$$x_{ik} + x_{jl} \leq x_{ij} + x_{kl} + x_{il} + x_{jk} \tag{46}$$

is facet inducing for $DCUT_n$ for all $n \geq 4$.

Proof. As DCUT_4 has dimension 9, listing 9 affinely independent roots proves that (46) is a facet of DCUT_4. The dicut vectors $\delta^+(S)$ where S ranges over the following subsets:

$$\emptyset, \{j\}, \{i,j\}, \{i,j,l\}, \{i,l\}, \{j,k\}, \{j,k,l\}, \{k,l\}, \{l\}$$

are such a set. Applying Theorem 15 gives the result that (46) is a facet inducing inequality for DCUT_n and $\mathrm{DCUT}_n^\square$, for $n \geq 4$. ■

To prove that the new non-homogeneous inequalities are facet defining inequalities, a zero lifting result for non-homogeneous inequalities can be used. As remarked earlier, such a result is contained in [12].

For completeness we conclude this section with a vertex description of the relaxation $3\mathrm{MET}_4^\square$ of $\mathrm{DCUT}_4^\square$. It has a total of 21 vertices, of which 15 correspond to dicuts and 6 are half-integral fractional vertices. The fractional vertices are as follows. Let $\{i,j\} \cup \{k,l\} = \{1,2,3,4\}$ be any partition of V_4. Then

$$x_{ik} = x_{jk} = x_{il} = x_{jl} = 0, \quad \text{else} \;\; x_{uv} = \frac{1}{2}, \;\; uv \in A_4,$$

is a vertex of $\mathrm{DCUT}_4^\square$.

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