On Combinatorial Properties of Linear Program Digraphs

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Abstract

The possible pivot operations of the simplex method to solve a linear program can be represented as a directed graph defined on the skeleton of the feasible region P. We consider the case that P is bounded, i.e., a convex polytope. The directed graph is called an LP digraph. LP digraphs are known to satisfy the following three properties: acyclicity, unique sink orientation(USO), and the Holt-Klee property. The three properties are not generally sufficient for a directed graph on the skeleton of P to be an LP digraph. In this paper, we first survey some previous results on LP digraphs, showing relationships among the three properties. Then we introduce a new necessary property for a directed graph on the skeleton of P to be an LP digraph, called the shelling property. We analyze the relationships between the shelling property and the three existing properties, showing that it is stronger than a combination of acyclicity and USO for non-simple polytopes in dimension at least four. In all other cases it is equivalent to the intersection of these two properties.

1 Introduction

Let P be a d-dimensional convex polytope (d-polytope) in $\Re^d$. We assume that the reader is familiar with polytopes, a standard reference being [10]. The vertices and the edges of P form an (abstract) undirected graph called the skeleton of P. Interest in such graphs stems from the fact that the simplex method with a given pivot rule can be viewed as an algorithm for finding a path in the skeleton of P to a vertex that maximizes a linear function $f(x) = c^T x$ over P. Research continues on pivot rules for the simplex method since they leave open the possibility of finding a strongly polynomial time algorithm for linear programming. Since the simplex method gives an orientation to edges traversed, it is of interest to study directed graphs based on the skeleton. We form a directed graph $G(P)$ by orienting each edge of the skeleton of P in some manner.

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We can distinguish four properties that the digraph $G(P)$ may have, each of which has been well studied:

- *Acyclicity*: $G(P)$ has no directed cycles.
- *Unique sink orientation (USO)* (Szabó and E. Welzl [9]): Each subdigraph of $G(P)$ induced by a face of P has a unique source and a unique sink.
- *Holt-Klee property* (Holt and Klee [5]): $G(P)$ has a unique sink orientation, and for every k-dimensional face (k-face) H of P there are k vertex disjoint paths from the unique source to the unique sink of H in the subdigraph $G(P, H)$ of $G(P)$ induced by H.
- *LP digraph*: There is a linear function f and a polytope P' combinatorially equivalent to P such that for each pair of vertices u and v of P' that form a directed edge (u, v) in $G(P')$, we have $f(u) < f(v)$. In this case, we denote $G(P')$ by LP(P', f). (This is the same as polytopal digraph in Mihalsin and Klee [6].)

The main property of interest is the fourth one, since the digraph LP(P, f) represents the possible pivot operations of the simplex method in maximizing f over P. The other three properties are necessary conditions for $G(P)$ to be an LP-digraph. We note here that Williamson Hoke [4] has defined a property called *complete unimodality* which is equivalent to a combination of acyclicity and unique sink orientation. A definition of complete unimodality and an important result of Williamson Hoke relating to our work are stated in Section 3.2. First, we survey known results on the relationships among the four properties, according to the dimension d of P, collecting them in the following theorem, whose proof along with bibliographic references, will be given in the next section.

Theorem 1. *For a digraph $G(P)$ based on a d-polytope P, the relationships among the properties acyclicity, USO, the Holt-Klee property and LP-digraph are as shown in Figure 1, where the regions A,B,...,J are non-empty. These relationships hold even for simple polytopes. They also hold for cubes, except for $d = 3$, when region E is empty.*

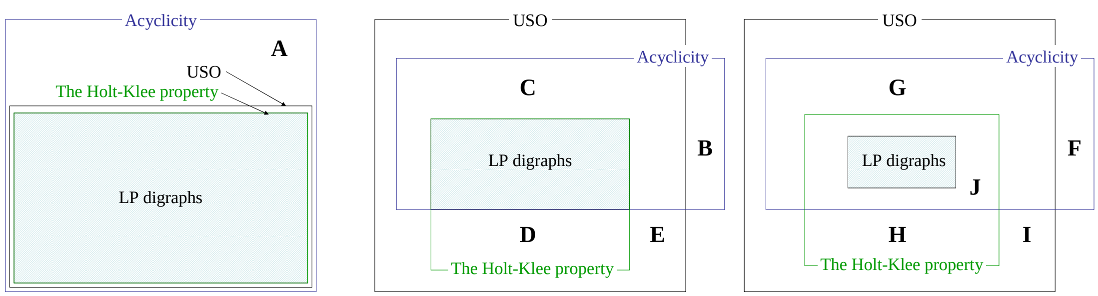

Figure 1: The relationships for d-polytopes: $d = 2$[left], $d = 3$[middle] and $d \geq 4$[right]

From Figure 1 it can be seen that LP digraphs are completely characterized when $d = 2, 3$ [6]. No such characterization is known yet for higher dimensions. The letters A,B,...,J in the figure refer to a set of ten non-empty regions, as will be demonstrated by specific examples that will be given in the next section.

In this paper we introduce another necessary property for $G(P)$ to be an LP digraph, based on *shelling*, which is one of the fundamental tools of polytope theory. A formal definition of shelling is given in Section 3. Suppose a polytope P has m vertices labelled $v_1, v_2, ..., v_m$, and let $G(P)$ be a digraph based on its skeleton. A permutation r of the vertices is a *topological sort* of $G(P)$ if, whenever (v_i, v_j) is a directed edge of $G(P)$, v_i precedes v_j in the permutation r. Let $L(P)$ be the face lattice of P. A polytope P^* whose face lattice is $L(P)$ "turned upside-down" is called a *combinatorially polar* polytope of P. Combinatorial polarity interchanges vertices of P with facets of P^*. We denote by r^* the facet ordering of P^* corresponding to the vertex ordering of P given by r.

- *Shelling property*: There exists a topological sort r of $G(P)$ such that the facets of P^* ordered by r^* are a shelling of P^*.

In Section 3 we analyze the relationship between the shelling property and the four previously defined properties. The results are summarized in the following theorem.

Theorem 2. *For digraphs $G(P)$ based on a d-polytope P, the relationships among the properties acyclicity, USO, the Holt-Klee property, LP-digraph and the shelling property are as shown in Figure 2, where the regions A,B,...,J,X,Y are non-empty.*

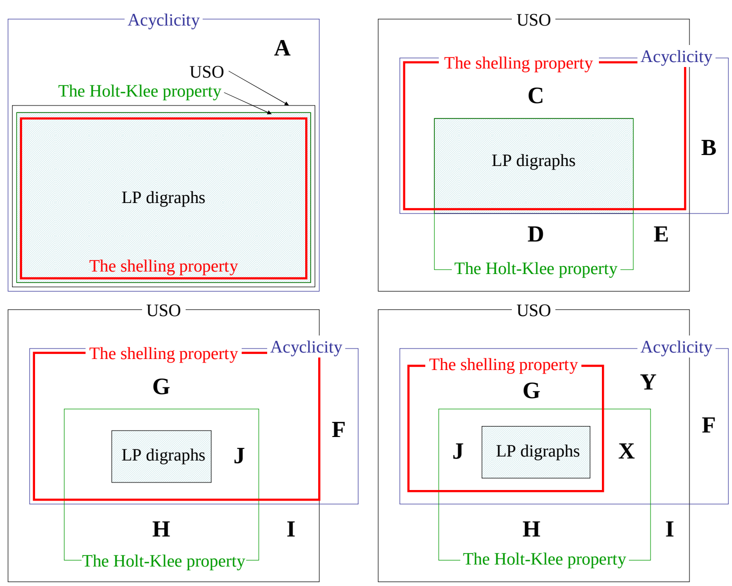

Figure 2: The relationships when $d = 2$[upperleft] and $d = 3$[upperright] and the relationships when for $d \geq 4$, P is simple[lowerleft] and general[lowerright]

2 Proof of Theorem 1

2.1 The case when P is a cube

Let P be the d-dimensional cube C_d. Specialized to C_d, Theorem 1 reduces to Figure 3.

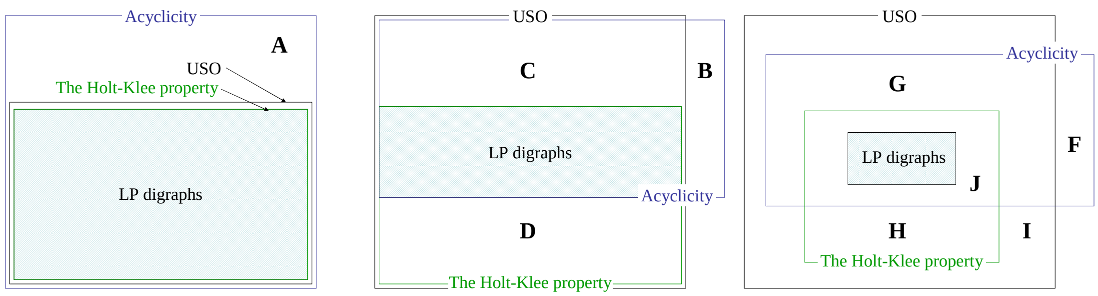

Figure 3: Possible orientations of the cube C_d: $d = 2$[left], $d = 3$[middle] and $d \geq 4$[right]

For each of the marked regions we will give a corresponding example based on a cube. When $d = 2$, it is immediate that the digraph $G(C_2)$ is a USO if and only if it satisfies the Holt-Klee property and is acyclic. On the other hand, there exists a $G(C_2)$ which is not a USO, does not satsify the Holt-Klee property, but is acyclic. This is example A of Figure 4. Based on this example, we also generate examples for the regions B and F, as shown in the same figure. These two examples are acyclic, but are not USOs.

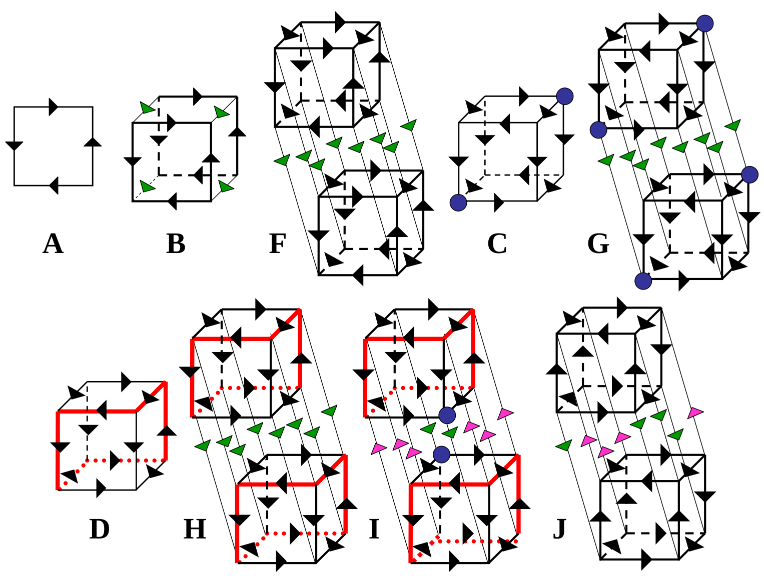

Figure 4: Examples corresponding to non-empty regions of Figure 1

When $d = 3$, there is no cyclic orientation of $G(C_3)$ which is a USO but does not satisfy the Holt-Klee property [8], so region E of Figure 1 is empty for cubes. On the other hand, there are two distinct orientations of C_3 which are acyclic USOs, but do not satisfy the Holt-Klee property [8]. Example C of Figure 4 is one such example, and can be used to generate example G, showing the corresponding region of Figure 3 is non-empty. In Figure 4 a cut set of size two separating source and sink of C is highlighted. There are two facets in G each isomorphic to C, the Holt-Klee property fails in each of them and so fails for G.

Furthermore, C_3 admits only one cyclic orientation which is a USO and satisfies the Holt-Klee property [8]. This is example D in Figure 4, which can be used to generate examples to show that regions H and I are non-empty, shown in the same figure. The cycles of examples D, H and I are highlighted. In I a cut set of size two between source and sink is highlighted, showing the Holt-Klee property fails.

On the other hand, there exists an orientation of C_4 which satisfies the three properties, but is not an LP digraph, see Morris [7]. This is example J of Figure 4.

The examples F,G,H,I and J readily extend to cubes in higher dimensions: duplicate the figure, add an edge between each pair of corresponding vertices and then direct them all from the first copy to the second.

2.2 The case when P is a simple polytope

Since C_d is simple, all of the preceeding examples apply to this case. It remains to show that the region E may be non-empty for simple 3-polytopes. Indeed, Figure 5 shows a 3-polytope with a cyclic USO which does not satisfy the Holt-Klee property. There is a cut set of size 2: top vertex and bottom left vertex, so the Holt-Klee property is violated. There is a cycle of length 6 along the boundary of the region left over when the top left vertex is deleted. These are highlighted in Figure 5.

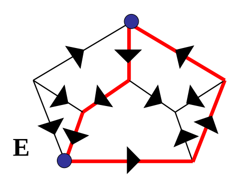

Figure 5: An example for region E of Figure 1

3 The shelling property and proof of Theorem 2

3.1 Shellings and Line shellings

We use the definition of shelling given in Ziegler [10, Definition 8.1] which is slightly more restrictive than the one used by Brugesser and Mani [1]. Let P be a d-polytope in $\Re^d$. A *shelling* of P is a linear ordering $F_1, F_2, \cdots, F_s$ of the facets of P such that either the facets are points, or it satisfies the following conditions [10, Definition 8.1]:

(i) the first facet F_1 has a linear ordering its facets which is a shelling of F_1.
(ii) For $1 < j \le m$ the intersection of the facet F_j with the previous facets is nonempty and is a beginning segment of a shelling of F_j, that is,

$$F_j \cap \bigcup_{i=1}^{j-1} F_i = G_1 \cup G_2 \cup \cdots \cup G_r$$

for some shelling $G_1, G_2, \cdots, G_r, \cdots, G_t$ of F_j, and $1 \le r \le t$. (In particular this requires that *all maximal faces included in* $F_j \cap \bigcup_{i=1}^{j-1} F_i$ *have the same dimension* $d-2$.)

Any polytope has at least one shelling because of the existence of *line shellings* [1], described below. Hence the condition (i) is in fact redundant [10, Remark 8.3 (i)].

It is easy to see that in general, not every sequence s of the m facets of P is a shelling. Consider a 3-dimensional cube C_3 with the facets F_i for $i = 1, \cdots, 6$ as in Figure 6. Let s be any sequence of the F_i's whose first three facets are F_6, F_2 and F_5. If the fourth facet of s is F_3, the intersection $(F_6 \cup F_2 \cup F_5) \cap F_3$ consists of two parallel line segments, which cannot be the beginning of a shelling of F_3.

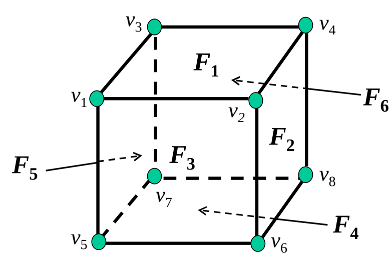

Figure 6: A 3-dimensional cube C_3 with the six facets

Let P be a d-polytope with m facets in $\Re^d$. A directed straight line L that intersects the interior of P and the affine hulls of the facets of P at distinct points is called *generic* with respect to P. We choose a generic line L and label a point interior to P on L as x. Starting at x, we number consecutively the intersection points along L with facets as $x_1, x_2, ..., x_m$, wrapping around at infinity, as in Figure 7. The ordering of the corresponding facets of P is the *line shelling* of P generated by L. Every line shelling is a shelling of P (see, e.g., [10]).

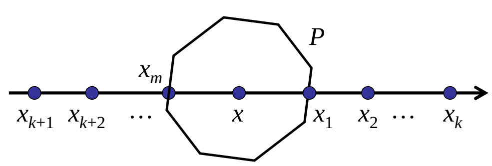

Figure 7: The intersection points along a directed straight line L

Using these definitions we can prove the following proposition.

Proposition 3. *If $G(P)$ is an LP-digraph, it satisfies the shelling property.*

Proof. Choose c such that $G(P) = LP(P, c^T x)$, and suppose P has m vertices. Consider the m distinct parallel hyperplanes with normal c that each contains a vertex of P. Choose a line L parallel to c that intersects the interior of P. Then the intersection of the m hyperplanes with L gives an ordering r of vertices which is a topological sort of $G(P)$. The corresponding ordering r^* of the facets of the polar P^* is a line shelling of P^*. Therefore $G(P)$ satsifies the shelling property. □

3.2 Acyclicity and Unique sink orientations

In this subsection we complete the proof of Theorem 2. We begin with the following proposition.

Proposition 4. *If $G(P)$ satisfies the shelling property, it is an acyclic USO.*

Proof. Since $G(P)$ satisfies the shelling property, there exists an ordering r of the vertices of P such that r^* is a shelling of P^*. It follows that $G(P)$ is acyclic. We prove that $G(P)$ is a USO by contradiction.

Let F_i be the i-th facet of r^*. The acyclicity of $G(P)$ implies that for any face H of P, the induced subdigraph $G(P, H)$ of $G(P)$ has no directed cycles. Therefore, there exists at least one source in $G(P, H)$. When $\dim(H) = 1$ trivially $G(P, H)$ has only one source. We prove that when $\dim(H) > 1$, $G(P, H)$ also has only one source. Let U be the set of the indices of the vertices of P included in H. Assume that there exist more than one source in $G(P, H)$. We denote the set of the indices of the sources in $G(P, H)$ by S and let k be the smallest index of S. In the polar, if $j \in S$ and $k < j$, no ridges are included in the intersection $F_j \cap \bigcup F_i$ for $i \in U$ and $i < j$. However, the intersection is not empty but includes at least H^*, where H^* is the face of P^* corresponding to H. The intersection $F_j \cap \bigcup F_i$ for $i \notin U$ and $i < j$ might include some ridges. However, these ridges do not include H^*, because the facets F_i for $i \notin U$ and $i < j$ do not. It follows that $F_j \cap \bigcup_{i=1}^{j-1} F_i$ contains a maximal face that is not $d - 2$ dimensional, contradicting the fact that r^* is a shelling of P^*. Therefore every face subdigraph for $G(P)$ has a unique source.

Finally, we show that every face subdigraph for $G(P)$ also has a unique sink. Here we reverse the orientation of $G(P)$ of each edge. We denote this digraph by $\hat{G}(P)$. One of topological sorts for $\hat{G}(P)$ is the reverse order of r^*. From the properties of a shelling of any polytope Q, the reverse order of a shelling of Q is also a shelling of Q [10, Lemma 8.10]. Hence the reverse order of r^* is also a shelling of P^*, and then $\hat{G}(P)$ satisfies the shelling property. Thus every face subdigraph for $\hat{G}(P)$ also has a unique source. It follows that every face subdigraph for $G(P)$ also has a unique sink as a source of $\hat{G}(P)$ corresponds to a sink of $G(P)$. Therefore, it is proved that $G(P)$ is a USO. This completes the proof. □

In the next two propositions we show that if P is simple, or if $d = 2, 3$, then the converse of this proposition is true. A *completely unimodal numbering* of the m vertices of a simple d-polytope P is a numbering $1, \cdots, m$ of the vertices such that there is exactly one local minimum on every k-face F of P, for $k = 2, 3, ..., d$. That is, on F there is only one vertex with no lower-numbered neighbours on F.

Proposition 5. *Let P be a simple d-polytope. If $G(P)$ is an acyclic USO then it satisfies the shelling property.*

Proof. Williamson Hoke [4] has shown that the completely unimodal numberings of P correspond to shellings of P^*. Since $G(P)$ is acyclic we may choose a topological sort of $G(P)$. Since $G(P)$ is a USO, it follows that this ordering of the vertices is completely unimodal. By Williamson Hoke's result it corresponds to a shelling of P^* and so $G(P)$ satisfies the shelling property. □

Proposition 6. *Let P be a d-polytope in $\Re^d$, with d=2,3. If $G(P)$ is an acyclic USO, then it satisfies the shelling property.*

Proof. Let P satisfy the conditions of the theorem. Acyclicity implies that there exists a topological sort of $G(P)$. When $d = 2$, it is easy to check that every topological sort r of $G(P)$ induces a shelling r^* of P^*.

Now suppose $d = 3$. Let F_i be the i-th facet of r^*. Since $G(P)$ is a USO, the intersection $I(j) := F_j \cap \bigcup_{i=1}^{j-1} F_i$ has at least one 1-face for every $j \geq 2$. We note that the intersection $I(j)$ for $j \geq 2$ is connected if and only if it has a shelling [10, Example 8.2(i)], which is just condition (ii) in the definition of shellings (see Section 3.1). Hence we prove that for $j \geq 2$, the intersection $I(j)$ consists of only one connected component. By way of contradiction, let us assume that for some k, $I(k)$ consists of $n \geq 2$ components, and for $2 \leq j \leq k-1$, $I(j)$ consists of only one connected component. Consider the *Euler characteristic* of P^* [10, Corollary 8.17]. The Euler characteristic of P^* is defined as $\chi(P^*) := v - e + f - 1$, where v, e and f are the number of vertices, edges and facets of P^*, respectively. We compute $\chi(P^*)$ by updating its value when each facet F_j is added according to r^*. The initial contribution of the facet F_1 to $\chi(P^*)$ is zero. Let $x(j)$ be the number of the vertices newly added by F_j. For $2 \leq j \leq k-1$, the contribution of F_j to $\chi(P^*)$ is equal to $x(j) - (x(j)+1) + 1 = 0$. Hence the contribution of the F_j's to $\chi(P^*)$ for $1 \leq j \leq k-1$ is equal to zero. On the other hand, the contribution of F_k to $\chi(P^*)$ is equal to $x(k) - (x(k)+n) + 1 = 1 - n$, hence the contribution of F_j's for $1 \leq j \leq k$ becomes $0 + 1 - n = 1 - n$. We note that $\chi(P^*) = 1$ when $d = 3$ [10, Corollary 8.17]. The only way to increase $\chi(P^*)$ is to add a facet F_h, for some h, all of whose vertices and edges are already contained in $I(h)$. In this case $\chi(P^*)$ increases by one. The last facet in the shelling order satisfies this property. However, we must have at least $n \geq 2$ such facets. Since each such facet F_h is a sink in $G(P)$, $G(P)$ is not a USO. This contradiction completes the proof. □

Proposition 6 cannot be extended to the case where $d \geq 4$ as Develin showed [3] with the following example. Let C_4^* be a 4-dimensional crosspolytope. Figure 8 shows an orientation $G(C_4^*)$ of the skeleton of C_4^*, which is an acyclic USO satisfying the Holt-Klee property, but not the shelling property, so the region X of Figure 2 is non-empty.

We show another example for the region X, which has fewer vertices and facets than Develin's example. Let Δ be a 4-dimensional polytope with the facets F_i for $i = 1, \cdots, 7$, as shown in Figure 9 and Table 2. We checked using the software PORTA [2] that Δ is convex when the coordinates of the 12 vertices of Δ are given as in Table 1. For the coordinates of the 12 vertices in Table 1, the supporting hyperplanes of the 7 facets of Δ are given as $a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 \leq b$ with the coefficients in Table 2. Let Δ^* be a combinatorial

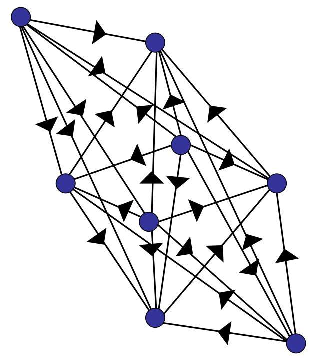

Figure 8: An example for region X of Figure 2

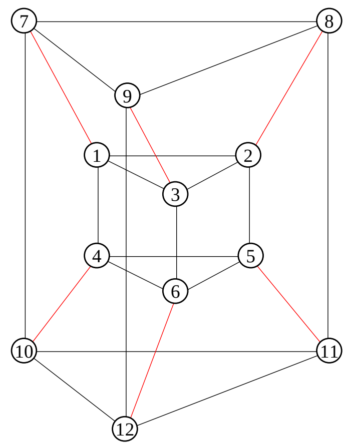

Figure 9: A 4-dimensional polytope Δ with 7 facets

polar of Δ. An orientation $G(\Delta^*)$ of the skeleton of Δ^* is given in Figure 10. One of topological sorts on the orientation is the sequence of all the 7 facets in order of indices. We have the following result.

Proposition 7. *$G(\Delta^*)$ is an acyclic USO that satisfies the Holt-Klee property, but not the shelling property.*

Proof. In $G(\Delta^*)$ all edges are directed from smaller index to larger index, so it is an acyclic USO. Δ^* is a simplicial polytope, i.e. all i-faces of Δ^* for $i \leq 3$ are i-dimensional simplices. Hence if an orientation of its skeleton is an acyclic USO, it also satisfies the Holt-Klee property. There exist five vertex-disjoint paths from F_1 to F_7 in Figure 10. Therefore, the graph $G(C_4^*)$ satisfies the Holt-Klee property. We prove that the graph $G(C_4^*)$ does not satisfy the shelling property.

There is a path $F_1, F_2, ..., F_7$ of all the seven vertices of $G(\Delta^*)$ in order of their indices, and so this ordering is the unique topological sort of the graph. By referring to Table 2 and

Figure 9, it can be verified that $F_3 \cap \bigcup_{i=1}^{2} F_i$ is the union of two disjoint 2-faces of Δ: one with vertices $2, 5, 8, 11$ and one with vertices $3, 6, 9, 12$. This union cannot be the beginning of a shelling of F_3, hence the unique topological sort of $G(\Delta^*)$ is not a shelling of Δ. This completes the proof. $\square$

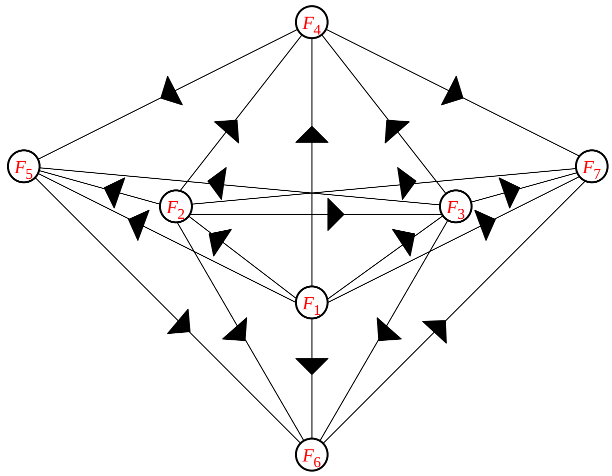

Figure 10: The graph $G(\Delta^*)$

In order to prove Proposition 8, we form a new 4-dimensional polytope Γ in Figure 11 by adding one new vertex to the 12 vertices of Δ and decomposing the facet F_4 of Δ into five new facets. The new 4-dimensional polytope Γ has 13 vertices and 11 facets, as shown in Table 4. We also checked using the software PORTA [2] that Γ is convex when the coordinates of the 13 vertices of Δ are given as in Table 3. For the coordinates of the 13 vertices in Table 3, the supporting hyperplanes of the 11 facets of Γ are given as $a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 \leq b$ with the coefficients in Table 4. Let Γ^* be a combinatorial polar of Γ. An orientation $G(\Gamma^*)$ of the skeleton of Γ^* is given in Figure 12. One of topological sorts on the orientation is the sequence of all the 11 facets in order of indices. We have the following result.

Proposition 8. *$G(\Gamma^*)$ is an acyclic USO that satisfies neither the Holt-Klee property nor the shelling property.*

Proof. By adding one new vertex to the 12 vertices of Δ, the new 4-polytope Γ^* includes 3-faces which are not simplices, as shown in Figure 14. We observe that the 3-faces with the indices 1 and 9 do not satisfy the Holt-Klee property (as the vertices F_2 and F_6 and the vertices F_4 and F_9 are two-point cut-sets respectively) while the other 3-faces in Figure 14 satisfy it. Moreover, $G(\Gamma^*)$ does not satisfy the shelling property by the same proof as that of Proposition 7. $\square$

Collecting the above propositions, we have verified the relationships among the five properties shown in Figure 13 for non-simple polytopes of dimension at least four. The examples G and J are orientations of a cube, hence they satisfy the shelling property [4]. The examples

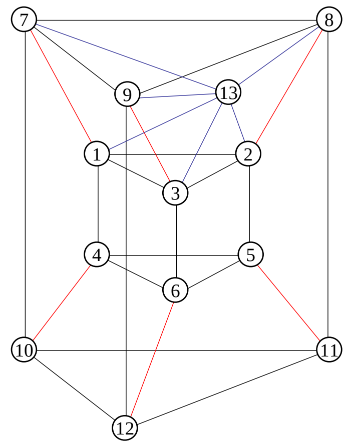

Figure 11: A 4-dimensional polytope Γ with 13 vertices and 11 facets

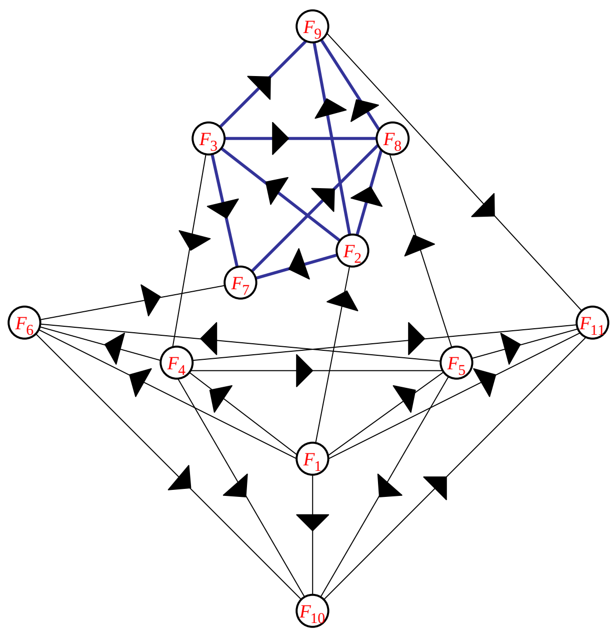

Figure 12: The graph $G(\Gamma^*)$

$G(C_4^*)$ and $G(\Delta^*)$ above are the examples X and Y, respectively. Combined with the results for $d = 2, 3$, we have proved Theorem 2.

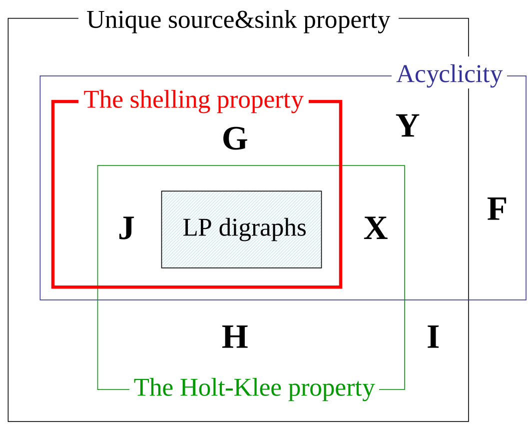

Figure 13: The relationships when P is a general polytope and $d \geq 4$

vertex	(x_1, x_2, x_3, x_4)	vertex	(x_1, x_2, x_3, x_4)	vertex	(x_1, x_2, x_3, x_4)	vertex	(x_1, x_2, x_3, x_4)
1	$(-2, 1, 2, 1)$	2	$(2, 1, 2, 1)$	3	$(0, -2, 2, 1)$	4	$(-2, 1, -2, 1)$
5	$(2, 1, -2, 1)$	6	$(0, -2, -2, 1)$	7	$(-2, 1, 2, -1)$	8	$(2, 1, 2, -1)$
9	$(0, -2, 2, -1)$	10	$(-2, 1, -2, -1)$	11	$(2, 1, -2, -1)$	12	$(0, -2, -2, -1)$

Table 1: The coordinates of the 12 vertices of the 4-dimensional polytope Δ

4 Conclusion

In this paper, we introduced a new necessary property for a directed graph of a d-polytope P to be an LP digraph, called the shelling property. We analyzed the relationships among the shelling property and the four existing propeties, acyclicity, unique sink orientation, the Holt-Klee property and LP digraphs. We proved that if $G(P)$ satisfies the shelling property, it is an acyclic USO. The converse is true for simple polytopes but not in general if the dimension is at least four. In showing this we gave one example not satisfying the Holt-Klee property in addition to Develin's example [3]. This establishes the independence of the shelling property from the other properties.

Moreover, when $d = 2, 3$, we also proved that if $G(P)$ is an acyclic USO then it also satisfies the shelling property. This is an extension of Williamson Hoke's result [4], which states that when P is simple, $G(P)$ is an acyclic USO if and only if it satisfies the shelling property.

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facet	vertices	(a_1, a_2, a_3, a_4, b)	facet	vertices	(a_1, a_2, a_3, a_4, b)
F_1	1, 2, 4, 5, 7, 8, 10, 11	(0, 1, 0, 0, 1)	F_2	1, 3, 4, 6, 7, 9, 10, 12	(−3,−2, 0, 0, 4)
F_3	2, 3, 5, 6, 8, 9, 11, 12	(3, −2, 0, 0, 4)	F_4	1, 2, 3, 7, 8, 9	(0, 0, 1, 0, 2)
F_5	1, 2, 3, 4, 5, 6	(0, 0, 0, 1, 1)	F_6	4, 5, 6, 10, 11, 12	(0, 0, −1, 0, 2)
F_7	7, 8, 9, 10, 11, 12	(0, 0, 0, −1,1)			

Table 2: The 7 facets of the 4-dimensional polytope Δ in Figure 9 and the coefficients of their supporting hyperplanes for the coordinates of the 12 vertices in Table 1

vertex	(x_1, x_2, x_3, x_4)	vertex	(x_1, x_2, x_3, x_4)	vertex	(x_1, x_2, x_3, x_4)	vertex	(x_1, x_2, x_3, x_4)
1	(−2, 1, 2, 1)	2	(2, 1, 2, 1)	3	(0, −2, 2, 1)	4	(−2, 1, −2, 1)
5	(2, 1, −2, 1)	6	(0, −2,−2, 1)	7	(−2, 1, 2, −1)	8	(2, 1, 2, −1)
9	(0, −2, 2, −1)	10	(−2, 1, −2,−1)	11	(2, 1, −2,−1)	12	(0, −2,−2,−1)
13	(0, 0, 4, 0)						

Table 3: The coordinates of the 13 vertices of the 4-dimensional polytope Γ

facet	vertices	(a_1, a_2, a_3, a_4, b)	facet	vertices	(a_1, a_2, a_3, a_4, b)
F_1	1, 2, 4, 5, 7, 8, 10, 11	(0, 1, 0, 0, 1)	F_2	1, 2, 7, 8, 13	(0, 2, 1, 0, 4)
F_3	1, 3, 7, 9, 13	(−3,−2, 2, 0, 8)	F_4	1, 3, 4, 6, 7, 9, 10, 12	(−3,−2, 0, 0, 4)
F_5	2, 3, 5, 6, 8, 9, 11, 12	(3, −2, 0, 0, 4)	F_6	1, 2, 3, 4, 5, 6	(0, 0, 0, 1, 1)
F_7	1, 2, 3, 13	(0, 0, 1, 2, 4)	F_8	2, 3, 8, 9, 13	(3, −2, 2, 0, 8)
F_9	7, 8, 9, 13	(0, 0, 1, −2,4)	F_{10}	4, 5, 6, 10, 11, 12	(0, 0, −1, 0, 2)
F_{11}	7, 8, 9, 10, 11, 12	(0, 0, 0, −1,1)			

Table 4: The 11 facets of the 4-dimensional polytope Γ in Figure 11 and the coefficients of their supporting hyperplanest for the coordinates of the 13 vertices in Table 3

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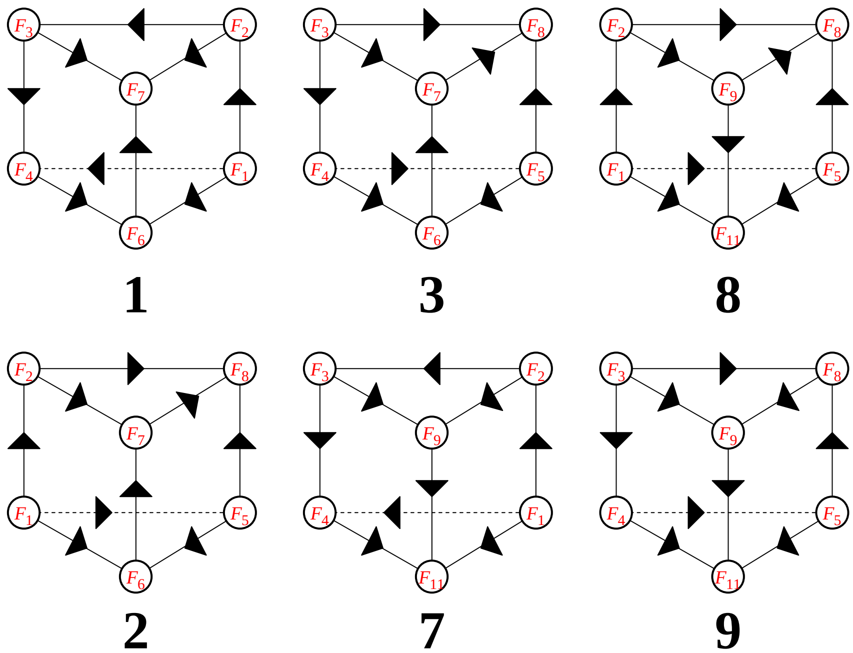

Figure 14: The 3-faces of the graph $G(\Gamma^*)$

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