

# Two-party Bell inequalities derived from combinatorics via triangular elimination

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## Abstract

Bell inequalities, originally introduced as a method to prove that some quantum states show nonlocal behavior, are now studied as a method to capture the extent of the nonlocality of quantum states. Tight Bell inequalities are considered to be more important than redundant ones. Despite the increasing importance of the study of Bell inequalities, few kinds of tight Bell inequalities have been found. Examples include the Clauser-Horne-Shimony-Holt inequality, the  $I_{mmvv}$  inequalities, the CGLMP inequalities, and the Bell inequalities in systems small enough to generate all the Bell inequalities by exhaustive search. In this paper, we establish a relation between the two-party Bell inequalities for two-valued measurements and a high-dimensional convex polytope called the cut polytope in polyhedral combinatorics. Using this relation, we propose a method, *triangular elimination*, to derive tight Bell inequalities from facets of the cut polytope. This method gives two hundred million inequivalent tight Bell inequalities from currently known results on the cut polytope. In addition, this method gives general formulas which represent families of infinitely many Bell inequalities. These results can be used to examine general properties of Bell inequalities.

## 1 Introduction

Entanglement and nonlocality have been important topics in quantum physics since the beginning of the discipline. Bell [3] formulated nonlocality of quantum states, which was the first consequence of the existence of entanglement. Using an inequality satisfied by the results of a correlation experiment using classical states, Bell showed that there exists a quantum state which produces results which do not satisfy the inequality. Clauser, Horne, Shimony and Holt [5] replaced Bell's inequality with another inequality, called the CHSH inequality, which admits a more realistic interpretation. Extending the CHSH inequality, a Bell inequality is a linear inequality which the results of a correlation experiment with two or more distant parties always satisfy. In this paper, we focus on the two-party case where one party has  $m_A$  choices of possible two-valued measurements and the other party has  $m_B$  choices. In this notation, the CHSH inequality is a Bell inequality in the case  $m_A = m_B = 2$ . The CHSH inequality is important both theoretically and experimentally to the extent that the terms "CHSH inequality" and "Bell inequality" are sometimes used synonymously.

Fine [12] proved that in the smallest case  $m_A = m_B = 2$ , there are only two kinds of Bell inequalities up to symmetric transformations: the inequalities representing nonnegativity of probability and the CHSH inequalities. After that, a connection between Bell inequalities and convex polytope theory proved by Peres [16] and progress in polyhedral computation have allowed us to compute the complete list of Bell inequalities in many small cases [7, 19, 21]. As a result, we know the complete list of Bell inequalities in the cases  $m_A = 2$ ,  $(m_A, m_B) = (3, 3)$  and  $(m_A, m_B) = (3, 4)$ . However, due to the complex structure of the polytope, it is unlikely that there exists a compact representation of the complete set of Bell inequalities for an arbitrarily large  $m_A$  and  $m_B$ , and the test whether a given result for Bell's experiment can be produced by some local state is NP-complete [2]. This leads to research to find a partial list of Bell inequalities which have some good properties. In this direction, Collins and Gisin [7] show a family  $I_{mm22}$  of Bell inequalities in the case

$m_A = m_B = m$  for general  $m$ . Besides, there are several extensions [7, 8, 14] of the CHSH inequality for multi-valued measurements. Readers are referred to a survey by Werner and Wolf [22] for an in-depth discussion of Bell inequalities.

Nonlocality is an important concept in quantum information science not only because it is evidence of entanglement, but also because it can sometimes be considered as a resource by itself. Examples include the gap of success probability between quantum and classical communication protocols [15] or cooperative games with incomplete information [6].

Polyhedral geometry plays an important role in study of Bell inequalities. The results of a correlation experiment with  $m_A$  choices of measurements for one party and  $m_B$  choices for the other are represented as a vector of  $m_A + m_B + m_A m_B$  probabilities. In classical mechanics, the set of vectors which are feasible as the results of a correlation experiment forms an  $(m_A + m_B + m_A m_B)$ -dimensional convex polytope. A Bell inequality is nothing but a linear inequality valid for this polytope, or a linear inequality satisfied by all the points in this polytope. A tight Bell inequality, or a Bell inequality which cannot be represented as a sum of other Bell inequalities, corresponds to a facet of this polytope. Interestingly, if we transform the coordinates of this high-dimensional space, the polytope becomes another convex polytope called the cut polytope  $\text{CUT}^\square(K_{1,m_A,m_B})$  of the complete tripartite graph  $K_{1,m_A,m_B}$ , which is studied in polyhedral combinatorics. This relation enables us to use rich results on cut polytope to study Bell inequalities. Related to this, Pironio [17] uses lifting, which is a common approach in study of cut polytope, to generate tight Bell inequalities in a larger system from those in a smaller system.

One obstacle to this approach is that many existing results on cut polytopes deal with the cut polytope  $\text{CUT}_n^\square = \text{CUT}^\square(K_n)$  of the complete graph  $K_n$  instead of the cut polytope  $\text{CUT}^\square(K_{1,m_A,m_B})$  of the complete tripartite graph. In particular, many results are known on valid inequalities and facets of  $\text{CUT}_n^\square$ . To overcome this gap, we introduce a method called triangular elimination to convert an inequality valid for  $\text{CUT}_n^\square$  to another inequality valid for  $\text{CUT}^\square(K_{1,m_A,m_B})$ , which is then converted to a Bell inequality by a transformation of coordinates. The CHSH inequality and some of the other previously known inequalities can be explained in this manner. More importantly, triangular elimination converts a facet of  $\text{CUT}_n^\square$  to a facet of  $\text{CUT}^\square(K_{1,m_A,m_B})$ , which corresponds to a tight Bell inequality.

A complete list of facets of  $\text{CUT}_n^\square$  for  $n \leq 7$  and a conjectured complete list for  $n = 8, 9$  are known. We apply triangular elimination to these facets to obtain 201,374,783 tight Bell inequalities. On the other hand, several formulas which represent many different inequalities valid for  $\text{CUT}_n^\square$  are known. We apply triangular elimination to these formulas to obtain new families of Bell inequalities. We discuss their properties such as tightness and inclusion of the CHSH inequality.

The rest of this paper is organized as follows. In Section 2, we introduce triangular elimination to derive tight Bell inequalities from facets of the cut polytope of the complete graph, and show its properties. We also give a computational result on the number of Bell inequalities obtained by triangular elimination. In Section 3 we apply triangular elimination to some of the known classes of facets of the cut polytope of the complete graph to obtain general formulas representing many Bell inequalities.

## 2 Triangular elimination

### 2.1 Bell inequalities and facets of cut polytopes

Consider a system composed of subsystems A (Alice) and B (Bob). Suppose that on both subsystems, one of  $m_A$  observables for Alice and one of  $m_B$  observables for Bob are measured. For each observable, the outcome is one of two values (in the rest of the paper, we label the outcomes as 0 or 1). The experiment is repeated a large number of times. The result of such a correlation experiment consists of the probability distribution of the  $m_A m_B$  joint measurements by both parties. Throughout this paper, we represent the experimental result as a vector  $\mathbf{q}$  in  $m_A + m_B + m_A m_B$  dimensional space in the following manner:  $q_{A_i}$ ,  $q_{B_j}$  and  $q_{A_i B_j}$  correspond to the probabilities  $\Pr[A_i = 1]$ ,  $\Pr[B_j = 1]$  and  $\Pr[A_i = 1 \wedge B_j = 1]$  respectively.

In classical mechanics, the result of a correlation experiment must correspond to a probability distribution over all *classical configurations*, where a classical configuration is an assignment of the outcomes  $\{0, 1\}$  to each of the  $m_A + m_B$  observables. The experimental result has a *local hidden variable model* if and only if a given experimental result can be interpreted as a result of such a classical correlation experiment.

*Bell inequalities* are valid linear inequalities for every experimental result which has a local hidden variable

model. Specifically using the above formulation, we represent a Bell inequality in the form

$$\sum_{1 \leq i \leq m_A} b_{A_i} q_{A_i} + \sum_{1 \leq j \leq m_B} b_{B_j} q_{B_j} + \sum_{1 \leq i \leq m_A, 1 \leq j \leq m_B} b_{A_i B_j} q_{A_i B_j} \leq b_0.$$

for suitably chosen constants  $b_x$ .

For example, Clauser, Horn, Shimony and Holt [5] have shown that the following CHSH inequality is a valid Bell inequality:

$$-q_{A_1} - q_{B_1} + q_{A_1 B_1} + q_{A_1 B_2} + q_{A_2 B_1} - q_{A_2 B_2} \leq 0.$$

In general, the set of all experimental results with a local hidden variable model forms a convex polytope with extreme points corresponding to the classical configurations. If the results of the experiment are in the above form, the polytope is called a *correlation polytope*, a name introduced by Pitowsky [18]. (Such polyhedra have been discovered and rediscovered several times, see for instance Deza and Laurent [11].) From such a viewpoint, Bell inequalities can be considered as the boundary, or face inequalities, of that polytope. Since every polytope is the intersection of finitely many half spaces represented by linear inequalities, every Bell inequality can be represented by a convex combination of finitely many extremal inequalities. Such extremal inequalities are called *tight* Bell inequalities. Non-extremal inequalities are called *redundant*.

In polytopal theory, the maximal extremal faces of a polytope are called *facets*. Therefore, tight Bell inequalities are facet inequalities of the polytope formed by experimental results with a local hidden variable model. Note that for a given linear inequality  $\mathbf{b}^T \mathbf{q} \leq b_0$  and  $d$  dimensional polytope, the face represented by the inequality is a facet of that polytope if and only if the dimension of the convex hull of the extreme points for which the equality holds is  $d - 1$ .

### 2.1.1 Cut polytope of complete tripartite graph

We introduce a simple representation of an experimental setting as a graph. Consider a graph which consists of vertices corresponding to observables  $A_i$  or  $B_j$  and edges corresponding to joint measurements between  $A_i$  and  $B_j$ . In addition, to represent probabilities which are the results of single (not joint) measurements, we introduce a vertex  $X$  (which represents the trace out operation of the other party) and edges between  $X$  and  $A_i$  for every  $1 \leq i \leq m_A$ , and between  $X$  and  $B_j$  for every  $1 \leq j \leq m_B$ . This graph is a complete tripartite graph since there exist edges between each party of vertices (observables)  $\{X\}$ ,  $\{A_i\}$  and  $\{B_j\}$ . Using this graph, we can conveniently represent either the result probabilities or the coefficients of a Bell inequality as edge labels. We denote this graph by  $K_{1, m_A, m_B}$ .

In polyhedral combinatorics, a polytope affinely isomorphic to the correlation polytope has been well studied. Specifically, if we consider the probabilities  $x_{A_i B_j} = \Pr[A_i \neq B_j]$  instead of  $q_{A_i B_j} = \Pr[A_i = 1 \wedge B_j = 1]$  for each edge, the probabilities form a polytope called the *cut polytope*. Thus, the cut polytope is another formulation of the polytope formed by Bell inequalities.

A *cut* in a graph is an assignment of  $\{0, 1\}$  to each vertex, 1 to an edge between vertices with different values assigned, and 0 to an edge between vertices with the same values assigned. In the above formulation, each cut corresponds to a classical configuration. Note that since the 0, 1 exchange of all values of vertices yields the same edge cut, we can without loss of generality assume that the vertex  $X$  is always assigned the label 0.

Let the *cut vector*  $\boldsymbol{\delta}'(S') \in \mathbb{R}^{\{X A_i\} \cup \{X B_j\} \cup \{A_i B_j\}}$  for some cut  $S'$  be  $\delta'_{uv}(S') = 1$  if vertices  $u$  and  $v$  are assigned different values, and 0 if assigned the same values. Then, the convex combination of all the cut vectors  $\text{CUT}^\square(K_{1, m_A, m_B}) = \{\mathbf{x} = \sum_{S': \text{cut}} \lambda_{S'} \boldsymbol{\delta}'(S') \mid \sum_{S': \text{cut}} \lambda_{S'} = 1 \text{ and } \lambda_{S'} \geq 0\}$  is called the cut polytope of the complete tripartite graph. The cut polytope has full dimension. Therefore,  $\dim(\text{CUT}^\square(K_{1, m_A, m_B})) = m_A + m_B + m_A m_B$ .

In this formulation, a tight Bell inequality  $\mathbf{b}^T \mathbf{q} \leq b_0$  corresponds to a facet inequality  $\mathbf{a}^T \mathbf{x} \leq a_0$  of the cut polytope. The affine isomorphisms between them are:

$$\begin{cases} x_{X A_i} = q_{X A_i}, \\ x_{X B_j} = q_{X B_j}, \\ x_{A_i B_j} = q_{X A_i} + q_{X B_j} - 2q_{A_i B_j}, \end{cases} \quad \begin{cases} q_{X A_i} = x_{X A_i}, \\ q_{X B_j} = x_{X B_j}, \\ q_{A_i B_j} = \frac{1}{2}(x_{X A_i} + x_{X B_j} - x_{A_i B_j}). \end{cases} \quad (1)$$

Actually, because cut polytopes are symmetric under the switching operation (explained in Section 2.4) we can assume that the right hand side of a facet inequality of the cut polytope is always 0. This means that a

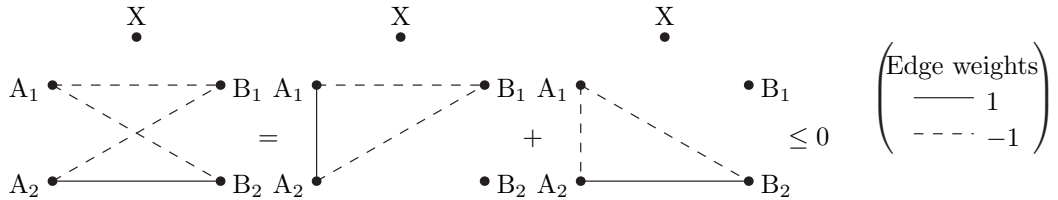


Figure 1: The most simple case of triangular elimination: The sum of two triangle inequalities is the CHSH inequality.

given Bell inequality is tight if and only if for the corresponding facet inequality  $\mathbf{a}^T \mathbf{x} \leq 0$  of the cut polytope, there exist  $m_A + m_B + m_A m_B - 1$  linearly independent cut vectors  $\delta'(S')$  for which  $\mathbf{a}^T \delta'(S') = 0$ .

For example, there exists a facet inequality  $-x_{A_1 B_1} - x_{A_1 B_2} - x_{A_2 B_1} + x_{A_2 B_2} \leq 0$  for  $\text{CUT}^\square(K_{1, m_A, m_B})$ ,  $1 \leq m_A, m_B$  which corresponds to the CHSH inequality. Therefore, the CHSH inequality is tight in addition to being valid.

A consequence of the above affine isomorphisms is that any theorem concerning facets of the cut polytope can be immediately translated to give a corresponding theorem for tight Bell inequalities. For example, the original proof of the validity of the CHSH inequality is rather long. However it is well known that the triangle inequality  $x_{uv} - x_{uw} - x_{vw} \leq 0$ , for any three vertices  $u, v, w$ , is valid for the cut polytope. In fact, Bell's original inequality [3] is essentially this inequality. Since the positive sum of valid inequalities is always a valid inequality, the validity of the inequality corresponding to the CHSH inequality is immediate:  $-x_{A_1 B_1} - x_{A_1 B_2} - x_{A_2 B_1} + x_{A_2 B_2} \leq 0$  is the sum of  $x_{A_1 A_2} - x_{A_1 B_1} - x_{A_2 B_1} \leq 0$  and  $x_{A_2 B_2} - x_{A_1 B_2} - x_{A_1 A_2} \leq 0$  (see Figure 1). Recently, Collins and Gisin [7] gave the following conjecture about the tightness of Bell inequalities: if a Bell inequality  $\mathbf{b}^T \mathbf{q} \leq b_0$  is tight in a given setting  $m_A, m_B$ , then for each  $m'_A \geq m_A$  and  $m'_B \geq m_B$ , the inequality  $\mathbf{b}'^T \mathbf{q}' \leq b_0$  is also tight. Here  $\mathbf{b}'$  is the vector  $b'_{uv} = b_{uv}$  if the vertices (observables)  $u, v$  appear in  $\mathbf{b}$  and is zero otherwise. They gave empirical evidence for this conjecture based on numerical experiments. In fact, a special case of the *zero-lifting theorem* by De Simone [9] gives a proof of their conjecture.

## 2.2 Triangular elimination

### 2.2.1 Cut polytope of complete graph

In the previous section we saw that the problem of enumerating tight Bell inequalities is equivalent to that of enumerating facet inequalities of the cut polytope of a corresponding complete tripartite graph. The properties of facet inequalities of the cut polytope of the complete graph  $K_n$  are well studied and there are rich results. For example, several general classes of facet inequalities with relatively simple representations are known.

For  $n \leq 7$  the complete list of facets is known [13], and for  $n = 8, 9$  a conjectured complete list is known [4, 20]. In addition, the symmetry of the polytope is also well-understood. We show how to apply such results to our complete tripartite graph case.

First, we introduce the cut polytope of complete graph. The graph is denoted by  $K_n$ , has  $n$  vertices, and has an edge between each pair of vertices. As before, a cut is an assignment of  $\{0, 1\}$  to each vertex, and an edge is labeled by 1 if the endpoints of the edge are labeled differently or 0 if labeled the same. The cut vectors  $\delta(S)$  of the complete graph are defined in the same manner as before. The set of all convex combinations of cut vectors  $\text{CUT}^\square(K_n) = \{\mathbf{x} = \sum_{S: \text{cut}} \lambda_S \delta(S) \mid \sum_{S: \text{cut}} \lambda_S = 1 \text{ and } \lambda_S \geq 0\}$  is called the cut polytope of the complete graph.  $\text{CUT}^\square(K_n)$  is also written as  $\text{CUT}_n^\square$ .

In contrast to the complete tripartite graph, the space on which the cut polytope of the complete graph exists has elements corresponding to probabilities of joint measurement by the same party. Because of the restrictions of quantum mechanics, such joint measurements are prohibited. Therefore, if we want to generate tight Bell inequalities from the known facet inequalities of the cut polytope of the complete graph, we must transform the inequalities to eliminate joint measurement terms. In polyhedral terms,  $\text{CUT}^\square(K_{1, m_A, m_B})$  is a projection of  $\text{CUT}^\square(K_n)$  onto a lower dimensional space.

### 2.2.2 Definition of triangular elimination

A well known method for projecting a polytope is called Fourier-Motzkin elimination. This is essentially the summation of two facet inequalities to cancel out the target term. For example, the CHSH inequality

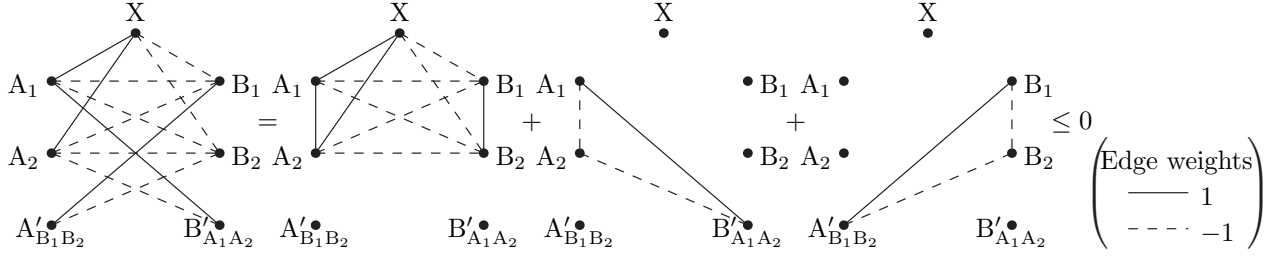


Figure 2: The  $I_{3322}$  inequality is generated by triangular elimination from the pentagonal inequality of  $\text{CUT}_5^\square$ .

$-x_{A_1B_1} - x_{A_1B_2} - x_{A_2B_1} + x_{A_2B_2} \leq 0$  is the sum of  $x_{A_1A_2} - x_{A_1B_1} - x_{A_2B_1} \leq 0$  and  $x_{A_2B_2} - x_{A_1B_2} - x_{A_1A_2} \leq 0$ .

In general, the result of Fourier-Motzkin elimination is not necessarily a facet. For example, it is known that the pentagonal inequality

$$x_{XA_1} + x_{XA_2} - x_{XB_1} - x_{XB_2} + x_{A_1A_2} - x_{A_1B_1} - x_{A_1B_2} - x_{A_2B_1} - x_{A_2B_2} + x_{B_1B_2} \leq 0 \quad (2)$$

is a facet inequality of  $\text{CUT}^\square(K_5)$ . If we eliminate joint measurement terms  $x_{A_1A_2}$  and  $x_{B_1B_2}$  by adding triangle inequalities  $x_{A_1B_2} - x_{A_1A_2} - x_{A_2B_2} \leq 0$  and  $x_{A_2B_1} - x_{B_1B_2} - x_{A_2B_2} \leq 0$ , the result is  $x_{XA_1} + x_{XA_2} - x_{XB_1} - x_{XB_2} - x_{A_1B_1} - 3x_{A_2B_2} \leq 0$ . Therefore, this inequality is a valid inequality for  $\text{CUT}^\square(K_{1,3,3})$ . However, the inequality is a summation of four valid triangle inequalities for  $\text{CUT}^\square(K_{1,3,3})$ , namely  $x_{XA_1} - x_{XB_1} - x_{A_1B_1} \leq 0$ ,  $x_{XA_2} - x_{XB_2} - x_{A_2B_2} \leq 0$ ,  $x_{XA_2} - x_{XB_2} - x_{A_2B_2} \leq 0$  and  $-x_{XA_2} + x_{XB_2} - x_{A_2B_2} \leq 0$ . This means that the inequality with eliminated terms is redundant.

Fourier-Motzkin elimination often produces large numbers of redundant inequalities, causing the algorithm to be computationally intractable when iterated many times. Therefore, it is important to find situations where the new inequalities found are guaranteed to be tight.

The difference between the two examples is that in the CHSH case, the second triangle inequality introduced a new vertex  $B_2$  where “new” means that the first triangle inequality had no term with subscript labeled  $B_2$ . Generalizing this operation, we will show that Fourier-Motzkin elimination by triangle inequalities which introduce new vertices, is almost always guaranteed to yield non-redundant inequalities. We call the operation *triangular elimination*.

**Definition 2.1** (triangular elimination). For a given valid inequality for  $\text{CUT}^\square(K_{1+n_A+n_B})$

$$\sum_{1 \leq i \leq n_A} a_{XA_i} x_{XA_i} + \sum_{1 \leq j \leq n_B} a_{XB_j} x_{XB_j} + \sum_{1 \leq i \leq n_A, 1 \leq j \leq n_B} a_{A_i B_j} x_{A_i B_j} + \sum_{1 \leq i < i' \leq n_A} a_{A_i A_{i'}} x_{A_i A_{i'}} + \sum_{1 \leq j < j' \leq n_B} a_{B_j B_{j'}} x_{B_j B_{j'}} \leq a_0, \quad (3)$$

the triangular elimination is defined as follows:

$$\sum_{1 \leq i \leq m_A} a_{XA_i} x_{XA_i} + \sum_{1 \leq j \leq m_B} a_{XB_j} x_{XB_j} + \sum_{1 \leq i \leq m_A, 1 \leq j \leq m_B} a_{A_i B_j} x_{A_i B_j} + \sum_{1 \leq i < i' \leq m_A} (a_{A_i A_{i'}} x_{A_i B_{A_i A_{i'}}} - |a_{A_i A_{i'}}| x_{A_{i'} B_{A_i A_{i'}}}) + \sum_{1 \leq j < j' \leq m_B} (a_{B_j B_{j'}} x_{A_{B_j B_{j'}} B_j} - |a_{B_j B_{j'}}| x_{A_{B_j B_{j'}} B_{j'}}) \leq a_0, \quad (4)$$

where  $m_A = n_A + \frac{n_B(n_B-1)}{2}$ ,  $m_B = n_B + \frac{n_A(n_A-1)}{2}$ . We denote (3) by  $\mathbf{a}^T \mathbf{x} \leq 0$ ,  $\mathbf{a}, \mathbf{x} \in \mathbb{R}^{\frac{(n_A+n_B)(n_A+n_B+1)}{2}}$  and (4) by  $\mathbf{a}'^T \mathbf{x}' \leq 0$ ,  $\mathbf{a}', \mathbf{x}' \in \mathbb{R}^{m_A+m_B+m_A m_B}$ , respectively.

Note that forbidden terms of the form  $x_{A_i A_{i'}}$  and  $x_{B_j B_{j'}}$  do not appear in (4).

As an example, let us see how the  $I_{3322}$  inequality is generated by triangular elimination (see Figure 2) of the pentagonal inequality (2). This inequality has two terms  $x_{A_1A_2}$  and  $x_{B_1B_2}$  which correspond to joint measurements of two observables in one subsystem and are not allowed. Therefore, we eliminate these terms by adding two new nodes  $A'_{B_1B_2}$  and  $B'_{A_1A_2}$  and adding two triangle inequalities  $-x_{A_1A_2} + x_{A_1B'_{A_1A_2}} - x_{A_2B'_{A_1A_2}} \leq 0$  and  $-x_{B_1B_2} + x_{A'_{B_1B_2}B_1} - x_{A'_{B_1B_2}B_2} \leq 0$ . If we rewrite the resulting inequality in terms of the vector  $\mathbf{q}$  instead

of the vector  $\mathbf{x}$  by using the isomorphism (1), this inequality becomes the  $I_{3322}$  inequality. As we will see in the next subsection, this gives another proof of the tightness of the  $I_{3322}$  inequality than directly checking the dimension of the face computationally.

### 2.3 Triangular elimination and facet

In this subsection, we show the main theorem of this paper: under a very mild condition, the triangular elimination of a facet is a facet.

**Theorem 2.1.** *The triangular elimination of a facet inequality  $\mathbf{a}^T \mathbf{x} \leq 0$  of  $\text{CUT}^\square(\mathbb{K}_{1+n_A+n_B})$  is facet inducing for  $\text{CUT}^\square(\mathbb{K}_{1,m_A,m_B})$  except for the cases that the inequality  $\mathbf{a}^T \mathbf{x} \leq 0$  is a triangle inequality labelled as either  $-x_{XA_1} - x_{XA_2} + x_{A_1A_2} \leq 0$  or  $-x_{A_1A_2} - x_{A_1A_3} + x_{A_2A_3} \leq 0$ .*

For example, as we saw, the CHSH inequality is the triangular elimination of Bell's original inequality, which is a triangle inequality. The  $I_{3322}$  inequality, found by Pitowsky and Svozil [19] and Collins and Gisin [7], is the triangular elimination of a pentagonal inequality.

*Proof.* Let  $r_F$  be the set of cut vectors on the hyperplane  $\mathbf{a}^T \mathbf{x}' = 0$ :  $r_F = \{\boldsymbol{\delta}'(S') \mid \mathbf{a}^T \boldsymbol{\delta}' = 0, S' : \text{cut}\}$  for  $\text{CUT}^\square(\mathbb{K}_{1,m_A,m_B})$ . We prove the theorem by exhibiting a linearly independent subset of these cut vectors with cardinality  $m_A + m_B + m_A m_B - 1$ .

In the following proof, we restrict ourselves to the case  $n_B = 1$  to avoid the existence of the term corresponding to the joint measurement of Bob's observables,  $x_{B_j B_{j'}}$ . If there exist such terms, apply the following proof in two steps, first for Alice's forbidden terms, and secondly for Bob's. In addition, we assume that  $a_{A_i A_{i'}} \leq 0$  for all eliminated terms. For the other cases, the proof is similar.

By the above restriction,  $m_A + m_B + m_A m_B - 1 = \frac{n_A(n_A-1)}{2} - 1 + \frac{n_A^2(n_A-1)}{2}$ .

A sketch of proof is as follows: first, we restrict  $r_F$  and decompose the whole space of  $\text{CUT}^\square(\mathbb{K}_{1,m_A,m_B})$  into two subspaces. For each subspace, we can pick a set of cut vectors which are linearly independent in that subspace. Next, we show that these sets of cut vectors are linearly independent in the whole space.

First, let the subset  $r'_F$  of  $r_F$  be those cuts such that, for any  $1 \leq i < i' \leq n_A$ , two vertices  $A_{i'}$  and  $B_{A_i A_{i'}}$  are assigned same value. Then, consider the intersection of the space spanned by  $\boldsymbol{\delta}'(S') \in r'_F$  and the subspace

$$W = \left\{ (x_{XA_i}, x_{XB_j}, x_{A_i B_j}, x_{A_i B_{A_i A_{i'}}})_{1 \leq i < i' \leq n_A, 1 \leq j \leq n_B}^T \right\}.$$

From the definition of  $r'_F$ ,  $\delta'_{A_{i'} B_{A_i A_{i'}}}(S') = 0$ . Therefore,

$$\begin{aligned} \mathbf{a}^T \boldsymbol{\delta}'(S') &= \sum_{1 \leq i \leq n_A} a_{XA_i} \delta_{XA_i}(S') + \sum_{1 \leq j \leq n_B} a_{XB_j} \delta_{XB_j}(S') \\ &\quad + \sum_{1 \leq i \leq n_A, 1 \leq j \leq n_B} a_{A_i B_j} \delta_{A_i B_j}(S') + \sum_{1 \leq i < i' \leq n_A} a_{A_i A_{i'}} \delta'_{A_i B_{A_i A_{i'}}}(S') = 0. \end{aligned}$$

This means that the intersection of space spanned by  $\boldsymbol{\delta}'(S') \in r'_F$  and  $W$  is equivalent to the space spanned by the cut vectors  $r_f = \{\boldsymbol{\delta}(S) \mid \mathbf{a}^T \boldsymbol{\delta} = 0, S : \text{cut}\}$  of  $\text{CUT}^\square(\mathbb{K}_{1+n_A+n_B})$ . Therefore, from the assumption that the inequality  $\mathbf{a}^T \mathbf{x} \leq 0$  is facet supporting, we can pick  $\frac{n_A(n_A-1)}{2} - 1$  linearly independent cut vectors and transform the cut vectors of  $\text{CUT}^\square(\mathbb{K}_{1+n_A+n_B})$  into corresponding cut vectors of  $\text{CUT}^\square(\mathbb{K}_{1,m_A,m_B})$ . Let this set of linearly independent cut vectors be  $D_0$ .

The remaining subspace of  $\text{CUT}^\square(\mathbb{K}_{1,m_A,m_B})$  is

$$V = \bigoplus_{i < i'} V_{A_i A_{i'}} = \bigoplus_{i < i'} \left\{ (x_{XB_{A_i A_{i'}}}, x_{A_{i'} B_{A_i A_{i'}}}, x_{A_{i''} B_{A_i A_{i'}}})_{i'' \neq i, i'}^T \right\}$$

for each eliminated term  $A_i A_{i'}$ ,  $1 \leq i < i' \leq n_A$ .

Instead of  $V$ , we consider the space

$$V' = \bigoplus_{i < i'} V'_{A_i A_{i'}} = \bigoplus_{i < i'} \left\{ (x_{XB_{A_i A_{i'}}} - x_{A_{i'} B_{A_i A_{i'}}}, x_{A_{i'} B_{A_i A_{i'}}}, x_{\alpha_{A_i, A_{i'}, A_{i''}}})_{i'' \neq i, i'}^T \right\}$$



where

$$x_{\alpha_{A_i, A_{i'}, A_{i''}}} = \begin{cases} \frac{1}{2}(x_{A_{i''} B_{A_i A_{i'}}} - x_{A_{i'} B_{A_i A_{i''}}} - x_{A_{i'} B_{A_i A_{i'}}} + 3x_{A_{i''} B_{A_i A_{i''}}}) & (i' < i'') \\ \frac{1}{2}(x_{A_{i''} B_{A_i A_{i'}}} - x_{A_{i''} B_{A_i A_{i'}}} - x_{A_{i'} B_{A_i A_{i'}}} - x_{A_{i'} B_{A_i A_{i'}}}) & (i'' < i') \end{cases}$$

in the following. Since the transform  $V$  to  $V'$  is linear, the linear independence of vectors in  $V$  is equivalent to that in  $V'$ .

Then, we consider the subset  $r''_{F, A_i A_{i'}}$  of  $r_F$  for each  $A_i A_{i'}$  restricted as follows:  $A_{i'}$  must be assigned 0 and both  $B_{A_i A_{i'}}$  and  $A_i$  must be assigned 1. For other terms  $A_{i''} A_{i'''} (1 \leq i'' < i''' \leq n_A)$ , vertices  $A_{i''''}$  and  $B_{A_{i''} A_{i'''}}$  must be assigned the same value. From that restriction, the equations

$$\begin{aligned} \delta'_{X B_{A_i A_{i'}}}(S'') - \delta'_{A_{i'} B_{A_i A_{i'}}}(S'') &= -\delta_{X A_{i'}}(S), \\ \delta'_{A_{i'} B_{A_i A_{i'}}}(S'') &= 1, \\ \delta'_{\alpha_{A_i, A_{i'}, A_{i''}}}(S'') &= -\delta_{A_{i''} A_{i'}}(S) \end{aligned}$$

hold for  $\delta'(S'') \in r''_{F, A_i A_{i'}}$ . This means that the intersection of the space spanned by  $\delta'(S'')$  and the subspace  $V'_{A_i A_{i'}}$  is equivalent to that of the space spanned by  $\delta(S) \in r_f$  and the subspace

$$U_{A_i A_{i'}} = \left\{ (x_{X A_{i'}}, 1, x_{A_{i''} A_{i'}})_{i'' \neq i, i'}^T \right\}.$$

Now, because  $r_f$  is on the hyperplane  $\mathbf{a}^T \mathbf{x} = 0$ , the above intersection has dimension  $n_A$  or  $n_A - 1$ . However, from the condition on the inequality  $\mathbf{a}^T \mathbf{x} \leq 0$ , the space spanned by  $r_f$  is not parallel to  $U_{A_i A_{i'}}$ . Therefore, the dimension is  $n_A$  and we can extract  $n_A$  cut vectors which are linearly independent in the subspace  $V'$  using the cut vectors from  $r_f$ . Let this set of cut vectors be  $D_{A_i A_{i'}}$ .

Finally, we show that  $D_0 \cup D_{A_i A_{i'}}$  is a linearly independent set of cut vectors. Suppose that the linear combination

$$\sum_{\delta'^T(S') \in D_0} \kappa_{S'} \delta'^T(S') + \sum_{1 \leq i < i' \leq n_A} \sum_{\delta'^T(S'') \in D_{A_i A_{i'}}} \lambda_{S''}^{A_i A_{i'}} \delta'^T(S'') = 0$$

holds. Consider the subspace  $V'_{A_i A_{i'}}$  of the above linear combination. From the construction, for  $D_0$  and  $D_{A_{i''} A_{i'''}}$ , the elements of cut vectors in that subspace are all zero. Therefore, for the linear combination to hold, it must be that  $\sum_{\delta'^T(S'') \in D_{A_i A_{i'}}} \lambda_{S''}^{A_i A_{i'}} \delta'^T(S'') = 0$ . However, the linear independence of  $D_{A_i A_{i'}}$  means that the coefficients are all zero. By repeating this argument, we can conclude that the coefficient  $\lambda_{S''}^{A_i A_{i'}}$  must be zero. So, from the linear independence of  $D_0$ , the coefficients  $\kappa_{S'}$  are also zero.  $\square$

## 2.4 Triangular elimination and symmetry

Many Bell inequalities are equivalent to each other due to the arbitrariness in the labelling of the party, observable and value identifiers. This corresponds to symmetries of the underlying polytope. We consider ways of representing nonequivalent Bell inequalities in this section.

The nonequivalence of Bell inequalities can be translated into two questions about facet inequalities  $f$  and  $f'$  of a given cut polytope of a complete graph, and their triangular eliminations  $F$  and  $F'$ , respectively:

1. does the equivalence of  $f$  and  $f'$  imply the equivalence of  $F$  and  $F'$ ?
2. does the equivalence of  $F$  and  $F'$  imply the equivalence of  $f$  and  $f'$ ?

The answers are both affirmative if we define equivalence appropriately, so equivalence before triangular elimination is logically equivalent to equivalence after triangular elimination. This means that, for example, to enumerate the nonequivalent Bell inequalities, we need only enumerate the facet inequalities of the cut polytope of the complete graph up to symmetry by party, observable and value exchange.

In  $\text{CUT}^\square(K_{1, m_A, m_B})$ , the relabelling of all vertices of Alice to that of Bob and vice versa corresponds to a party exchange. On the other hand, the local relabelling of some vertices of Alice (or Bob) corresponds to an observable exchange. Thus by the observable exchange of Alice represented by the permutation  $\sigma$  over  $\{A_1, \dots, A_{m_A}\}$ , an inequality  $\mathbf{a}^T \mathbf{x} \leq a_0$  is transformed into  $\mathbf{a}'^T \mathbf{x} \leq a_0$  where  $a'_{\sigma(A_i)V} = a_{A_i V}$  for any vertex  $V$ .

In addition, there is an operation which corresponds to a value exchange of some observables, called a *switching* in the theory of cut polytopes. By the switching corresponding to the value exchange of an Alice's observable  $A_{i_0}$ , an inequality  $\mathbf{a}^T \mathbf{x} \leq a_0$  is transformed into  $\mathbf{a}'^T \mathbf{x} \leq a_0 - \sum_V a_{A_{i_0}V}$  where  $a'_{A_{i_0}V} = -a_{A_{i_0}V}$ , and  $a'_{A_iV} = a_{A_iV}$  for any  $i \neq i_0$  and any vertex  $V \neq A_{i_0}$  (definitions for Bob's exchange are similar).

It is well known, and easily shown, that by repeated application of the switching operation we may reduce the right hand side of any facet inequality to zero.

Let  $n_A \leq n_B$  and  $n = 1 + n_A + n_B$ . Let  $f$  and  $f'$  be facets of  $\text{CUT}_n^\square$  where the  $n$  nodes of  $K_n$  is labelled by  $V = \{A_1, \dots, A_{n_A}, B_1, \dots, B_{n_B}, X\}$ . The two facets  $f$  and  $f'$  are said to be *equivalent* and denoted  $f \sim f'$  if  $f$  can be transformed to  $f'$  by applying zero or more of the following operations: (1) (only applicable in the case  $n_A = n_B$ ) swapping labels of nodes  $A_i$  and  $B_i$  for all  $1 \leq i \leq n_A$ , (2) relabelling the nodes within  $A_1, \dots, A_{n_A}$ , (3) relabelling the nodes within  $B_1, \dots, B_{n_B}$ , and (4) switching.<sup>1</sup>

Two facets  $F$  and  $F'$  of  $\text{CUT}^\square(K_{1,m_A,m_B})$  are said to be *equivalent* and denoted  $F \sim F'$  if  $F$  can be transformed to  $F'$  by applying permutation which fixes node  $X$ , switching, or both. This notion of equivalence of facets of  $\text{CUT}^\square(K_{1,m_A,m_B})$  corresponds to equivalence of tight Bell inequalities up to party, observable and value exchange.

**Theorem 2.2.** *Let the triangular elimination of facet inequalities  $f$  and  $f'$  be  $F$  and  $F'$ , respectively. Then,  $f \sim f' \iff F \sim F'$ .*

*Proof.* A sketch of the proof is as follows. Since the permutation and switching operations are commutative, it is sufficient to prove the proposition under each operation separately. Because the  $\Rightarrow$  direction is straightforward for both permutation and switching, we concentrate on the proof of  $\Leftarrow$  direction.

First, consider switching. Suppose  $F$  is obtained from a switching of  $F'$ . The switching could involve either (i) a new observable introduced by the triangular elimination, or (ii) an observable which had a joint measurement term eliminated. Since a switching of type (i) has no effect on  $f$  and  $f'$ , we need only consider type (ii). We can view the triangular elimination of the term  $A_i A_{i'}$  as addition of triangle inequality  $x_{A_i A_{i'}} - x_{A_i B_{A_i A_{i'}}} - x_{A_{i'} B_{A_i A_{i'}}} \leq 0$  or its switching equivalent inequality  $-x_{A_i A_{i'}} - x_{A_i B_{A_i A_{i'}}} + x_{A_{i'} B_{A_i A_{i'}}} \leq 0$  according to the sign of the coefficient  $a_{A_i A_{i'}}$ . Thus, if  $F$  is switching of  $F'$  of vertices  $A_i$  and  $B_{A_i A_{i'}}$  then  $f$  is switching of  $f'$  of  $A_i$ .

Next, consider the permutation corresponding to an observable exchange. Observe that for any vertex  $A_i (1 \leq i \leq n_A)$ , triangular elimination does not change the number of terms  $A_i V$  with non-zero coefficient. In addition, it can be shown that for any facet inequality  $f$  of the cut polytope of the complete graph other than the triangle inequality, there is no vertex satisfying the following conditions: (a) there are exactly two terms  $A_i V$ , with non-zero coefficients, and (b) for those non-zero coefficients  $a_{A_i W}$  and  $a_{A_i U}$ ,  $|a_{A_i W}| = |a_{A_i U}|$  [2]. This means that if  $F \sim F'$ , then the corresponding permutation  $\sigma$  is always in the following form: for permutations  $\tau_A$  over  $\{A_1, \dots, A_{n_A}\}$  and  $\tau_B$  over  $\{B_1, \dots, B_{n_B}\}$ ,  $\sigma(A_i) = \tau_A(A_i)$  and  $\sigma(B_{A_i A_{i'}}) = B_{\tau_A(A_i)\tau_A(A_{i'})}$ . The situation is the same for Bob.

Therefore,  $f$  and  $f'$  are equivalent under the permutations  $\tau_A$  and  $\tau_B$ . □

## 2.5 Computational results

By Theorem 2.2, we can compute the number of the classes of facets of  $\text{CUT}^\square(K_{1,m_A,m_B})$  of the same type obtained by applying triangular elimination to non-triangular facets of  $\text{CUT}_n^\square$ . We consulted De Simone, Deza and Laurent [10] for the H-representation of  $\text{CUT}_7$ , and the ‘‘conjectured complete description’’ of  $\text{CUT}_8$  and the ‘‘description possibly complete’’ of  $\text{CUT}_9$  in SMAPO [20]. The result is summarized in Table 1. A program is available on request to generate these Bell inequalities from the list in [20]. For  $n = 8$  and 9, the number is a lower bound since the known list of the facets of  $\text{CUT}_n^\square$  is not proved to be complete.

## 3 Families of Bell inequalities

While a large list of individual tight Bell inequalities is useful in some applications, a few formulas which give many different Bell inequalities for different values of parameters are easier to treat theoretically. The cut polytope of the complete graph has several classes of valid inequalities whose subclasses of facet-inducing

<sup>1</sup>The two facets  $f$  and  $f'$  are said to be *equivalent* and denoted  $f \sim f'$  if  $f$  can be transformed to  $f'$  by permutation and switching where the permutation  $\tau$  on  $V$  satisfies: (1)  $\tau(X) = X$  and (2)  $\tau$  either fixes two sets  $\{A_1, \dots, A_{n_A}\}$  and  $\{B_1, \dots, B_{n_B}\}$  setwise or (in the case  $n_A = n_B$ ) swaps these two sets.



Table 1: The number of inequivalent facets of  $\text{CUT}_n^\square$  and the number of inequivalent tight Bell inequalities obtained as the triangular eliminations of the facets of  $\text{CUT}_n^\square$ . Asterisk (\*) indicates the value is a lower bound.

$n$	Facets of $\text{CUT}_n^\square$	Tight Bell ineqs. via triangular elimination
3	1	2
4	1	2
5	2	8
6	3	22
7	11	323
8	147*	40,399*
9	164,506*	201,374,783*

inequalities are partially known (see [11, Chapters 27–30] for details). In this section, we apply triangular elimination to two typical examples of such classes to obtain two general formulas for Bell inequalities. In addition, we prove sufficient conditions for these formulas to give a tight Bell inequality.

In this section, terms of the left hand side of an inequality are arrayed in the format introduced by Collins and Gisin [7]; each row corresponds to coefficients of each observable of party  $A$  and each column corresponds to that of party  $B$ . Because of switching equivalence, we can assume that the right hand side of inequality are always zero. The example of the CHSH  $-q_{A_1} - q_{B_1} + q_{A_1 B_1} + q_{A_1 B_2} + q_{A_2 B_1} - q_{A_2 B_2} \leq 0$  is arrayed in the form as follows:

$$\left( \begin{array}{c|cc} & -1 & 0 \\ \hline -1 & 1 & 1 \\ 0 & 1 & -1 \end{array} \right) \leq 0.$$

### 3.1 Bell inequalities derived from hypermetric inequalities

*Hypermetric inequalities* are a fundamental class of inequalities valid for the cut polytope of the complete graph. Here we derive a new family of Bell inequalities by applying triangular elimination to the hypermetric inequalities. A special case of this family, namely the triangular eliminated pure hypermetric inequality, contains four previously known Bell inequalities: the trivial inequalities like  $q_{A_1} \leq 1$ , the well known CHSH inequality found by Clauser, Horne, Shimony and Holt [5], the inequality named  $I_{3322}$  by Collins and Gisin [7], originally found by Pitowsky and Svozil [19], and the  $I_{3422}^2$  inequality by Collins and Gisin [7].

Let  $s$  and  $t$  be nonnegative integers and  $b_{A_1}, \dots, b_{A_s}, b_{B_1}, \dots, b_{B_t}$  be integers. We define  $b_X = 1 - \sum_{i=1}^s b_{A_i} - \sum_{j=1}^t b_{B_j}$ . Then it is known that  $\sum_{uv} b_u b_v x_{uv} \leq 0$ , where the sum is taken over the  $\binom{s+t+1}{2}$  edges of the complete graph on nodes  $X, A_1, \dots, A_s, B_1, \dots, B_t$ , is valid for  $\text{CUT}_{s+t+1}^\square$ . This inequality is called the *hypermetric inequality* defined by the weight vector  $\mathbf{b} = (b_X, b_{A_1}, \dots, b_{A_s}, b_{B_1}, \dots, b_{B_t})$ .

We apply triangular elimination to this hypermetric inequality. Let  $s_+$  and  $t_+$  be the number of positive entries of the form  $b_{A_i}$  and of the form  $b_{B_j}$ , respectively. Without loss of generality, we assume that  $b_{A_1}, \dots, b_{A_{s_+}}, b_{B_1}, \dots, b_{B_{t_+}} > 0$ , and  $b_{A_{s_++1}}, \dots, b_{A_s}, b_{B_{t_++1}}, \dots, b_{B_t} \leq 0$ . By assigning  $a_{uv} = b_u b_v$  in the formula (4), the Bell inequality obtained by triangular elimination is:

$$\begin{aligned} & \sum_{i=1}^{s_+} b_{A_i} \left( \frac{1-b_{A_i}}{2} - \sum_{i'=1}^{i-1} b_{A_{i'}} \right) q_{A_i} + \sum_{i=s_++1}^s b_{A_i} \left( \frac{1-b_{A_i}}{2} - \sum_{i'=s_++1}^{i-1} b_{A_{i'}} \right) q_{A_i} \\ & + \sum_{j=1}^{t_+} b_{B_j} \left( \frac{1-b_{B_j}}{2} - \sum_{j'=1}^{j-1} b_{B_{j'}} \right) q_{B_j} + \sum_{j=t_++1}^t b_{B_j} \left( \frac{1-b_{B_j}}{2} - \sum_{j'=t_++1}^{j-1} b_{B_{j'}} \right) q_{B_j} \\ & + \sum_{j=1}^{t_+} \sum_{j'=t_++1}^t b_{B_j} b_{B_{j'}} q_{A_{j'} B_{j'}} + \sum_{i=1}^{s_+} \sum_{i'=s_++1}^s b_{A_i} b_{A_{i'}} q_{B_{i'} A_{i'}} - \sum_{i=1}^s \sum_{j=1}^t b_{A_i} b_{B_j} q_{A_i B_j} \\ & - \sum_{1 \leq i < i' \leq s} b_{A_i} b_{A_{i'}} q_{A_i B_{i'}} + \sum_{1 \leq i < i' \leq s} |b_{A_i} b_{A_{i'}}| q_{A_{i'} B_{i'}} - \sum_{1 \leq j < j' \leq t} b_{B_j} b_{B_{j'}} q_{A_{j'} B_j} + \sum_{1 \leq j < j' \leq t} |b_{B_j} b_{B_{j'}}| q_{A_{j'} B_{j'}} \leq 0. \end{aligned} \tag{5}$$

Though the formula (5) represents a Bell inequality for any choice of weight vector  $\mathbf{b}$ , this Bell inequality is not always tight. Many sufficient conditions for a hypermetric inequality to be facet-inducing are known in study of cut polytopes. By Theorem 2.1, these sufficient conditions give sufficient conditions for the Bell inequality (5) to be tight. The sufficient conditions stated in [11, Corollary 27.2.5] give the following theorem.

**Theorem 3.1.** *The Bell inequality (5) is tight if one of the following conditions is satisfied.*

- (i) *For some  $l > 1$ , the integers  $b_{A_1}, \dots, b_{A_s}, b_{B_1}, \dots, b_{B_t}$  and  $b_X$  contain  $l+1$  entries equal to 1 and  $l$  entries equal to  $-1$ , and the other entries (if any) are equal to 0.*
- (ii) *At least 3 and at most  $n-3$  entries in  $b_{A_1}, \dots, b_{A_s}, b_{B_1}, \dots, b_{B_t}$  and  $b_X$  are positive, and all the other entries are equal to  $-1$ .*

Now we consider some concrete cases when the formula (5) represents a tight Bell inequality. If we let  $s+t=2l$ ,  $s \leq l$ ,  $l > 1$ ,  $b_{A_1} = \dots = b_{A_s} = b_{B_1} = \dots = b_{B_{l-s}} = 1$ , and  $b_{B_{l-s+1}} = \dots = b_{B_t} = -1$ , then  $b_X = 1$  and by case (i) of Theorem 3.1, the Bell inequality (5) is tight. In this case, the Bell inequality (5) is in the following form.

$$\begin{aligned}
& - \sum_{i=1}^s (i-1)q_{A_i} - \sum_{j=1}^{l-s} \sum_{j'=l-s+1}^t q_{A'_{jj'}} - \sum_{j=1}^{l-s} (j-1)q_{B_j} - \sum_{j=l-s+1}^t (j-(l-s))q_{B_j} \\
& \quad - \sum_{i=1}^s \sum_{j=1}^{l-s} q_{A_i B_j} + \sum_{i=1}^s \sum_{j=l-s+1}^t q_{A_i B_j} - \sum_{1 \leq i < i' \leq s} q_{A_i B'_{i'}} + \sum_{1 \leq i < i' \leq s} q_{A_{i'} B'_{ii'}} \\
& \quad - \sum_{1 \leq j < j' \leq l-s} q_{A'_{jj'} B_j} - \sum_{l-s+1 \leq j < j' \leq t} q_{A'_{jj'} B_j} + \sum_{j=1}^{l-s} \sum_{j'=l-s+1}^t q_{A'_{jj'} B_j} + \sum_{1 \leq j < j' \leq t} q_{A'_{jj'} B_{j'}} \leq 0. \quad (6)
\end{aligned}$$

Examples of tight Bell inequality in the form (6) are  $I_{3322}$  and  $I_{3422}^2$  inequalities [7].

In case of  $l=1$ , Theorem 3.1 does not guarantee that the Bell inequality (6) is tight. However, in cases of  $(l, s, t) = (1, 1, 1)$  and  $(1, 1, 2)$ , the Bell inequality (6) becomes trivial and CHSH inequalities, respectively, both of which are tight.

Letting  $(l, s, t) = (2, 2, 2)$  in (6) gives:

$$-q_{A_2} - q_{B_1} - 2q_{B_2} + q_{A_1 B_1} + q_{A_1 B_2} + q_{A_2 B_1} + q_{A_2 B_2} - q_{A_1 B'_{12}} + q_{A_2 B'_{12}} - q_{A'_{12} B_1} + q_{A'_{12} B_2} \leq 0. \quad (7)$$

Following the notation in [7], we write the inequality (7) by arraying its coefficients:

$$\left( \begin{array}{c|ccc} & (A_2) & (A_1) & (A'_{12}) \\ \hline & -1 & 0 & 0 \\ \hline (B_2) & -2 & 1 & 1 & 1 \\ (B_1) & -1 & 1 & 1 & -1 \\ (B'_{12}) & 0 & 1 & -1 & 0 \end{array} \right) \leq 0. \quad (7')$$

Now it is clear that the Bell inequality (7) is  $I_{3322}$  inequality.

Letting  $(l, s, t) = (2, 1, 3)$  in (6) gives:

$$\left( \begin{array}{c|ccc} & (B_2) & (B_3) & (B_1) \\ \hline & -1 & -2 & 0 \\ \hline (A_1) & 0 & 1 & 1 & -1 \\ (A'_{13}) & -1 & 0 & 1 & 1 \\ (A'_{12}) & -1 & 1 & 0 & 1 \\ (A'_{23}) & 0 & -1 & 1 & 0 \end{array} \right) \leq 0. \quad (8)$$

After exchanging the two values 1 and 0 of the observable  $A_1$ , and doing the same to the two values of the observable  $B_3$ , the Bell inequality (8) becomes:

$$\left( \begin{array}{c|ccc} & (B_2) & (\overline{B_3}) & (B_1) \\ \hline & 0 & 1 & -1 \\ \hline (\overline{A_1}) & -1 & -1 & 1 & 1 \\ (A'_{13}) & 0 & 0 & -1 & 1 \\ (A'_{12}) & -1 & 1 & 0 & 1 \\ (A'_{23}) & 1 & -1 & -1 & 0 \end{array} \right) \leq 1, \quad (8')$$

which is  $I_{3422}^2$  inequality [7]. This means that the Bell inequality (8) is equivalent to  $I_{3422}^2$  inequality.

### 3.2 Bell inequalities derived from pure clique-web inequalities

Clique-web inequalities [11, Chapter 29] are generalization of hypermetric inequalities. One of the important subclasses of clique-web inequalities are the pure clique-web inequalities, which are always facet-inducing. Here we introduce an example of Bell inequalities derived from some pure clique-web inequalities.

For nonnegative integers  $s$ ,  $t$  and  $r$  with  $s \geq t \geq 2$  and  $s - t = 2r$ , we consider the pure clique-web inequality with parameters  $n = s + t + 1$ ,  $p = s + 1$ ,  $q = t$  and  $r$ . After relabelling the  $n$  vertices of  $K_n$  by  $A_1, \dots, A_s, X, B_1, \dots, B_t$  in this order, the Bell inequality (4) corresponding to the clique-web inequality is:

$$\begin{aligned}
& - \sum_{i=r+1}^{s-r} (i-r-1)q_{A_i} - 2t \sum_{i=s-r+1}^s q_{A_i} - \sum_{j=1}^t (j-r)q_{B_j} \\
& + \sum_{i=1}^s \sum_{j=1}^t q_{A_i B_j} + \sum_{\substack{1 \leq i < i' \leq s \\ r+1 \leq j < j' \leq t}} (-q_{A_i B_{i'}} + q_{A_{i'} B_{i'}}) + \sum_{1 \leq j < j' \leq t} (-q_{A'_{jj'} B_j} + q_{A'_{jj'} B_{j'}}) \leq 0. \quad (9)
\end{aligned}$$

The next theorem is a direct consequence of Theorem 2.1.

**Theorem 3.2.** *For any nonnegative integers  $s$ ,  $t$  and  $r$  with  $s \geq t \geq 2$  and  $s - t = 2r$ , the Bell inequality (9) is tight.*

### 3.3 Inclusion relation

Collins and Gisin [7] pointed out that the following  $I_{3322}$  inequality becomes the CHSH inequality if we fix two measurements  $A_3$  and  $B_1$  to a deterministic measurement whose result is always 0.

$$\begin{aligned}
I_{3322}: & \left( \begin{array}{cc|ccc} & & (A_1) & (A_2) & (A_3) \\ & & -1 & 0 & 0 \\ \hline (B_1) & -2 & 1 & 1 & 1 \\ (B_2) & -1 & 1 & 1 & -1 \\ (B_3) & 0 & 1 & -1 & 0 \end{array} \right) \leq 0, \\
\text{CHSH:} & \left( \begin{array}{cc|cc} & & (A_1) & (A_2) \\ & & -1 & 0 \\ \hline (B_2) & -1 & 1 & 1 \\ (B_3) & 0 & 1 & -1 \end{array} \right) \leq 0.
\end{aligned}$$

As stated in [7], this fact implies the CHSH inequality is irrelevant if the  $I_{3322}$  inequality is given. In other words, if a quantum state satisfies the  $I_{3322}$  inequality with every set of measurements, then it also satisfies the CHSH inequality with every set of measurements.

We generalize this argument and define *inclusion relation* between two Bell inequalities: A Bell inequality  $\mathbf{a}^T \mathbf{q} \leq 0$  includes another Bell inequality  $\mathbf{b}^T \mathbf{q} \leq 0$  if we can obtain the inequality  $\mathbf{b}^T \mathbf{q} \leq 0$  by fixing some measurements in the inequality  $\mathbf{a}^T \mathbf{q} \leq 0$  to deterministic ones.

We do not know whether all the Bell inequalities (except positive probability) include the CHSH inequality. However, we can prove that many Bell inequalities represented by (5) or (9) include the CHSH inequality.

**Theorem 3.3.** *If  $b_{A_1} = b_{A_2} = 1$  and  $b_{B_{t+1}} = -1$ , then the Bell inequality represented by (5) contains the CHSH inequality.*

*Proof.* The Bell inequality (5) contains  $s + \binom{t}{2}$  observables of Alice and  $t + \binom{s}{2}$  observables of Bob. By fixing all but 4 observables  $A_1$ ,  $A_2$ ,  $B_{t+1}$  and  $B'_{12}$  to the one whose value is always 0, we obtain the following CHSH inequality:  $-q_{A_2} - q_{B_{t+1}} + q_{A_1 B_{t+1}} + q_{A_2 B_{t+1}} - q_{A_1 B'_{12}} + q_{A_2 B'_{12}} \leq 0$ .  $\square$

**Theorem 3.4.** *All the Bell inequalities in the form (9) include the CHSH inequality.*

*Proof.* By fixing all but 4 observables  $A_{r+1}$ ,  $A_{r+2}$ ,  $B_{r+1}$  and  $B'_{r+1, r+2}$  to the one whose value is always 0, the Bell inequality (9) becomes the following CHSH inequality:  $-q_{A_{r+2}} - q_{B_{r+1}} + q_{A_{r+1} B_{r+1}} + q_{A_{r+2} B_{r+1}} - q_{A_{r+1} B'_{r+1, r+2}} + q_{A_{r+2} B'_{r+1, r+2}} \leq 0$ .  $\square$

### 3.4 Relationship between $I_{mm22}$ and triangular eliminated Bell inequality

Collins and Gisin [7] proposed a family of tight Bell inequalities obtained by the extension of CHSH and  $I_{3322}$  as  $I_{mm22}$  family, and conjectured that  $I_{mm22}$  is always facet supporting (they also confirmed that for  $m \leq 7$ ,  $I_{mm22}$  is actually facet supporting by computation). Therefore, whether their  $I_{mm22}$  can be obtained by triangular elimination of some facet class of  $\text{CUT}^\square(\mathbb{K}_n)$  is an interesting question.

The  $I_{mm22}$  family has the structure as follows:

$$\left( \begin{array}{c|cccccc} & -1 & 0 & \cdots & 0 & 0 & 0 \\ \hline -(m-1) & 1 & 1 & \cdots & 1 & 1 & 1 \\ -(m-2) & 1 & 1 & \cdots & 1 & 1 & -1 \\ -(m-3) & 1 & 1 & \cdots & 1 & -1 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ -1 & 1 & 1 & -1 & 0 & \cdots & 0 \\ 0 & 1 & -1 & 0 & 0 & \cdots & 0 \end{array} \right) \leq 0. \quad (10)$$

From its structure, it is straightforward that if  $I_{mm22}$  can be obtained by triangular elimination of some facet class of  $\text{CUT}_n^\square$ , then only  $A_m$  and  $B_m$  are new vertices introduced by triangular elimination, since the other vertices have degree more than 2. For  $m = 2, 3, 4$ , the  $I_{mm22}$  inequality is the triangular elimination of the triangle, pentagon and Grishukhin inequality  $\sum_{1 \leq i < j \leq 4} x_{ij} + x_{56} + x_{57} - x_{67} - x_{16} - x_{36} - x_{27} - x_{47} - 2 \sum_{1 \leq i \leq 4} x_{i5} \leq 0$ , respectively. In general,  $I_{mm22}$  inequality is the triangular elimination of a facet-inducing inequality of  $\text{CUT}_{2m-1}^\square$  and it is tight [1].

### 3.5 Known tight Bell inequalities other than the triangular elimination of $\text{CUT}^\square(\mathbb{K}_n)$

Since we have obtained a large number of tight Bell inequalities by triangular elimination of  $\text{CUT}^\square(\mathbb{K}_n)$ , the next question is whether they are complete i.e., whether all families and their equivalents form the whole set of facets of  $\text{CUT}^\square(\mathbb{K}_{1,m_A,m_B})$ .

For the case  $m_A = m_B = 3$ , the answer is affirmative. Both Śliwa [21] and Collins and Gisin [7] showed that there are only three kinds of inequivalent facets: positive probabilities, CHSH and  $I_{3322}$ , corresponding to the triangle facet, the triangular elimination of the triangle facet and the triangular elimination of the pentagonal facet of  $\text{CUT}^\square(\mathbb{K}_n)$ , respectively.

On the other hand, in the case  $m_A = 3$  and  $m_B = 4$ , the answer is negative. Collins and Gisin enumerated all of the tight Bell inequalities and classified them into 6 families of equivalent inequalities [7]. While positive probabilities, CHSH,  $I_{3322}$  and  $I_{3422}^2$  inequalities are either facets of  $\text{CUT}^\square(\mathbb{K}_n)$  or their triangular eliminations, the other two are not:

$$I_{3422}^1 = \left( \begin{array}{c|ccc} & 1 & 1 & -2 \\ \hline 1 & -1 & -1 & 1 \\ 0 & -1 & 1 & 1 \\ 0 & 1 & -1 & 1 \\ 1 & -1 & -1 & -1 \end{array} \right) \leq 2, \quad I_{3422}^3 = \left( \begin{array}{c|ccc} & 1 & 0 & -1 \\ \hline 0 & -2 & 1 & 1 \\ 0 & 0 & -1 & 1 \\ -1 & 1 & 1 & 1 \\ 2 & -1 & -1 & -1 \end{array} \right) \leq 2. \quad (11)$$

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