

Deriving Tight Bell Inequalities for 2 Parties with Many 2-valued Observables from Facets of Cut Polytopes

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Abstract

Relatively few families of Bell inequalities have previously been identified. Some examples are the trivial, CHSH, I_{mm22} , and CGLMP inequalities. This paper presents a large number of new families of tight Bell inequalities for the case of many observables. For example, 44,368,793 inequivalent tight Bell inequalities other than CHSH are obtained for the case of 2 parties each with 10 2-valued observables. This is accomplished by first establishing a relationship between the Bell inequalities and the facets of the cut polytope, a well studied object in polyhedral combinatorics. We then prove a theorem allowing us to derive new facets of cut polytopes from facets of smaller polytopes by a process derived from Fourier-Motzkin elimination, which we call triangular elimination. These new facets in turn give new tight Bell inequalities. We give additional results for projections, liftings, and the complexity of membership testing for the associated Bell polytope.

1 Introduction

Quantum nonlocality and Bell inequalities. The power of quantum information theory over the classical one, such as unconditionally secure secret communication, is based on a clever use of the quantum nonlocality of states. To explore what is possible in quantum information theory, it is important to distinguish the quantum states which have nonlocality from those which do not.

A quantum state has nonlocality if it produces a non-classical correlation table as a result of the correlation experiment when each party is given some set of observables. This leads to the significance of the problem of testing whether a given correlation table in a setting with n parties each of which has m v -valued observables is classical or not.

A linear inequality is called a *Bell inequality* if it is satisfied by all the classical correlation tables. Bell inequalities are used to test a correlation table because each of them is a necessary condition for a correlation table to be classical.

Bell polytopes. Peres [14] showed that for $n, m, v \geq 1$, all the classical correlation tables in the n -party m -observable v -value setting form a convex polytope, which we call the *Bell polytope* $\mathcal{B}^\square(n, m, v)$. An inequality is a Bell inequality if and only if it is valid for $\mathcal{B}^\square(n, m, v)$.

The membership test for $\mathcal{B}^\square(n, m, v)$ corresponds to the problem of testing whether a given correlation table is classical or not. There is evidence suggesting that the membership test for $\mathcal{B}^\square(n, m, v)$ is computationally intractable [16]. Nevertheless a knowledge of valid inequalities for $\mathcal{B}^\square(n, m, v)$ allows us to demonstrate *non-membership* efficiently: it is sufficient to give a single violated Bell inequality to show that the corresponding correlation table exhibits non-classical behavior. Among valid inequalities, those which support facets of $\mathcal{B}^\square(n, m, v)$ are the most useful because all other valid inequalities can be derived from them. A Bell inequality is said to be *tight* if and only if it supports a facet of $\mathcal{B}^\square(n, m, v)$.

Historically, Clauser, Horne, Shimony and Holt [3] introduced an inequality valid for $\mathcal{B}^\square(2, 2, 2)$, which is known as the CHSH inequality. Fine [9] proved that the trivial inequalities and inequalities equivalent to the CHSH inequality together form the complete list of facets of $\mathcal{B}^\square(2, 2, 2)$. Because computing the facets of a high-dimensional polytope from its vertices is a very difficult problem, the complete list of facets of $\mathcal{B}^\square(n, m, v)$ is currently known only for small n , m and v , namely for $(n, m, v) = (2, 2, 2)$ [9] and for $(n, m, v) = (2, 3, 2), (3, 2, 2)$ [17].

Known Bell inequalities. The difficulty of finding a complete facet characterization for $\mathcal{B}^\square(n, m, v)$ opens two directions of study. One is to find some, instead of all, of the facets of $\mathcal{B}^\square(n, m, v)$. In this direction, Collins and Gisin [4] show a family I_{mm22} of inequalities valid for $\mathcal{B}^\square(2, m, 2)$ for general m , which are confirmed to support a facet for $m \leq 7$. Masanes [13] shows that the CGLMP inequalities [5] valid for $\mathcal{B}^\square(2, 2, v)$ actually support facets of $\mathcal{B}^\square(2, 2, v)$.

The other direction is to study the complete list of facets of an affinely projected image of $\mathcal{B}^\square(n, m, v)$ and lift them to a valid inequalities for $\mathcal{B}^\square(n, m, v)$. Werner and Wolf [21] and Śliwa [19] both consider the polytope formed by the full correlation functions, which is viewed as a kind of projection (see Section 7.4).

Correlation polytopes and cut polytopes. In [16], Pitowsky introduced correlation polytopes as the set of possible joint correlations of events in a probabilistic space, and showed the equivalence of $\mathcal{B}^\square(2, 2, 2)$ to the correlation polytope $\text{COR}^\square(K_{2,2})$ of a complete bipartite graph $K_{2,2}$. Cut polytopes, which have essentially the same structure as the correlation polytopes, were independently introduced in combinatorial optimization and have been extensively studied. Many results are known for cut polytopes, including their relation to correlation polytopes, which suggests we consider Bell polytopes in the context of cut polytopes. The book by Deza and Laurent [8] is a comprehensive study of these polyhedra, and their applications. Ziegler [22] is a good source for basic definitions and results on convex polyhedra.

Results in the case $n = v = 2$. In the direction of finding a partial list of facets of $\mathcal{B}^\square(n, m, v)$, we consider the case of $n = v = 2$. Restricting the setting to $n = v = 2$ has the advantage that $\mathcal{B}^\square(2, m, 2)$ is affinely isomorphic to the cut polytope $\text{CUT}^\square(K_{1,m,m})$ of a complete tripartite graph $K_{1,m,m}$, which we prove in Section 3. We give an operation named triangular elimination which transforms a facet of a cut polytope of one graph to a facet of a cut polytope of a larger graph. This operation transforms a facet of a cut polytope of the complete graph to a facet of $\text{CUT}^\square(K_{1,m,m})$, which can be transformed to a facet of $\mathcal{B}^\square(2, m, 2)$ through an affine isomorphism. By using this operation and the list of known facets of CUT_9^\square [18], we have 44,368,793 different non-CHSH facets of $\mathcal{B}^\square(2, 10, 2)$ of which no two inequalities are equivalent up to the exchange of parties, observables or values. In addition, two additional results are proved: (1) A facet-supporting inequality for $\mathcal{B}^\square(2, m, 2)$ also supports a facet of $\mathcal{B}^\square(2, m', 2)$ for any $m' > m$. (2) The membership test of $\mathcal{B}^\square(2, m, 2)$ is NP-complete, which strengthens the unlikelihood of the complete facet characterization.

Results on projections and liftings of $\mathcal{B}^\square(n, m, v)$. In the direction of studying an affinely projected image of $\mathcal{B}^\square(n, m, v)$, we consider projections other than that to the polytope formed by the full correlation functions. We prove that a facet-supporting inequality for $\mathcal{B}^\square(n, m, v)$ never supports a facet of $\mathcal{B}^\square(n', m, v)$ for any $n' > n$, in contrast to the case of $\mathcal{B}^\square(2, m', 2)$.

Result on dimension of $\mathcal{B}^\square(n, m, v)$. In addition, we identify the dimension of $\mathcal{B}^\square(n, m, v)$ in Section 2 by proving that the only linear equations valid for $\mathcal{B}^\square(n, m, v)$ are the normalization condition and the no-signaling condition.

2 Definition and dimension of Bell polytope

Peres [14] shows that for $n, m, v \geq 1$, all the classical correlation tables in a setting with n parties each of which has m v -valued observables form a polytope defined as follows.

Definition 2.1 (Bell polytope of (n, m, v) -setting). The *Bell polytope* of (n, m, v) -setting is defined as the convex hull of v^{nm} points $\beta(\mathbf{c})$:

$$\mathcal{B}^\square(n, m, v) = \text{conv}\{\beta(\mathbf{c}) \in \mathbb{R}^{(mv)^n} \mid \mathbf{c} \in \{1, \dots, v\}^{n \times m}\},$$

where

$$\beta(\mathbf{c})_{(j_1, k_1), \dots, (j_n, k_n)} = \begin{cases} 1 & \text{if } c_{ij_i} = k_i \text{ for all } 1 \leq i \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

Here the $(mv)^n$ coordinates of vectors in $\mathbb{R}^{(mv)^n}$ are indexed by $(j_1, k_1), \dots, (j_n, k_n) \in (\{1, \dots, m\} \times \{1, \dots, v\})^n$.

$\mathcal{B}^\square(n, m, v)$ is not of full dimension. It is straightforward to show that any point $\mathbf{q} \in \mathcal{B}^\square(n, m, v)$ satisfies the following linear equations.

- *Normalization condition:* For each $\mathbf{j} \in \{1, \dots, m\}^n$,

$$\sum_{\mathbf{k} \in \{1, \dots, v\}^n} q_{(j_1, k_1), \dots, (j_n, k_n)} = 1. \quad (1)$$

- *No-signaling condition:* For each $i^* \in \{1, \dots, n\}$, $j_1, \dots, j_{i^*-1}, j_{i^*+1}, \dots, j_n, j, j' \in \{1, \dots, m\}$, $k_1, \dots, k_{i^*-1}, k_{i^*+1}, \dots, k_n \in \{1, \dots, v\}$,

$$\sum_{k=1}^v q_{(j_1, k_1), \dots, (j, k), \dots, (j_n, k_n)} = \sum_{k=1}^v q_{(j_1, k_1), \dots, (j', k), \dots, (j_n, k_n)}. \quad (2)$$

The following theorem states that $\mathcal{B}^\square(n, m, v)$ is of full dimensional in the affine subspace defined by these equations.

Theorem 2.1. For $n, m, v \geq 1$,

$$\dim \mathcal{B}^\square(n, m, v) = (m(v-1) + 1)^n - 1.$$

Proof. Equations (1) and (2) together define an $((m(v-1) + 1)^n - 1)$ -dimensional affine subspace of $\mathbb{R}^{(mv)^n}$ in which $\mathcal{B}^\square(n, m, v)$ lies. This means $\dim \mathcal{B}^\square(n, m, v) \leq (m(v-1) + 1)^n - 1$.

On the other hand, we can find $(m(v-1) + 1)^n$ affinely independent points in $\mathcal{B}^\square(n, m, v)$ as follows. Let $L = (\{1, \dots, m\} \times \{1, \dots, v-1\}) \cup \{*\}$. Note that $|L| = m(v-1) + 1$. For any $l_1, \dots, l_n \in L$, define $\mathbf{c}^{(l_1, \dots, l_n)} \in \{1, \dots, v\}^{n \times m}$ by

$$c_{ij}^{(l_1, \dots, l_n)} = \begin{cases} k_i & \text{if } l_i = (j, k_i), \\ v & \text{if } l_i = * \text{ or } l_i = (j', k_i) \text{ for some } j' \neq j. \end{cases}$$

Let $\mathbf{q}^{(l_1, \dots, l_n)} = \beta(\mathbf{c}^{(l_1, \dots, l_n)})$. By definition of $\mathcal{B}^\square(n, m, v)$, all of the $(m(v-1) + 1)^n$ points $\mathbf{q}^{(l_1, \dots, l_n)}$ belong to $\mathcal{B}^\square(n, m, v)$. It is straightforward that these $(m(v-1) + 1)^n$ points are affinely independent. This means $\dim \mathcal{B}^\square(n, m, v) \geq (m(v-1) + 1)^n - 1$, hence $\dim \mathcal{B}^\square(n, m, v) = (m(v-1) + 1)^n - 1$. \square

3 Affine isomorphism between $\mathcal{B}^\square(2, m, 2)$ and $\text{CUT}^\square(K_{1, m, m})$

In this section, we restrict our focus to the case of $n = v = 2$. In this setting we can give an affine isomorphism between the Bell polytope $\mathcal{B}^\square(2, m, 2)$ and the cut polytope $\text{CUT}^\square(K_{1, m, m})$. Cut polytopes have been extensively studied in combinatorial geometry both theoretically and computationally, and we can use these results to study Bell polytopes $\mathcal{B}^\square(2, m, 2)$.

We begin by giving definitions of the correlation polytopes and cut polytopes.

Definition 3.1 (Correlation polytope [8, Section 5.1]). Let $G = (V, E)$ be a graph with vertex set V and edge set E where an edge connecting vertices u and v is denoted by uv . Consider an \mathbb{R} -vector space $\mathbb{R}^{V \cup E}$, that is, a $(|V| + |E|)$ -dimensional vector space over \mathbb{R} whose coordinates are labeled by $V \cup E$.

We define the *correlation polytope* of the graph G as the convex hull of the points $\pi(I)$:

$$\text{COR}^\square(G) = \text{conv}\{\pi(I) \mid I \subseteq V\},$$

where

$$\text{for } v \in V, \pi_v(I) = \begin{cases} 1 & \text{if } v \in I, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and for } uv \in E, \pi_{uv}(I) = \begin{cases} 1 & \text{if } \{u, v\} \subseteq I, \\ 0 & \text{otherwise.} \end{cases}$$

Definition 3.2 (Cut polytope [8, Section 4.1]). Let $G = (V, E)$ be a graph. Consider an \mathbb{R} -vector space \mathbb{R}^E . The *cut polytope* of the graph G is defined as the convex hull of the points $\delta(I)$:

$$\text{CUT}^\square(G) = \text{conv}\{\delta(I) \mid I \subseteq V\},$$

where for $uv \in E$,

$$\delta_{uv}(I) = \begin{cases} 1 & \text{if exactly one of } u \text{ and } v \text{ is in } I, \\ 0 & \text{otherwise.} \end{cases}$$

For simplicity, we denote the cut polytope $\text{CUT}^\square(K_n)$ of a complete graph K_n by CUT_n^\square .

The cut polytope is highly symmetric. For example, it looks the ‘‘same’’ at each vertex. This notion is formalized by the switching operation.

Definition 3.3 (Switching). For a vector $\mathbf{x} \in \mathbb{R}^E$ and a set $S \subseteq V$, the *switching* of \mathbf{x} by S is the vector $\mathbf{x}' \in \mathbb{R}^E$ defined by

$$x'_{uv} = \begin{cases} -x_{uv} & (\delta_{uv}(S) = 1) \\ x_{uv} & (\delta_{uv}(S) = 0), \end{cases}$$

and denoted by \mathbf{x}^S .

For a vector $\mathbf{a} \in \mathbb{R}^E$, a scalar $a_0 \in \mathbb{R}$ and a set $S \subseteq V$, the *switching* of the inequality $\mathbf{a}^\top \mathbf{x} \leq a_0$ by S is the inequality $(\mathbf{a}^S)^\top \mathbf{x} \leq a_0 - \mathbf{a}^\top \delta(S)$. This switching is valid for $\text{CUT}^\square(G)$ if and only if the original inequality $\mathbf{a}^\top \mathbf{x} \leq a_0$ is valid for $\text{CUT}^\square(G)$. Similarly, the switching supports a facet of $\text{CUT}^\square(G)$ if and only if the original inequality supports a facet of $\text{CUT}^\square(G)$.

For a facet f of $\text{CUT}^\square(G)$ supported by the inequality $\mathbf{a}^\top \mathbf{x} \leq a_0$, the *switching* of f by S is the facet of $\text{CUT}^\square(G)$ supported by the switching of $\mathbf{a}^\top \mathbf{x} \leq a_0$ by S , and denoted by $\gamma(S) \cdot f$.

Pitowsky [15, pp. 27–29] shows that the Bell polytope $\mathcal{B}^\square(2, 2, 2)$ is affinely isomorphic to the correlation polytope $\text{COR}^\square(K_{2,2})$ of a complete bipartite graph $K_{2,2}$. The same applies to $\mathcal{B}^\square(2, m, 2)$ and $\text{COR}^\square(K_{m,m})$ for general m as the next theorem states.

Theorem 3.1. $\mathcal{B}^\square(2, m, 2)$ is affinely isomorphic to $\text{COR}^\square(K_{m,m})$, where $K_{m,m}$ is the complete bipartite graph with m left vertices and m right vertices. The isomorphism maps every $\mathbf{q} \in \mathcal{B}^\square(2, m, 2)$ to $\mathbf{p} \in \text{COR}^\square(K_{m,m})$ given by:

$$p_{j_1, j_2+m} = q_{(j_1,2), (j_2,2)} \tag{3}$$

$$p_{j_1} = q_{(j_1,2), (j_2,2)} + q_{(j_1,2), (j_2,1)} \tag{3}$$

$$p_{j_2+m} = q_{(j_1,2), (j_2,2)} + q_{(j_1,1), (j_2,2)}, \tag{4}$$

where $1 \leq j_1, j_2 \leq m$. Note that the value of the right hand side of equation (3) does not depend on j_2 , and the value of the right hand side of equation (4) does not depend on j_1 .

Definition 3.4 (Suspension graph). Let $G = (V, E)$ be a graph with n vertices. Then the *suspension graph* ∇G of G is the graph $G' = (V \cup \{0\}, E \cup \{0v \mid v \in V\})$ with $n + 1$ vertices obtained from G by adding one new vertex and connecting it to each of n existing vertices.

Theorem 3.2 (Affine isomorphism of $\text{COR}^\square(G)$ to $\text{CUT}^\square(\nabla G)$ [8, Section 5.2]). For any graph $G = (V, E)$, the correlation polytope $\text{COR}^\square(G)$ of G is affinely isomorphic to the cut polytope $\text{CUT}^\square(\nabla G)$ of the suspension graph of G . The isomorphism maps every $\mathbf{p} \in \text{COR}^\square(G)$ to $\mathbf{x} \in \text{CUT}^\square(\nabla G)$ given by:

$$x_{0v} = p_v \quad (v \in V),$$

$$x_{uv} = p_u + p_v - 2p_{uv} \quad (uv \in E).$$

Combining Theorems 3.1 and 3.2 give the following result immediately. Note that $\nabla K_{m,m} = K_{1,m,m}$.

Theorem 3.3. $\mathcal{B}^\square(2, m, 2)$ is affinely isomorphic to $\text{CUT}^\square(\mathbf{K}_{1,m,m})$, where $\mathbf{K}_{1,m,m}$ is the complete tripartite graph with one partition with one vertex and two partitions with m vertices each. The isomorphism maps every $\mathbf{q} \in \mathcal{B}^\square(2, m, 2)$ to $\mathbf{x} \in \text{CUT}^\square(\mathbf{K}_{1,m,m})$ given by:

$$x_{0,j_1} = q_{(j_1,2),(j_2,2)} + q_{(j_1,2),(j_2,1)}, \quad (5)$$

$$x_{0,j_2+m} = q_{(j_1,2),(j_2,2)} + q_{(j_1,1),(j_2,2)}, \quad (6)$$

$$x_{j_1,j_2+m} = q_{(j_1,2),(j_2,1)} + q_{(j_1,1),(j_2,2)},$$

where $1 \leq j_1, j_2 \leq m$. Note that the value of the right hand side of equation (5) does not depend on j_2 , and the value of the right hand side of equation (6) does not depend on j_1 .

Thus, the study of the Bell polytope $\mathcal{B}^\square(2, m, 2)$ is equivalent to that of the cut polytope $\text{CUT}^\square(\mathbf{K}_{1,m,m})$. Both correlation polytopes and cut polytopes are of full dimension. This means each of their facets has a unique representation by a supporting linear inequality.

The cut polytopes of complete graphs have been extensively studied and large classes of their facets are known. In addition, conjectured complete lists of facets of CUT_n^\square is known for $n \leq 9$ [18], of which the lists for $n \leq 7$ are known to be complete. It is also known that the problem of testing membership in CUT_n^\square is known NP-complete [2], so a complete facet characterization for general n is unlikely.

4 Triangular elimination

In this section, we establish a method called *triangular elimination* to construct facets of $\text{CUT}^\square(\mathbf{K}_{1,m,m})$ from facets of CUT_n^\square where $m \geq \frac{1}{2} \lfloor \frac{n-2}{2} \rfloor \lfloor \frac{n-4}{2} \rfloor$.

When $2m+1 > n$, the zero-lifting theorem [6] guarantees that any facet-supporting inequality for CUT_n^\square supports a facet of $\text{CUT}_{2m+1}^\square$. Since $\text{CUT}^\square(\mathbf{K}_{1,m,m})$ is a projected image of $\text{CUT}_{2m+1}^\square$, Fourier-Motzkin elimination [22, Lecture 1] can be used to convert facet-supporting inequalities for $\text{CUT}_{2m+1}^\square$ to valid inequalities for $\text{CUT}^\square(\mathbf{K}_{1,m,m})$. The problem is that Fourier-Motzkin elimination does not always produce facet-supporting inequalities. The triangular elimination procedure we introduce here is a sufficient condition for Fourier-Motzkin elimination to produce facet-supporting inequalities.

Theorem 4.9 is the main theorem in this section which guarantees triangular elimination always produce a facet of $\text{CUT}^\square(\mathbf{K}_{1,m,m})$ from a non-triangle facet of CUT_n^\square . It relies strongly on Theorem 4.3, which is likely to be of independent interest to researchers interested in polyhedral theory.

4.1 Definition of triangular elimination

Consider a graph $G = (V, E)$. Let $u, u' \in V$, $uu' \in E$, and $A \subseteq N_G(u) \cap N_G(u')$, where $N_G(v)$ is the set of vertices adjacent to $v \in V$ in G . We create a graph $G^+ = (V^+, E^+)$ from G by removing the edge uu' and instead adding a new vertex v adjacent to each vertex of $\{u, u'\} \cup A$:

$$V^+ = V \cup \{v\}, \quad E^+ = (E \setminus \{uu'\}) \cup \{uv, u'v\} \cup \{vw \mid w \in A\}.$$

Definition 4.1 (Detour extension). We say such G^+ is a *detour extension* of G with removed edge uu' , added vertex v and adjacent vertex set A .

We would like to construct a facet of $\text{CUT}^\square(G^+)$ from a facet of $\text{CUT}^\square(G)$. Let $G' = (V^+, E^+ \cup \{uu'\})$. Note that we can always construct a facet of $\text{CUT}^\square(G')$ from a facet f of $\text{CUT}^\square(G)$ by the following zero-lifting theorem.

Theorem 4.1 (Zero-lifting theorem [6]). Let $G = (V, E)$ be a graph, $u \in V$ and $A \subseteq N_G(u)$. We create a graph $G' = (V', E')$ from G by adding a new vertex v adjacent to each vertex of $\{u\} \cup A$:

$$V' = V \cup \{v\}, \quad E' = E \cup \{uv\} \cup \{vw \mid w \in A\}.$$

For an inequality $\mathbf{a}^\top \mathbf{x} \leq a_0$ in \mathbb{R}^E , define its zero-lifting $\bar{\mathbf{a}}^\top \mathbf{x} \leq a_0$ by

$$\bar{a}_{ww'} = \begin{cases} a_{ww'} & \text{if } ww' \in E, \\ 0 & \text{otherwise.} \end{cases}$$

If the inequality $\mathbf{a}^\top \mathbf{x} \leq a_0$ supports a facet of $\text{CUT}^\square(G)$, then its zero-lifting supports a facet of $\text{CUT}^\square(G')$.

Let $\mathbf{a}^\top \mathbf{x} \leq a_0$ be the inequality supporting f . If $a_{uu'} = 0$, the zero-lifting of $\mathbf{a}^\top \mathbf{x} \leq a_0$ is not only an inequality in $\mathbb{R}^{E'}$ but also an inequality in \mathbb{R}^{E^+} . In this case, the zero-lifting theorem guarantees it supports a facet of $\text{CUT}^\square(G^+)$. How can we extend this construction to the case of $a_{uu'} \neq 0$? The answer is to *eliminate* the term $a_{uu'}x_{uu'}$ from the inequality $\mathbf{a}^\top \mathbf{x} \leq a_0$ by adding an appropriate valid inequality to it. We consider the simplest case of adding a triangle inequality $-a_{uu'}x_{uu'} + a_{uu'}x_{uv} - |a_{uu'}|x_{u'v} \leq 0$. We call this operation the *triangular elimination*.

The formal definition is as follows.

Definition 4.2 (Triangular elimination). Let $G = (V, E)$ be a graph, and let $G^+ = (V^+, E^+)$ be the detour extension of G with removed edge $uu' \in E$, added vertex v and adjacent vertex set A . Let $\mathbf{a}^\top \mathbf{x} \leq a_0$ be an inequality in \mathbb{R}^E . The *triangular elimination* of the inequality $\mathbf{a}^\top \mathbf{x} \leq a_0$ is the inequality in \mathbb{R}^{E^+} defined by

$$\mathbf{a}^\top \mathbf{x} - a_{uu'}x_{uu'} + a_{uu'}x_{uv} - |a_{uu'}|x_{u'v} \leq a_0. \quad (7)$$

Note that (7) is indeed an inequality in \mathbb{R}^{E^+} because it does not have the term of $x_{uu'}$.

4.2 Properties of inequality produced by triangular elimination

The cut cone $\text{CUT}(G)$ of a graph G is a polyhedral cone closely related to the cut polytope $\text{CUT}^\square(G)$. The set of facets of $\text{CUT}(G)$ consists of the facets of $\text{CUT}^\square(G)$ containing the coordinate origin. In this subsection, we begin by considering the case of cut cones because they are easier for theoretical handling, and then make use of the switching operation to extend the results to cut polytopes.

Definition 4.3 (Cut cone [8, Section 4.1]). Let $G = (V, E)$ be a graph. Consider an \mathbb{R} -vector space \mathbb{R}^E . The *cut cone* of the graph G is defined as the conic hull of the vectors $\boldsymbol{\delta}(I)$:

$$\begin{aligned} \text{CUT}(G) &= \text{cone}\{\boldsymbol{\delta}(I) \mid I \subseteq V\} \\ &= \left\{ \sum_{I \subseteq V} \lambda_I \boldsymbol{\delta}(I) \mid \lambda_I \geq 0 \ (\forall I \subseteq V) \right\}, \end{aligned}$$

where $\boldsymbol{\delta}(I)$ is the same as in Definition 3.2. Like the cut polytope, we denote the cut cone $\text{CUT}(K_n)$ of a complete graph by CUT_n .

First we show that the inequality produced by triangular elimination from a valid inequality for $\text{CUT}(G)$ is valid for $\text{CUT}(G^+)$.

Theorem 4.2. *Let $G = (V, E)$ be a graph, and let $G^+ = (V^+, E^+)$ be the detour extension of G with removed edge $uu' \in E$, added vertex v and adjacent vertex set A . Let $\mathbf{a}^\top \mathbf{x} \leq a_0$ be an inequality in \mathbb{R}^E , and $\mathbf{b}^\top \mathbf{x} \leq a_0$ be its triangular elimination. Then, the following two conditions are equivalent.*

(i) *The inequality $\mathbf{a}^\top \mathbf{x} \leq a_0$ is valid for $\text{CUT}(G)$.*

(ii) *The inequality $\mathbf{b}^\top \mathbf{x} \leq a_0$ is valid for $\text{CUT}(G^+)$.*

Proof. ((i) \implies (ii)) Let $G' = (V^+, E^+ \cup \{uu'\})$. The inequality $\mathbf{b}^\top \mathbf{x} \leq a_0$ is the sum of two inequalities

$$\mathbf{a}^\top \mathbf{x} \leq a_0, \quad -a_{uu'}x_{uu'} + a_{uu'}x_{uv} - |a_{uu'}|x_{u'v} \leq 0,$$

both of which are valid for $\text{CUT}(G')$. This means the inequality $\mathbf{b}^\top \mathbf{x} \leq a_0$ is also valid for $\text{CUT}(G')$, hence valid for $\text{CUT}(G^+)$.

((ii) \implies (i)) $\mathbf{a}^\top \mathbf{x}$ is obtained from $\mathbf{b}^\top \mathbf{x}$ by collapsing two vertices u' and v . This implies (ii) \implies (i). \square

Remark 4.1. As far as the validity of inequalities is concerned, we do not need the condition $A \subseteq N_G(u) \cap N_G(u')$ for detour extension. However, this condition is needed to preserve the facet-supporting property of the inequalities, which we consider next.

Theorem 4.3. *Let $G = (V, E)$ be a graph, and let $G^+ = (V^+, E^+)$ be the detour extension of G with removed edge $uu' \in E$, added vertex v and adjacent vertex set A . Let $\mathbf{a}^\top \mathbf{x} \leq 0$ be an inequality supporting a facet of $\text{CUT}(G)$. If there exists an edge $e \in E \setminus (\{uu'\} \cup \{uw, u'w \mid w \in A\})$ such that $a_e \neq 0$, then the triangular elimination of $\mathbf{a}^\top \mathbf{x} \leq 0$ supports a facet of $\text{CUT}(G^+)$.*

Note that Lemma 4.4 states essentially the same thing as Lemma 26.5.2 (ii) of [8].

Lemma 4.4. *Let \mathbb{R}^E be a vector space of a finite dimension, and let $D \subseteq E$. Let $\pi: \mathbb{R}^E \rightarrow \mathbb{R}^D$ be the orthogonal projection. Let P be a full-dimensional polyhedron in \mathbb{R}^E . Let f be a facet of P supported by an inequality $\mathbf{a}^\top \mathbf{x} \leq a_0$. If there exists $e \in E \setminus D$ such that $a_e \neq 0$, then the projected image $\pi(f)$ is of full dimension in \mathbb{R}^D .*

Proof. Let H be the hyperplane defined by $\mathbf{a}^\top \mathbf{x} = a_0$, so that $f = P \cap H$. We prove $\pi(H) = \mathbb{R}^D$. For any $\mathbf{y} \in \mathbb{R}^D$, define $\mathbf{x} \in \mathbb{R}^E$ as follows. Let $x_{e'} = y_{e'}$ for each $e' \in D$. Set any values to $x_{e'}$ for $e' \in E \setminus (D \cup \{e\})$. Set x_e by

$$x_e = a_0 - \sum_{e' \in E \setminus \{e\}} a_{e'} x_{e'}.$$

This \mathbf{x} is on H and satisfies $\pi(\mathbf{x}) = \mathbf{y}$. This means $\mathbf{y} \in \pi(H)$, which means $\pi(H) = \mathbb{R}^D$.

Therefore, $\pi(f) = \pi(P) \cap \pi(H) = \pi(P)$ is of full dimension in \mathbb{R}^D . \square

Proof of Theorem 4.3. Let $\mathbf{b}^\top \mathbf{x} \leq 0$ be the triangular elimination of $\mathbf{a}^\top \mathbf{x} \leq 0$. Let f be the facet of $\text{CUT}(G)$ supported by the inequality $\mathbf{a}^\top \mathbf{x} \leq 0$, and F be the face of $\text{CUT}(G^+)$ supported by $\mathbf{b}^\top \mathbf{x} \leq 0$.

We prove the case of $a_{uu'} \leq 0$. The case of $a_{uu'} > 0$ is proved by applying the case of $a_{uu'} \leq 0$ to the switching of f with respect to $\{u\}$.

Let $|E| = d$ and $|A| = d'$. Because f is a facet of $\text{CUT}(G)$, there exist $d-1$ subsets $S_1, \dots, S_{d-1} \subseteq V \setminus \{u\}$ such that $\delta_G(S_1), \dots, \delta_G(S_{d-1})$ are linearly independent roots of f .

Let $D = \{uu'\} \cup \{ij \mid i \in \{u, u'\}, j \in A\} \subseteq E$, and let $\pi: \mathbb{R}^E \rightarrow \mathbb{R}^D$ be the orthogonal projection. Because there exists an edge $e \in E \setminus D$ such that $a_e \neq 0$, the projected image $\pi(f)$ of f to \mathbb{R}^D is of full dimension by Lemma 4.4. This means that there exist $2d' + 1$ subsets $T_1, \dots, T_{2d'+1} \subseteq V \setminus \{u\}$ such that $\delta_G(T_1), \dots, \delta_G(T_{2d'+1})$ are roots of f and $\pi(\delta_G(T_1)), \dots, \pi(\delta_G(T_{2d'+1}))$ are linearly independent. Note that the intersection of $\pi(\text{CUT}(G))$ and the hyperplane $x_{uu'} = 0$ has a dimension d' , which means at most d' out of $2d' + 1$ subsets $T_1, \dots, T_{2d'+1}$ satisfy $u' \notin T_i$. Therefore, at least $d' + 1$ out of $2d' + 1$ subsets $T_1, \dots, T_{2d'+1}$ contain u' . Without loss of generality, we assume that $T_1, \dots, T_{d'+1} \ni u'$. For $1 \leq i \leq d' + 1$, let $T'_i = T_i \cup \{v\}$.

Let

$$C = \{\delta_{G^+}(S_1), \dots, \delta_{G^+}(S_{d-1}), \delta_{G^+}(T'_1), \dots, \delta_{G^+}(T'_{d'+1})\}.$$

The $d + d'$ cut vectors in C are roots of F . We prove that these $d + d'$ cut vectors are linearly independent. Let M be a matrix of size $(d + d' + 1) \times (d + d')$ whose column vectors are these $d + d'$ cut vectors. We group the rows of M into 4 groups E_1, E_2, E_3 and E_4 , where

$$\begin{aligned} E_1 &= E \setminus (E_2 \cup E_3 \cup E_4), \\ E_2 &= \{ui \mid i \in A\}, \\ E_3 &= \{vi \mid i \in A\}, \\ E_4 &= \{uv\}. \end{aligned}$$

Then M is in the form

$$M = \begin{matrix} E_1 \\ E_2 \\ E_3 \\ E_4 \end{matrix} \begin{pmatrix} X & * \\ Y & Z \\ Y & \mathbf{1} - Z \\ O & \mathbf{1} \end{pmatrix},$$

where $\mathbf{1}$ represents a matrix whose elements are all 1. The first $d-1$ columns of M represent $\delta_{G^+}(S_1), \dots, \delta_{G^+}(S_{d-1})$ and the last $d' + 1$ columns represent $\delta_{G^+}(T'_1), \dots, \delta_{G^+}(T'_{d'+1})$. Because $\delta_{G^+}(S_1), \dots, \delta_{G^+}(S_{d-1})$ are linearly independent,

$$\text{rank} \begin{pmatrix} X \\ Y \end{pmatrix} = d - 1.$$

Similarly, because $\pi(\delta_{G^+}(T'_1)), \dots, \pi(\delta_{G^+}(T'_{d'+1}))$ are linearly independent,

$$\text{rank} \begin{pmatrix} Z \\ \mathbf{1} - Z \\ \mathbf{1} \end{pmatrix} = d' + 1,$$

which means

$$\text{rank} \begin{pmatrix} Z \\ \mathbf{1} \end{pmatrix} = d' + 1.$$

Therefore,

$$\text{rank } M = \text{rank} \begin{pmatrix} X & * \\ Y & Z \\ O & -2Z \\ O & \mathbf{1} \end{pmatrix} = \text{rank} \begin{pmatrix} X \\ Y \end{pmatrix} + \text{rank} \begin{pmatrix} -2Z \\ \mathbf{1} \end{pmatrix} = (d-1) + (d'+1) = d+d'.$$

This means the $d+d'$ cut vectors in C are linearly independent roots of F , which means F is a facet of $\text{CUT}(G^+)$. \square

Now we show that Theorems 4.2 and 4.3 hold also in the case of cut polytopes. We will present two lemmas to establish the relation between $\text{CUT}(G)$ and $\text{CUT}^\square(G)$. The first lemma contains well known facts (see, e.g. [8, Section 26.3]). We include the proof here for completeness. Recall that the switching of the inequality $\mathbf{a}^\top \mathbf{x} \leq a_0$ by the cut $\delta_G(S)$ is

$$(\mathbf{a}^S)^\top \mathbf{x} \leq a_0 - \mathbf{a}^\top \delta_G(S). \quad (8)$$

Lemma 4.5. *Let $G = (V, E)$ be a graph, and $\mathbf{a}^\top \mathbf{x} \leq a_0$ be an inequality in \mathbb{R}^E . Let S be a subset of V such that the linear function $\mathbf{a}^\top \mathbf{x}$ takes the maximum at $\delta_G(S)$ in $\text{CUT}^\square(G)$. Then the following conditions are equivalent.*

- (i) *The inequality $\mathbf{a}^\top \mathbf{x} \leq a_0$ is valid for $\text{CUT}^\square(G)$.*
- (ii) *The switching of the inequality $\mathbf{a}^\top \mathbf{x} \leq a_0$ by the cut $\delta_G(S)$ is valid for $\text{CUT}^\square(G)$.*
- (iii) *The switching of the inequality $\mathbf{a}^\top \mathbf{x} \leq a_0$ by the cut $\delta_G(S)$ is valid for $\text{CUT}(G)$.*

Similarly, the following conditions are equivalent.

- (i) *The inequality $\mathbf{a}^\top \mathbf{x} \leq a_0$ supports a facet of $\text{CUT}^\square(G)$.*
- (ii) *The switching of the inequality $\mathbf{a}^\top \mathbf{x} \leq a_0$ by the cut $\delta_G(S)$ supports a facet of $\text{CUT}^\square(G)$.*
- (iii) *The switching of the inequality $\mathbf{a}^\top \mathbf{x} \leq a_0$ by the cut $\delta_G(S)$ supports a facet of $\text{CUT}(G)$.*

Proof. We first show the claim about the validity of the inequalities. The equivalence between (i) and (ii) is trivial because switching by any cut of G maps $\text{CUT}^\square(G)$ onto itself. (iii) \implies (ii) is also trivial because $\text{CUT}^\square(G) \subseteq \text{CUT}(G)$. To show (ii) \implies (iii), assume that (8) is valid for $\text{CUT}^\square(G)$. This means that for any $S' \subseteq V$,

$$(\mathbf{a}^S)^\top \delta_G(S') \leq a_0 - \mathbf{a}^\top \delta_G(S). \quad (9)$$

Letting $S' = \emptyset$ gives

$$a_0 - \mathbf{a}^\top \delta_G(S) \geq 0. \quad (10)$$

By the definition of S and the relation

$$(\mathbf{a}^S)^\top \delta_G(S') = \mathbf{a}^\top \delta_G(S \triangle S') - \mathbf{a}^\top \delta_G(S),$$

the left hand side of (9) takes the maximum when $S \triangle S' = S$, or equivalently $S' = \emptyset$. This means

$$(\mathbf{a}^S)^\top \delta_G(S') \leq 0 \quad (11)$$

for any $S' \subseteq V$. Inequalities (10) and (11) give

$$(\mathbf{a}^S)^\top (\lambda \delta_G(S')) \leq a_0 - \mathbf{a}^\top \delta_G(S)$$

for any $S' \subseteq V$ and $\lambda \geq 0$, which means (8) is valid for $\text{CUT}(G)$.

For the facet-supporting property, the argument is similar. (i) \iff (ii) and (iii) \implies (ii) are trivial. To show (ii) \implies (iii), assume that (8) supports a facet of $\text{CUT}^\square(G)$. From the argument above, especially (10) and (11), it is necessary that $a_0 - \mathbf{a}^\top \delta_G(S) = 0$ for (8) to support a nonempty face of $\text{CUT}^\square(G)$. Because $\text{CUT}(G)$ has every facet of $\text{CUT}^\square(G)$ that contains the coordinate origin, the inequality (8) supports a facet of $\text{CUT}(G)$. \square

Lemma 4.6. Let $G = (V, E)$ be a graph, and let $G^+ = (V^+, E^+)$ be the detour extension of G with removed edge $uu' \in E$, added vertex v and adjacent vertex set A . Let $\mathbf{a}^\top \mathbf{x} \leq a_0$ be an inequality in \mathbb{R}^E , and $\mathbf{b}^\top \mathbf{x} \leq a_0$ be its triangular elimination. Then, there exists a subset S of V such that the following conditions hold:

- (i) The switching of $\mathbf{b}^\top \mathbf{x} \leq a_0$ by the cut $\delta_{G^+}(S)$ is the triangular elimination of the switching of $\mathbf{a}^\top \mathbf{x} \leq a_0$ by the cut $\delta_G(S)$.
- (ii) The switching of $\mathbf{a}^\top \mathbf{x} \leq a_0$ by the cut $\delta_G(S)$ is valid (resp. facet-supporting) for $\text{CUT}(G)$ if and only if $\mathbf{a}^\top \mathbf{x} \leq a_0$ is valid (resp. facet-supporting) for $\text{CUT}^\square(G)$.
- (iii) The switching of $\mathbf{b}^\top \mathbf{x} \leq a_0$ by the cut $\delta_{G^+}(S)$ is valid (resp. facet-supporting) for $\text{CUT}(G^+)$ if and only if $\mathbf{b}^\top \mathbf{x} \leq a_0$ is valid (resp. facet-supporting) for $\text{CUT}^\square(G^+)$.

Proof. First note that by definition, the inequality $\mathbf{b}^\top \mathbf{x} \leq a_0$ is written as

$$\mathbf{a}^\top \mathbf{x} - a_{uu'}x_{uu'} + a_{uu'}x_{uv} - |a_{uu'}|x_{u'v} \leq a_0. \quad (12)$$

Let S be a subset of $V \setminus \{u'\}$ such that the linear function $\mathbf{a}^\top \mathbf{x}$ gives the maximum in $\text{CUT}^\square(G)$ at the point $\delta(S)$. Then the switching of $\mathbf{a}^\top \mathbf{x} \leq a_0$ by the cut $\delta_G(S)$ is

$$(\mathbf{a}^S)^\top \mathbf{x} \leq a_0 - \mathbf{a}^S \delta_G(S), \quad (13)$$

and the switching of (12) by the cut $\delta_{G^+}(S)$ is

$$(\mathbf{a}^S)^\top \mathbf{x} - (-1)^{\chi_u(S)} a_{uu'}x_{uu'} + (-1)^{\chi_u(S)} a_{uu'}x_{uv} - |a_{uu'}|x_{u'v} \leq a_0 - \mathbf{a}^S \delta_G(S), \quad (14)$$

where $\chi_u(S)$ is 1 if $u \in S$, or 0 otherwise. We will check the conditions claimed in the lemma are satisfied.

- (i) By definition, the triangular elimination of (13) is the inequality (14).
- (ii) This is proved by Lemma 4.5.
- (iii) Let $G' = (V^+, E^+ \cup \{uu'\})$. Both $\mathbf{a}^\top \mathbf{x}$ and $-a_{uu'}x_{uu'} + a_{uu'}x_{uv} - |a_{uu'}|x_{u'v}$ are linear functions on $\mathbb{R}^{E^+ \cup \{uu'\}}$. By definition of S , $\mathbf{a}^\top \mathbf{x}$ takes the maximum at $\delta_{G'}(S)$ in $\text{CUT}^\square(G')$. In addition, the linear function $-a_{uu'}x_{uu'} + a_{uu'}x_{uv} - |a_{uu'}|x_{u'v}$ takes the maximum value 0 at $\delta_{G'}(S)$ in $\text{CUT}^\square(G')$. Therefore, the left hand side of (12) takes the maximum at $\delta_{G'}(S)$ in $\text{CUT}^\square(G')$, which means it takes the maximum at $\delta_{G^+}(S)$ in $\text{CUT}^\square(G^+)$. Then the claim is proved by Lemma 4.5. \square

Theorem 4.7. Let $G = (V, E)$ be a graph, and let $G^+ = (V^+, E^+)$ be the detour extension of G with removed edge $uu' \in E$, added vertex v and adjacent vertex set A . Let $\mathbf{a}^\top \mathbf{x} \leq a_0$ be an inequality in \mathbb{R}^E , and $\mathbf{b}^\top \mathbf{x} \leq a_0$ be its triangular elimination. Then, the following two conditions are equivalent.

- (i) The inequality $\mathbf{a}^\top \mathbf{x} \leq a_0$ is valid for $\text{CUT}^\square(G)$.
- (ii) The inequality $\mathbf{b}^\top \mathbf{x} \leq a_0$ is valid for $\text{CUT}^\square(G^+)$.

Proof. Let S be the subset of V stated in Lemma 4.6. Then condition (i) holds if and only if the switching of the inequality $\mathbf{a}^\top \mathbf{x} \leq a_0$ by the cut $\delta_G(S)$ is valid for $\text{CUT}(G)$, and condition (ii) holds if and only if the switching of the inequality $\mathbf{b}^\top \mathbf{x} \leq a_0$ by the cut $\delta_{G^+}(S)$ is valid for $\text{CUT}(G^+)$. By Theorem 4.2, the two conditions are equivalent. \square

Theorem 4.8. Let $G = (V, E)$ be a graph, and let $G^+ = (V^+, E^+)$ be the detour extension of G with removed edge $uu' \in E$, added vertex v and adjacent vertex set A . Let $\mathbf{a}^\top \mathbf{x} \leq a_0$ be an inequality supporting a facet of $\text{CUT}^\square(G)$. If there exists an edge $e \in E \setminus (\{uu'\} \cup \{uw, u'w \mid w \in A\})$ such that $a_e \neq 0$, then the triangular elimination of $\mathbf{a}^\top \mathbf{x} \leq a_0$ supports a facet of $\text{CUT}^\square(G^+)$.

Proof. Let $\mathbf{b}^\top \mathbf{x} \leq a_0$ be the triangular elimination of $\mathbf{a}^\top \mathbf{x} \leq a_0$. Let S be the subset of V stated in Lemma 4.6. Then the switching of the inequality $\mathbf{a}^\top \mathbf{x} \leq a_0$ by the cut $\delta_G(S)$ supports a facet of $\text{CUT}(G)$. By Theorem 4.3, the switching of the inequality $\mathbf{b}^\top \mathbf{x} \leq a_0$ by the cut $\delta_{G^+}(S)$ supports a facet of $\text{CUT}(G^+)$. This means the inequality $\mathbf{b}^\top \mathbf{x} \leq a_0$ supports a facet of $\text{CUT}^\square(G^+)$. \square

Note that in case of $a_{uu'} = -c < 0$, we can consider, instead of (7), the inequality

$$\mathbf{a}^T \mathbf{x} + cx_{uu'} + cx_{uv} + cx_{u'v} \leq a_0 + 2c, \quad (15)$$

which is obtained by adding a triangle inequality $c(x_{uu'} + x_{uv} + x_{u'v}) \leq 2c$ to the inequality $\mathbf{a}^T \mathbf{x} \leq a_0$. The inequality (15) is the switching of the inequality (7) with respect to $\{v\}$. This means two things:

- The inequality (15) is valid for $\text{CUT}^\square(G^+)$ if and only if the inequality (7) is valid for $\text{CUT}^\square(G^+)$.
- The inequality (15) supports a facet of $\text{CUT}^\square(G^+)$ if and only if the inequality (7) supports a facet of $\text{CUT}^\square(G^+)$.

4.3 Constructing facets of $\text{CUT}^\square(K_{1,m,m})$ by iterative triangular elimination

Let $n > 1$ be an integer and $m = \frac{1}{2} \lfloor \frac{n-2}{2} \rfloor \lfloor \frac{n-4}{2} \rfloor + n - 2$. The complete tripartite graph $K_{1,m,m}$ is obtained from the complete graph K_n by repeating the operation of detour extension. The next theorem follows from Theorems 4.7 and 4.8.

Theorem 4.9. *Let $n > 1$ be an integer and $m = \frac{1}{2} \lfloor \frac{n-2}{2} \rfloor \lfloor \frac{n-4}{2} \rfloor + n - 2$.*

- Any valid inequality for CUT_n^\square can be converted to a valid inequality for $\text{CUT}^\square(K_{1,m,m})$ by repeating the operation of triangular elimination.
- Any facet of CUT_n^\square except for the triangle inequalities can be converted to a facet of $\text{CUT}^\square(K_{1,m,m})$ by repeating the operation of triangular elimination.

Proof. (i) This is proved by applying Theorem 4.7 repeatedly.

- For any facet of CUT_n^\square except for the triangle inequalities, there exists at least four relevant vertices. Therefore, we can apply Theorem 4.8 to every single operation of triangular elimination, and so we obtain the theorem. □

Remark 4.2. As stated in Remark 4.1, any valid inequality for $\text{CUT}(G)$ is converted to a valid inequality for $\text{CUT}(G')$ by triangular elimination even if we do not put the condition $A \subseteq N_G(u) \cap N_G(u')$ in the definition of detour extension. In this way, we can obtain a valid inequality for $\text{CUT}^\square(K_{1,n-2,n-2})$ from any valid inequality for CUT_n^\square . This “compact” construction allows much smaller m for the same n than Theorem 4.9. In other words, this construction allows much greater n for the same m , resulting in much more facets to apply this construction to. Without the condition on A , it can be proved that the dimension of the face obtained by triangular elimination is greater than or equal to the dimension of the original face. However, numerical tests shows that applying this conversion to most of the facets of CUT_n^\square give faces of $\text{CUT}^\square(K_{1,n-2,n-2})$ that are not facets. Because we are interested in computing facets of $\text{CUT}^\square(K_{1,m,m})$, we stick to detour extension with the condition on A .

Example 4.1. We show two facet-defining triangle inequalities for CUT_4^\square can be transformed into valid inequalities for $\text{CUT}^\square(K_{1,2,2})$. First, consider the triangle inequality

$$x_{01} \leq x_{02} + x_{12}. \quad (16)$$

The variable x_{12} is not a valid variable for $\text{CUT}^\square(K_{1,2,2})$, so we eliminate it by adding (16) to the triangle inequality

$$x_{12} \leq x_{14} + x_{24}$$

to obtain

$$x_{01} \leq x_{02} + x_{14} + x_{24}. \quad (17)$$

By Theorem 4.9 (i), this is a valid inequality for $\text{CUT}^\square(K_{1,2,2})$, but the face it supports is not a facet of $\text{CUT}^\square(K_{1,2,2})$. The inequality (17) gives a valid inequality

$$q_{(1,1),(2,2)} \leq 2q_{(2,1),(2,1)} + q_{(2,1),(2,2)}$$

for $\mathcal{B}^\square(2, 2, 2)$. Next consider the triangle inequality

$$x_{12} \leq x_{13} + x_{23}. \quad (18)$$

We again eliminate x_{12} adding

$$x_{14} \leq x_{12} + x_{24}$$

to obtain

$$x_{14} \leq x_{13} + x_{23} + x_{24}. \quad (19)$$

This time, (19) is not only valid for $\text{CUT}^\square(\mathcal{K}_{1,2,2})$, but it supports a facet of $\text{CUT}^\square(\mathcal{K}_{1,2,2})$. The inequality (17) gives a facet-defining inequality

$$q_{(1,1),(2,1)} \leq q_{(1,1),(1,1)} + q_{(2,1),(1,1)} + q_{(2,1),(2,1)}$$

for $\mathcal{B}^\square(2, 2, 2)$, which is known as the CHSH inequality [3].

Example 4.2. We show how a pentagon inequality, which defines a facet of CUT_5^\square can be transformed to a valid inequality for $\text{CUT}^\square(\mathcal{K}_{1,3,3})$. The pentagon inequality

$$x_{01} + x_{12} + x_{02} + x_{45} \leq x_{04} + x_{14} + x_{24} + x_{05} + x_{15} + x_{25}$$

is transformed by adding to it the triangle inequalities

$$x_{16} \leq x_{12} + x_{26} \quad \text{and} \quad x_{34} \leq x_{35} + x_{45}$$

to give the inequality

$$x_{01} + x_{02} + x_{16} + x_{34} \leq x_{04} + x_{14} + x_{24} + x_{05} + x_{15} + x_{25} + x_{26} + x_{35}. \quad (20)$$

This inequality gives the I_{3322} inequality [17] for $\mathcal{B}^\square(2, 3, 2)$.

Example 4.3. In Example 4.1, we considered two triangle inequalities. There is another case of triangle inequality which appears only in CUT_n^\square for $n \geq 6$. Let us consider the lifting of the triangle inequality

$$x_{13} \leq x_{12} + x_{23}. \quad (21)$$

for CUT_6^\square to $\text{CUT}^\square(\mathcal{K}_{1,5,5})$. We have to eliminate all of the three variables in (21) to obtain a valid inequality for $\text{CUT}^\square(\mathcal{K}_{1,5,5})$. To do this, we add the triangle inequalities

$$\begin{aligned} x_{17} &\leq x_{13} + x_{37}, \\ x_{12} &\leq x_{18} + x_{28}, \\ x_{23} &\leq x_{29} + x_{39} \end{aligned}$$

to obtain a valid inequality

$$x_{17} \leq x_{37} + x_{18} + x_{28} + x_{29} + x_{39} \quad (22)$$

for $\text{CUT}^\square(\mathcal{K}_{1,5,5})$. The face it supports is not a facet of $\text{CUT}^\square(\mathcal{K}_{1,5,5})$.

4.4 Equivalence of facets obtained by triangular elimination

Cut polytopes have many symmetries. If we know one facet of $\text{CUT}^\square(\mathcal{K}_{1,m,m})$, we can apply symmetric transformations to it to obtain many different facets. This leads to the question: ‘‘How many different classes of facets of $\text{CUT}^\square(\mathcal{K}_{1,m,m})$ are obtained by applying triangular elimination to facets of CUT_n^\square ?’’ In this subsection, we answer to this question by establishing a relation between the equivalence of facets of CUT_n^\square and the equivalence of their triangular eliminations.

4.4.1 Definitions on symmetry of cut polytopes

We need formal definitions to describe the symmetry of $\text{CUT}^\square(\mathcal{K}_{1,m,m})$.

Automorphism of graph An *automorphism* of a graph $G = (V, E)$ is a permutation σ on V such that

$$uv \in E \iff \sigma(u)\sigma(v) \in E.$$

The set of all the automorphisms of G is called the automorphism group of G and denoted by $\text{Aut}(G)$. For example, if G is a complete graph K_n , then its automorphism group $\text{Aut}(K_n)$ is the symmetric group \mathcal{S}_n of degree n .

In Section 3 we introduced the switching operation, which is one of the symmetries of the cut polytope. Another is permutation.

Permutation For a vector $\mathbf{x} \in \mathbb{R}^E$ and an automorphism $\sigma \in \text{Aut}(G)$ of G , the *permutation* of \mathbf{x} by σ is the vector $\mathbf{x}' \in \mathbb{R}^E$ defined by

$$x'_{uv} = x_{\sigma(u)\sigma(v)},$$

and denoted by $\sigma \cdot \mathbf{x}$.

For a vector $\mathbf{a} \in \mathbb{R}^E$, a scalar $a_0 \in \mathbb{R}$ and an automorphism $\sigma \in \text{Aut}(G)$, the *permutation* of the inequality $\mathbf{a}^T \mathbf{x} \leq a_0$ by σ is the inequality $(\sigma \cdot \mathbf{a})^T \mathbf{x} \leq a_0$. This permutation is valid for $\text{CUT}^\square(G)$ if and only if the original inequality $\mathbf{a}^T \mathbf{x} \leq a_0$ is valid for $\text{CUT}^\square(G)$. Similarly, the permutation supports a facet of $\text{CUT}^\square(G)$ if and only if the original inequality supports a facet of $\text{CUT}^\square(G)$.

For a facet f of $\text{CUT}^\square(G)$ supported by the inequality $\mathbf{a}^T \mathbf{x} \leq a_0$, the *permutation* of f by σ is the facet of $\text{CUT}^\square(G)$ supported by the permutation of $\mathbf{a}^T \mathbf{x} \leq a_0$ by σ , and denoted by $\sigma \cdot f$.

Facets of the same type Two facets f and f' of $\text{CUT}^\square(G)$ are *switching equivalent*, denoted by $f \approx f'$, if and only if there exists a set $S \subseteq V$ such that $\gamma(S) \cdot f = f'$.

Let \mathcal{G} be a subgroup of $\text{Aut}(G)$. Two facets f and f' of $\text{CUT}^\square(G)$ are \mathcal{G} -*permutation equivalent*, denoted by $f \sim_{\mathcal{G}} f'$, if and only if there exists an automorphism $\sigma \in \mathcal{G}$ of G such that $\sigma\gamma(S) \cdot f \approx f'$. In case of $\mathcal{G} = \text{Aut}(G)$, we say f and f' are *of the same type* instead of $\text{Aut}(G)$ -permutation equivalent, and denote this fact by $f \sim f'$.

Notation To keep the notations simple, we focus on the cases where n is odd in most of the rest of this subsection.

Let k be a natural number, $n = 2k + 1$ and $m = k + \binom{k}{2}$. Label the n vertices of $K_n = (V, E)$ by $X, A_1, \dots, A_k, B_1, \dots, B_k$, and the $2m + 1$ vertices of $K_{1,m,m}$ by

$$X; A_1, \dots, A_k, A'_1, \dots, A'_{\binom{k}{2}}; B_1, \dots, B_k, B'_1, \dots, B'_{\binom{k}{2}}.$$

Let $\langle \cdot \rangle: \left\{ \binom{1, \dots, k}{2} \right\} \rightarrow \left\{ 1, \dots, \binom{k}{2} \right\}$ be a bijection. Define the bijection $\iota: \left\{ \binom{A_1, \dots, A_k}{2} \right\} \cup \left\{ \binom{B_1, \dots, B_k}{2} \right\} \rightarrow \left\{ A'_1, \dots, A'_{\binom{k}{2}}, B'_1, \dots, B'_{\binom{k}{2}} \right\}$ by

$$\begin{aligned} \iota(A_i A_j) &= B'_{\langle i, j \rangle}, \\ \iota(B_i B_j) &= A'_{\langle i, j \rangle}. \end{aligned}$$

4.4.2 Switching equivalence

Theorem 4.10. *Let $G = (V, E)$ be a graph, and let $G^+ = (V^+, E^+)$ be the detour extension of G with removed edge $uw' \in E$, added vertex v and adjacent vertex set A . Let f and f' be facets of $\text{CUT}^\square(G)$, and F and F' be the facets of $\text{CUT}^\square(G^+)$ obtained as their triangular eliminations, respectively. Then,*

$$f \approx f' \iff F \approx F'.$$

Proof. Let the inequality supporting f, f', F and F' be

$$\begin{aligned} f: \mathbf{a}^T \mathbf{x} &\leq a_0, \\ f': \mathbf{a}'^T \mathbf{x} &\leq a'_0, \\ F: \mathbf{b}^T \mathbf{x} &\leq c_0, \\ F': \mathbf{b}'^T \mathbf{x} &\leq c'_0, \end{aligned}$$

respectively. By the definition of triangular elimination,

$$\begin{aligned}\mathbf{b}^T \mathbf{x} &= \mathbf{a}^T \mathbf{x} - a_{uu'}x_{uu'} + a_{uu'}x_{uv} - |a_{uu'}|x_{u'v}, \\ b_0 &= a_0, \\ \mathbf{b}'^T \mathbf{x} &= \mathbf{a}'^T \mathbf{x} - a'_{uu'}x_{uu'} + a'_{uu'}x_{uv} - |a'_{uu'}|x_{u'v}, \\ b'_0 &= a'_0.\end{aligned}$$

(\implies) Assume $f \approx f'$. Then there exists a set $S \subseteq V \setminus \{u'\}$ such that $f' = \gamma(S) \cdot f$. We will prove that $F' = \gamma(S) \cdot F$.

Since $f' = \gamma(S) \cdot f$, we have $\mathbf{a}' = \mathbf{a}^S$ and $a'_0 = a_0 - \mathbf{a}^T \delta(S)$. Now it is sufficient if we prove $\mathbf{b}' = \mathbf{b}^S$. First, for $ww' \in E'$ such that $w, w' \neq v$, note that $ww' \in E$ and

$$b'_{ww'} = a'_{ww'} = a_{ww'}^S = b_{ww'}^S.$$

Next,

$$\begin{aligned}b'_{uv} = a'_{uv} = a_{uv}^S &= \begin{cases} a_{uu'} = b_{uv} = b_{uv}^S & \text{if } u \notin S, \\ -a_{uu'} = -b_{uv} = b_{uv}^S & \text{if } u \in S, \end{cases} \\ b'_{u'v} = -|a'_{uu'}| &= -|a_{uu'}^S| = -|a_{uu'}| = b_{u'v} = b_{u'v}^S.\end{aligned}$$

Finally, for $w \in A$, we have $b_{vw} = b'_{vw} = 0$ which means $b'_{vw} = b_{vw}^S$. Putting these equations together, we conclude that $\mathbf{b}' = \mathbf{b}^S$.

(\Leftarrow) Assume $F' \approx F$. Then there exists a set $S' \subseteq V' \setminus \{u'\}$ such that $F' = \gamma(S') \cdot F$. Let $S = S' \setminus \{v\}$. Now we prove $f' = \gamma(S) \cdot f$.

Since $F' = \gamma(S') \cdot F$, we have $\mathbf{b}' = \mathbf{b}^{S'}$ and $a'_0 = a_0 - \mathbf{b}^T \delta(S') = a_0 - \mathbf{a}^T \delta(S)$. It is sufficient if we prove $\mathbf{a}' = \mathbf{a}^S$.

For each $ww' \in E \setminus \{uu'\}$, we have $ww' \in E'$ and

$$a'_{ww'} = b'_{ww'} = b_{ww'}^{S'} = a_{ww'}^S.$$

In addition,

$$a'_{uu'} = b'_{uv} = b_{uv}^S = \begin{cases} b_{uv} = a_{uu'} = a_{uu'}^S & \text{if } u \notin S', \\ -b_{uv} = -a_{uu'} = a_{uu'}^S & \text{if } u \in S', \end{cases}$$

and so $\mathbf{a}' = \mathbf{a}^S$. □

By applying Theorem 4.10 repeatedly, we obtain the following corollary.

Corollary 4.11. *Let f and f' be facets of CUT_n^\square , and F and F' be the facets of $\text{CUT}^\square(\mathbb{K}_{1,m,m})$ obtained by triangular elimination of f and f' , respectively. Then,*

$$f \approx f' \iff F \approx F'.$$

4.4.3 Switching permutation equivalence

Here we consider the switching permutation equivalence of the facets of $\text{CUT}^\square(\mathbb{K}_{1,m,m})$ obtained by triangular elimination of facets of CUT_n^\square .

Note that in relation to Bell polytopes, the switching operation in $\text{CUT}^\square(\mathbb{K}_{1,m,m})$ corresponds to the value exchange in $\mathcal{B}^\square(2, m, 2)$, and the permutation operation in $\text{CUT}^\square(\mathbb{K}_{1,m,m})$ corresponds to the party and observable exchange in $\mathcal{B}^\square(2, m, 2)$.

Let \mathcal{G}_1 be the subgroup of $\text{Aut}(\mathbb{K}_n)$ generated by

$$\mathcal{S}(\{A_1, \dots, A_k\}) \cup \mathcal{S}(\{B_1, \dots, B_k\}).$$

Define $\sigma_0 \in \text{Aut}(\mathbb{K}_n)$ by

$$\sigma_0 = \begin{pmatrix} A_1 & \cdots & A_k & B_1 & \cdots & B_k \\ B_1 & \cdots & B_k & A_1 & \cdots & A_k \end{pmatrix},$$

and let \mathcal{G} be the subgroup of $\text{Aut}(\mathbb{K}_n)$ generated by $\mathcal{G}_1 \cup \{\sigma_0\}$.

Theorem 4.12. *Let $n \geq 5$ be an odd number. Let f and f' be non-triangle facets of CUT_n^\square , and F and F' be the facets of $\text{CUT}^\square(K_{1,m,m})$ obtained by triangular elimination of f and f' , respectively. Then,*

$$f \sim_{\mathcal{G}} f' \iff F \sim F'.$$

First, we consider the case of $k = 1$. In this case, $n = 2k + 1 = 3$ and $m = k + \binom{k}{2} = 1$. This means that $\text{CUT}_n^\square = \text{CUT}_{1,m,m}^\square$ and triangular elimination does nothing. Theorem 4.12 is trivial in case of $k = 1$. In the rest of this section, we consider the case of $k > 1$.

Let $\mathcal{H} = \text{Aut}(K_{1,m,m})$, and let \mathcal{H}_1 be the subgroup of \mathcal{H} generated by

$$\mathcal{S}(\{A_1, \dots, A_k, A'_1, \dots, A'_{\binom{k}{2}}\}) \cup \mathcal{S}(\{B_1, \dots, B_k, B'_1, \dots, B'_{\binom{k}{2}}\}).$$

If we define $\tau_0 \in \mathcal{H}$ by

$$\tau_0 = \begin{pmatrix} A_1 & \cdots & A_k & A'_1 & \cdots & A'_{\binom{k}{2}} & B_1 & \cdots & B_k & B'_1 & \cdots & B'_{\binom{k}{2}} \\ B_1 & \cdots & B_k & B'_1 & \cdots & B'_{\binom{k}{2}} & A_1 & \cdots & A_k & A'_1 & \cdots & A'_{\binom{k}{2}} \end{pmatrix},$$

then \mathcal{H} is generated by $\mathcal{H}_1 \cup \{\tau_0\}$ since $k > 1$.

For $\sigma \in \mathcal{G}$, define $\tau \in \mathcal{H}$ by

$$\tau(u) = \begin{cases} \sigma(u) & (\text{if } u \in \{X, A_1, \dots, A_k, B_1, \dots, B_k\}), \\ \iota(\sigma(v)\sigma(w)) & (\text{if } u = \iota(vw)). \end{cases}$$

The mapping $\varphi: \mathcal{G} \rightarrow \mathcal{H}$ which maps each $\sigma \in \mathcal{G}$ to $\tau \in \mathcal{H}$ defined in this way is a homomorphism between groups. Let $\mathcal{H}' = \text{im } \varphi$ be the image of φ .

For now, we prove the following claim.

Claim 4.1. $F \sim_{\mathcal{H}} F' \implies F \sim_{\mathcal{H}'} F'$.

To prove this claim, we need some definitions to classify the vertices of the graph G according to their role with respect to any given facet of $\text{CUT}^\square(G)$.

Definition 4.4 (Irrelevant vertex of graph with respect to a facet). Let $G = (V, E)$ be a graph. A vertex $u \in V$ is *irrelevant* with respect to a facet $f: \mathbf{a}^\top \mathbf{x} \leq a_0$ of $\text{CUT}^\square(G)$ if and only if for any $v \in V$ such that $uv \in E$, we have $a_{uv} = 0$.

Definition 4.5 (Triangular facet at vertex of graph). Let $G = (V, E)$ be a graph, and let $u \in V$. A facet $f: \mathbf{a}^\top \mathbf{x} \leq a_0$ is *triangular* at u if and only if there exists two different vertices $v, v' \in V$ such that the following conditions are satisfied.

- (i) $uv, uv' \in E$, and $|a_{uv}| = |a_{uv'}| \neq 0$.
- (ii) For any $w \in V \setminus \{v, v'\}$ such that $uw \in E$, we have $a_{uw} = 0$.

In such a case, we call the two vertices v and v' the vertices *adjacent* to u .

Lemma 4.13. *Let f be a facet of CUT_n^\square . If f is not a triangle inequality, then f is not triangular at any vertex of K_n .*

Proof. The proof is by contradiction.

Let $K_n = (V, E)$. Suppose a facet $f: \mathbf{a}^\top \mathbf{x} \leq a_0$ of CUT_n^\square is not a triangle inequality, and it is triangular at $u \in V$. Let $v, v' \in V$ be the vertices adjacent to u . Then $|a_{uv}| = |a_{uv'}| \neq 0$. By switching f by an appropriate subset of $\{u, v, v'\}$, we can assume $a_{uv} = a_{uv'} = -\lambda < 0$ without loss of generality.

A triangle inequality

$$\lambda(-x_{uv} - x_{uv'} + x_{vv'}) \leq 0 \tag{23}$$

supports a facet of CUT_n^\square . Let $\mathbf{a}' \in \mathbb{R}^E$ be the vector which makes

$$\mathbf{a}'^\top \mathbf{x} = \mathbf{a}^\top \mathbf{x} + \lambda(x_{uv} + x_{uv'} - x_{vv'})$$

an identity. Since f is not a triangle inequality, $\mathbf{a}' \neq 0$.

We prove the inequality $\mathbf{a}'^T \mathbf{x} \leq a_0$ is valid for CUT_n^\square by showing that for any $S \subseteq V$, $\mathbf{a}'^T \delta(S) \leq a_0$. Let $\mathbf{x} = \delta(S)$. Define $S' \subseteq V$ by

$$S' = \begin{cases} S \cup \{u\} & (\text{if } v \in S, \\ S \setminus \{u\} & (\text{if } v \notin S, \end{cases}$$

and let $\mathbf{x}' = \delta(S')$. Since f is triangular at u adjacent to v and v' , the inequality $\mathbf{a}'^T \mathbf{x} \leq a_0$ does not have any terms that correspond to edges incident to u , which means $\mathbf{a}'^T \mathbf{x} = \mathbf{a}'^T \mathbf{x}'$. Because $x'_{uv} = x'_{uv'} - x'_{vv'} = 0$,

$$\mathbf{a}'^T \mathbf{x} = \mathbf{a}'^T \mathbf{x}' = \mathbf{a}^T \mathbf{x}' + \lambda(x'_{uv} + x'_{uv'} - x'_{vv'}) = \mathbf{a}^T \mathbf{x}' \leq a_0,$$

which means the inequality $\mathbf{a}'^T \mathbf{x} \leq a_0$ is valid for CUT_n^\square .

The inequality $\mathbf{a}^T \mathbf{x} \leq a_0$ is the sum of the inequality (23) and the inequality $\mathbf{a}'^T \mathbf{x} \leq a_0$. This means that f is not a facet, hence a contradiction. Therefore, f is not triangular at any vertex u . \square

Claim 4.1 is proved from Lemma 4.13 as follows.

Proof of Claim 4.1. Before the main part of the proof, consider the case when F is a triangle inequality. By $F \sim_{\mathcal{H}} F'$, F' is also a triangle inequality. Then $F \approx F'$, which trivially implies $F \sim_{\mathcal{H}'} F'$.

Now assume that F is not a triangle inequality. By $F \sim_{\mathcal{H}} F'$, F' is not a triangle inequality. First, we will prove the case of $\tau \in \mathcal{H}_1$.

Let $K_n = (V, E)$. We classify the elements of V by their roles in the facet F into three groups:

$$\begin{aligned} V_1 &= \{u \in V \mid u \text{ is irrelevant with respect to } F\}, \\ V_2 &= \{u \in V \mid F \text{ is triangular at } u\}, \\ V_3 &= V \setminus (V_1 \cup V_2). \end{aligned}$$

Note that $V = V_1 \cup V_2 \cup V_3$ is a partition of V into disjoint union. Similarly, let

$$\begin{aligned} V'_1 &= \{u \in V \mid u \text{ is irrelevant with respect to } F'\}, \\ V'_2 &= \{u \in V \mid F' \text{ is triangular at } u\}, \\ V'_3 &= V \setminus (V'_1 \cup V'_2). \end{aligned}$$

Since $F' \approx \tau \cdot F$,

$$\tau(V_1) = V'_1, \quad \tau(V_2) = V'_2, \quad \tau(V_3) = V'_3.$$

By Lemma 4.13,

$$V_2, V'_2 \subseteq \{A'_1, \dots, A'_{\binom{k}{2}}, B'_1, \dots, B'_{\binom{k}{2}}\}. \quad (24)$$

By the definition of triangular elimination,

$$\begin{aligned} \{A'_1, \dots, A'_{\binom{k}{2}}, B'_1, \dots, B'_{\binom{k}{2}}\} &\subseteq V_1 \cup V_2, \\ \{A'_1, \dots, A'_{\binom{k}{2}}, B'_1, \dots, B'_{\binom{k}{2}}\} &\subseteq V'_1 \cup V'_2, \end{aligned}$$

which means

$$V_3, V'_3 \subseteq \{A_1, \dots, A_k, B_1, \dots, B_k\}. \quad (25)$$

From the relations (24) and (25), there exist two permutations $\sigma \in \mathcal{G}_1$ and $\bar{\sigma} \in \mathcal{S}(\{A'_1, \dots, A'_{\binom{k}{2}}\}) \times \mathcal{S}(\{B'_1, \dots, B'_{\binom{k}{2}}\})$ such that

$$\begin{aligned} \sigma(u) &= \tau(u) \quad (\forall u \in V_3), \\ \bar{\sigma}(u) &= \tau(u) \quad (\forall u \in V_2). \end{aligned}$$

Since $(\sigma\bar{\sigma})(u) = \tau(u)$ for any $u \in V \setminus V_1$, $F' \approx \sigma\bar{\sigma} \cdot F$.

By comparing the coefficients of F' and $\varphi(\sigma) \cdot F$ in a similar way to the proof of Lemma 4.13, we have $F' \approx \varphi(\sigma) \cdot F$. This completes the proof in case of $\tau \in \mathcal{H}_1$.

In case of $\tau \notin \mathcal{H}_1$, τ can be written as $\tau = \tau' \tau_0$ by using some $\tau' \in \mathcal{H}_1$. Note that

$$F' \approx \tau \cdot F = \tau' \tau_0 \cdot F.$$

From what we already proved, there exists $\sigma' \in \mathcal{G}$ such that $F' \approx \varphi(\sigma') \tau_0 \cdot F$. Since $\tau_0 = \varphi(\sigma_0)$, we have $F' \approx \varphi(\sigma' \sigma_0) \cdot F$. This means $F \sim_{\mathcal{H}'} F'$ in case of $\tau \notin \mathcal{H}_1$. \square

Table 1: The number C_n of the classes of facets of $\text{CUT}^\square(\mathbb{K}_{1,m,m})$ of the same type obtained by applying triangular elimination to non-triangular facets of CUT_n^\square . The values of C_8 and C_9 depend on the conjecture that the lists of facets of CUT_8 and CUT_9 on the Web site [18] are complete.

n	5	6	7	8	9
m	3	5	6	9	10
C_n	1	6	63	16,234	44,368,793

The following claim is straightforward from the definition of triangular elimination.

Claim 4.2. For $\sigma \in \mathcal{G}$, $f' \approx \sigma \cdot f \iff F' \approx \varphi(\sigma) \cdot F$.

Theorem 4.12 immediately follows Claims 4.1 and 4.2. It answers the question we posed at the beginning of Section 4.4 in cases where n is odd. For example, CUT_7^\square has 67 different classes of \mathcal{G} -permutation equivalent facets, where 4 out of them are triangle inequalities:

- $x_{XA_1} - x_{XB_1} - x_{A_1B_1} \leq 0$, which is itself a facet-supporting inequality of $\text{CUT}^\square(\mathbb{K}_{1,m,m})$ and corresponds to the trivial inequality of $\mathcal{B}^\square(2, m, 2)$.
- $x_{A_1A_2} - x_{A_1B_1} - x_{A_2B_1} \leq 0$, like (18) in Example 4.1, whose triangular elimination gives a facet-supporting inequality of $\text{CUT}^\square(\mathbb{K}_{1,m,m})$ and corresponds to the CHSH inequality of $\mathcal{B}^\square(2, m, 2)$.
- $x_{XA_1} - x_{XA_2} - x_{A_1A_2} \leq 0$, like (16) in Example 4.1, whose triangular elimination does not support a facet of $\text{CUT}^\square(\mathbb{K}_{1,m,m})$.
- $x_{A_1A_3} - x_{A_1A_2} - x_{A_2A_3} \leq 0$, like (21) in Example 4.3, whose triangular elimination does not support a facet of $\text{CUT}^\square(\mathbb{K}_{1,m,m})$.

This means $\text{CUT}^\square(\mathbb{K}_{1,6,6})$ has 63 different classes of facets of the same type which can be obtained by applying triangular elimination to non-triangular facets of CUT_7^\square .

In cases where n is even, we need special care to define what corresponds to the subgroup \mathcal{G} of $\text{Aut}(\mathbb{K}_n)$. Let $n = 2k$ and label the n vertices of \mathbb{K}_n by $X, A_1, \dots, A_k, B_1, \dots, B_{k-1}$. We can define \mathcal{H}_1 and τ_0 in the same way as the cases where n is odd, and $\mathcal{H}_1 \cup \{\tau_0\}$ generates the group $\text{Aut}(\mathbb{K}_{1,m,m})$. The problem is that when n is even, there does not exist $\sigma_0 \in \text{Aut}(\mathbb{K}_n)$ such that $\varphi(\sigma_0) = \tau_0$. Therefore, we take a different approach. We regard \mathbb{K}_n as a subgraph of \mathbb{K}_{n+1} which has an extra vertex B_k . For any facet f of CUT_n^\square , its 0-lifting \bar{f} is a facet of CUT_{n+1}^\square by the 0-lifting theorem [6]. We say two facets f and f' of CUT_n^\square are *equivalent* if and only if their 0-lifting \bar{f} and \bar{f}' satisfy $\bar{f} \sim_{\mathcal{G}} \bar{f}'$, where \mathcal{G} is the subgroup of $\text{Aut}(\mathbb{K}_{n+1})$ defined above. Let F and F' be the facet of $\text{CUT}^\square(\mathbb{K}_{1,m,m})$ obtained by applying triangular elimination to f and f' , respectively. Similarly, let \bar{F} and \bar{F}' be the facet of $\text{CUT}^\square(\mathbb{K}_{1,m+1,m+1})$ obtained by applying triangular elimination to \bar{f} and \bar{f}' , respectively. Then \bar{F} and \bar{F}' is the 0-lifting of F and F' . This means that F and F' are of the same type if and only if \bar{F} and \bar{F}' are of the same type. Therefore, the following fact holds.

Corollary 4.14. Let $n \geq 4$ be an even number. Let f and f' be non-triangle facets of CUT_n^\square , and F and F' be the facets of $\text{CUT}^\square(\mathbb{K}_{1,m,m})$ obtained by triangular elimination of f and f' , respectively. Then,

$$f \text{ and } f' \text{ are equivalent} \iff F \sim F'.$$

By Theorem 4.12 and Corollary 4.14, we can compute the number of the classes of facets of $\text{CUT}^\square(\mathbb{K}_{1,m,m})$ of the same type obtained by applying triangular elimination to non-triangular facets of CUT_n^\square . We consulted De Simone, Deza and Laurent [7] for the H-representation of CUT_7 , and the ‘‘conjectured complete description’’ of CUT_8 and the ‘‘description possibly complete’’ of CUT_9 in SMAPO [18]. The result is summarized in Table 1.

5 Tight Bell inequalities from triangular elimination

As stated in previous section, triangular elimination preserves facet supporting property (Theorem 4.9 (ii)) and inequivalence property under known isomorphisms (Corollary 4.14), which correspond to party, observable and value exchanges [20]. As a consequence, we can obtain a large number of tight Bell inequalities.

In this section, we compile the results of triangular elimination in the form of Bell inequalities. Throughout the rest of this section, we use the term “family” as set of Bell inequalities, on the other hand, the term “class” as set of facets of cut polytope. In addition, we denote $q_{(j,2),(j',2)}$ and $q_{(j,2),(j',2)} + q_{(j,2),(j',1)}$ as $q_{A_j B_{j'}}$ and q_{A_j} respectively, and define $q_{B_{j'}}$ similarly. Then, terms of the left hand side of inequality are arrayed in the format introduced by Collins and Gisin [4]; each row corresponds to coefficients of each observable of party A and each column corresponds to that of party B . Because of switching equivalence, we can assume that the right hand side of inequality are always zero. The example of the CHSH $q_{A_1 B_1} + q_{A_1 B_2} + q_{A_2 B_1} - q_{A_2 B_2} \leq 0$ is arrayed in the form as follows:

$$\left(\begin{array}{c|c|c} \hline \hline 1 & 1 & \\ \hline 1 & -1 & \\ \hline \hline \end{array} \right). \quad (26)$$

Note that the complete graph K_{2k+1} for some k has symmetric group \mathcal{S}_{2k+1} as its automorphism group, on the other hand, the complete tripartite graph $K_{1,k,k}$ has only the subgroup of \mathcal{S}_{2k+1} . Therefore, some classes of facets which are originally equivalent as a facet class of $\text{CUT}_{2k+1}^\square$ under permutation and switching can define inequivalent families of Bell inequalities.

5.1 Family of tight Bell inequalities obtained from triangular elimination of hypermetric facet

In the case of cut polytope of complete graph, some explicit classes of valid inequalities are known, for example, the hypermetric, clique-web and gap inequalities [8, Part V]. For these classes of inequalities, some sufficient conditions to be facet supporting are also known. Therefore through triangular elimination, we can obtain families of tight Bell inequalities from them.

First, we give a new family of tight Bell inequalities found by applying triangular elimination to the hypermetric inequality class [8, Chapter 28]. One of this family, namely the triangular eliminated $(2k+1)$ -gonal inequality, contains three previously known Bell inequalities: the trivial inequalities like $q_{A_1} \leq 1$, the well known CHSH inequality found by Clauser, Horne, Shimony and Holt [3] and the inequality named I_{3322} by Collins and Gisin [4], originally found by Pitowsky and Svozil [17].

Let $n = 2k + 1$ for some k , $\mathbf{a} \in \mathbb{Z}^{\{A_1, \dots, A_k\}}$, $\mathbf{b} \in \mathbb{Z}^{\{B_1, \dots, B_k\}}$ be integer weight vectors for each observable and $c \in \mathbb{Z}$ satisfying $c + \sum_{j=1}^k a_{A_j} + \sum_{j'=1}^k b_{B_{j'}} = 1$. Because of equivalence under observable exchange, we can assume that without loss of generality, the elements of \mathbf{a}, \mathbf{b} are sorted in some manner. Similarly, exchange of \mathbf{a} and \mathbf{b} does not yields new family.

Then the following inequality is always a valid Bell inequality:

$$\begin{aligned} & \sum_{j=1}^k (1 - a_{A_j} - 2 \sum_{j'=1}^{j-1} a_{A_{j'}}) a_{A_j} q_{A_j} + \sum_{j=1}^k (1 - b_{B_j} - 2 \sum_{j'=1}^{j-1} b_{B_{j'}}) b_{B_j} q_{B_j} \\ & \quad - 2 \sum_{1 \leq j, j' \leq k} a_{A_j} b_{B_{j'}} q_{A_j B_{j'}} \\ & \quad - 2 \sum_{uu' \in \binom{k}{2}} a_{A_u} a_{A_{u'}} (q_{A_u B_{v_{uu'}}} - q_{A_{u'} B_{v_{uu'}}}) - 2 \sum_{uu' \in \binom{k}{2}} b_{B_u} b_{B_{u'}} (q_{A_{v_{uu'}} B_u} - q_{A_{v_{uu'}} B_{u'}}) \leq 0, \quad (27) \end{aligned}$$

where $v_{uu'}$ is the added vertex in each step of detour extension corresponding to uu' .

The following $\mathbf{a}, \mathbf{b}, c$ are known to be facet supporting [8, Section 28.2]:

$$(\mathbf{a}^T, \mathbf{b}^T, c) =$$

1. (the triangular elimination of $(2k+1)$ -gonal inequality)
 - (a) $((1, 1, \dots, 1), (-1, -1, \dots, -1), 1)$
 - (b) $((1, 1, \dots, 1), (0, -1, -1, \dots, -1), 0)$
2. (a) $((a_1, \dots, a_{p_a}, -1, \dots, -1), (b_1, \dots, b_{p_b}, -1, \dots, -1), -1)$
for $3 \leq p_a + p_b \leq n - 3$, $a_1, \dots, a_{p_a}, b_1, \dots, b_{p_b} \geq 0$.
 - (b) $((a_1, \dots, a_{p_a}, -1, \dots, -1), (b_1, \dots, b_{p_b}, -1, \dots, -1), c)$
for $2 \leq p_a + p_b \leq n - 4$, $a_1, \dots, a_{p_a}, b_1, \dots, b_{p_b}, c \geq 0$.

Note that in the case of the triangular elimination of $(2k + 1)$ -gonal inequality, each element of weight vectors is always 0, 1 or -1 . From 0-lifting theorem and permutation-switching equivalence, it is straightforward that there is only above two families which can be the inequivalent triangular elimination of $(2k + 1)$ -gonal inequality. Moreover, for these two families, family with $c \neq 0$ (1(a)) contains the trivial ($k = 1$) and I_{3322} ($k = 2$), on the other hand, family with $c = 0$ (1(b)) contains the CHSH ($k = 2$).

For example, $\mathcal{B}^\square(2, 6, 2)$ has the following facet:

$$\left(\begin{array}{c|ccc|cc} & -1 & -2 & -3 & & \\ \hline & 1 & 1 & 1 & -1 & -1 \\ -1 & 1 & 1 & 1 & 1 & -1 \\ -2 & 1 & 1 & 1 & & 1 & 1 \\ \hline -3 & -1 & 1 & & & & \\ & -1 & & 1 & & & \\ & & -1 & -1 & & & \end{array} \right). \quad (28)$$

5.2 Other families of tight Bell inequality

There are more general classes of facets in cut polytope of the complete graph. Of these classes, the clique-web inequalities contains hypermetric inequalities as a special case. There are also known sufficient conditions for clique-web inequalities be facet supporting.

For example, the *pure* clique-web inequality is facet supporting [8, Section 29.4]. Using triangular elimination, for $m \geq 7$ we can also obtain families of tight Bell inequalities like

$$\left(\begin{array}{c|cccc|cccc} & & & & & -1 & -2 & -2 & -2 \\ \hline -3 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \hline & -1 & & & 1 & & & & \\ & -1 & & & & 1 & & & \\ & & -1 & & & & 1 & & \\ & & -1 & & & & & 1 & \\ & & & -1 & & & & & 1 \\ & & & & -1 & & & & 1 \end{array} \right). \quad (29)$$

Note that because of the complexity of the structure, there are large number of triangular eliminated clique-web facets which are equivalent in the CUT_n^\square but not equivalent as families of tight Bell inequalities. These separated families are induced by the original classes and the embedding into the $\text{CUT}^\square(K_{1,m,m})$.

5.3 Relationship between I_{mm22} and triangular eliminated Bell inequality

Collins and Gisin [4] proposed a family of tight Bell inequalities obtained by the extension of CHSH and I_{3322} as I_{mm22} family, and conjectured that I_{mm22} is always facet supporting (they also confirmed that for $m \leq 7$, I_{mm22} is actually facet supporting by computation). Therefore, whether their I_{mm22} can be obtained by triangular elimination of some facet class of CUT_n^\square is an interesting question.

The I_{mm22} family has the structure as follows:

$$\left(\begin{array}{c|ccc|ccc} & & & -1 & & & \\ \hline -(m-1) & 1 & \cdots & 1 & 1 & 1 & \\ -(m-2) & & \cdots & 1 & 1 & -1 & \\ -(m-3) & 1 & \cdots & 1 & -1 & & \\ \vdots & \vdots & \cdots & \ddots & & & \\ 0 & 1 & -1 & & & & \end{array} \right). \quad (30)$$

From its structure, it is straightforward that if I_{mm22} can be obtained by triangular elimination of some facet class of CUT_n^\square , then only A_m and B_m are detour vertices, since other vertices have degree more than 2. However, there is no known corresponding facet class of cut polytope in this form. For specific values, by computation, we found that corresponding facets for $m = 2, 3, 4$ are the triangle, pentagonal and Grishukhin [11] inequality, respectively.

5.4 Facet of $\mathcal{B}^\square(2, m, 2)$ other than the triangular elimination of CUT_n^\square

Since we have obtained a large number of tight Bell inequalities by triangular elimination of CUT_n^\square , the next question is whether they are complete i.e., whether all families and its equivalents form whole facets of $\mathcal{B}^\square(2, m, 2)$.

For $m = 3$ case, this is affirmative. There is only three kinds of facet families: the trivial, CHSH and I_{3322} [4, 17], corresponding to the triangle facet, the triangular elimination of the triangle facet and the triangular elimination of the pentagonal facet of CUT_n^\square , respectively.

On the other hand, for $m \geq 4$, the answer is negative because there is facet, found by facet enumeration of $\mathcal{B}^\square(2, 4, 2)$ by lrs [1], such as

$$\left(\begin{array}{c|ccc} & & -1 & 1 & -1 \\ \hline & 2 & 1 & 1 & \\ \hline 1 & & 1 & -1 & -1 \\ 1 & 1 & & -1 & 1 \\ 1 & -1 & 1 & & 1 \end{array} \right). \quad (31)$$

This facet have no vertex with degree 2, therefore, it is not the triangular elimination of any facet.

6 NP-completeness of membership testing

In this section, we consider the computational complexity of the problem to determine whether a given correlation table $\mathbf{q} \in \mathbb{R}^{(mv)^n}$ is induced by some classical (n, m, v) -system or not. This problem is rephrased in polytopal terminology as follows.

Membership test for Bell polytope

Instance: A positive integer m and a vector $\mathbf{q} \in \mathbb{Q}^{(2m)^2}$.

Question: Is $\mathbf{q} \in \mathcal{B}^\square(2, m, 2)$?

Theorem 6.1. *The membership test for the Bell polytope is NP-complete in the sense of polynomial-time Turing reducibility¹.*

The membership test for Bell polytope is in NP by Carathéodory's theorem. The proof of NP-hardness can be sketched as follows. First, we prove the affine isomorphism between Bell polytopes $\mathcal{B}^\square(2, m, 2)$ and cut polytopes of tri-partite graphs $K_{1,m,m}$. Next, we prove the weighted maximum cut problem on $K_{1,m,m}$ is NP-complete. Finally, we prove the weighted maximum cut problem on $K_{1,m,m}$ is polynomial-time Turing reducible to the membership test of cut polytopes of $K_{1,m,m}$ in a similar way to the proof of the NP-hardness of the membership test of cut polytopes of complete graphs.

6.1 NP-completeness of weighted maximum cut on $K_{1,m,m}$

Weighted maximum cut

Instance: A graph $G = (V, E)$, an integer vector $\mathbf{w} \in \mathbb{Z}^E$, and an integer k .

Question: Is there a subset $C \subseteq V$ that satisfies the condition

$$\sum_{\substack{u \in C \\ v \in V \setminus C}} w_{uv} \geq k?$$

Weighted maximum cut is NP-complete. See Garey and Johnson [10] for a proof.

Let $G = (V, E)$ be a graph and $H = (V', E')$ be a minor of G . Weighted maximum cut on H is easier than on G because we can reduce the former to the latter by assigning 0 to removed edges and $-M$ to contracted edges, where M is any integer greater than $\sum_{e \in E'} t_e$.

Any graph with m vertices is a subgraph (therefore a minor) of K_m , which is a minor of $K_{m,m}$, which is a subgraph (therefore a minor) of $K_{1,m,m}$. This means that weighted maximum cut is still NP-complete if we restrict G to be in the form of $K_{1,m,m}$.

¹Pitowsky [16] shows that the membership test for correlation polytope of a given graph G is NP-complete in the sense of polynomial-time Karp reducibility by reduction from one-in-three SAT.

6.2 Turing reduction from weighted maximum cut to membership test of cut polytopes

Avis and Deza [2] show the membership test to cut polytopes of complete graphs is NP-hard. This implies the NP-hardness of the membership test to correlation polytopes of complete graphs.

In a similar way, we can prove the following theorem.

Theorem 6.2. *The membership test to $\text{CUT}^\square(K_{1,m,m})$ is NP-hard.*

Proof. Let $K_{1,m,m} = (V, E)$ and $P = \text{CUT}^\square(K_{1,m,m})$. The optimization of a linear function on P is equivalent to the weighted maximum cut problem on $K_{1,m,m}$, so it is NP-complete. We know P is included in a hypercube $[0, 1]^E$ and P includes a hypercube $[\frac{1}{\binom{2m+1}{2}+1}, \frac{1}{\binom{2m+1}{2}}]^E$. The proof is completed by the polynomial-time Turing reduction from optimization problem to membership problem given by Corollary 4.3.12 and Theorem 6.3.2 (a) in [12]. \square

Now the proof of Theorem 6.1 is obtained by combining Theorems 3.3 and 6.2.

7 Projections of Bell polytope and lifting their faces back

Let $\varphi: V \rightarrow U$ be an affine mapping between two affine spaces U and V . If two polytopes $P \subseteq U$ and $Q \subseteq V$ satisfy $\varphi(Q) = P$, an inequality

$$\mathbf{a}^\top \mathbf{u} \leq a_0 \quad (32)$$

is valid for P if and only if the inequality

$$\mathbf{a}^\top \varphi(\mathbf{v}) \leq a_0 \quad (33)$$

is valid for Q . When (32) is valid for P , we call the valid inequality (33) for Q the *lifting* of (32) by the affine mapping φ .

For (33) to support a facet of Q , it is necessary for (32) to support a facet of P . Whether it is sufficient or not depends on P , Q and φ . If it is, we can obtain some of the facets of Q by lifting the facets of the polytope P with lower dimension, which are hopefully computed easily. Note that we cannot obtain all the facets of Q this way; if Q has at least one vertex², Q has at least one facet which is not in the form of (33).

In this section, we consider lifting by several specific projections for the case where Q is a Bell polytope.

7.1 Projection from $\mathcal{B}^\square(n+1, m, v)$ to $\mathcal{B}^\square(n, m, v)$

Fix any $j = 1, \dots, m$. Let $\varphi: \mathbb{R}^{(mv)^{n+1}} \rightarrow \mathbb{R}^{(mv)^n}$ be a linear mapping which maps every point $\mathbf{q}' \in \mathbb{R}^{(mv)^{n+1}}$ to a point $\mathbf{q} \in \mathbb{R}^{(mv)^n}$ defined by

$$q_{(j_1, k_1), \dots, (j_n, k_n)} = \sum_{k=1}^v q'_{(j_1, k_1), \dots, (j_n, k_n), (j, k)}.$$

Note that if $\mathbf{q}' \in \mathcal{B}^\square(n+1, m, v)$, $\varphi(\mathbf{q}')$ does not depend on the choice of j ³. By the definition of $\mathcal{B}^\square(n, m, v)$, $\varphi(\mathcal{B}^\square(n+1, m, v)) = \mathcal{B}^\square(n, m, v)$.

This means that if an inequality

$$\mathbf{a}^\top \mathbf{q} \leq a_0 \quad (34)$$

is valid for $\mathcal{B}^\square(n, m, v)$, the inequality

$$\mathbf{a}^\top \varphi(\mathbf{q}') \leq a_0 \quad (35)$$

obtained as the lifting of (34) by φ is valid for $\mathcal{B}^\square(n+1, m, v)$. For example, let us consider the CHSH inequality [3]:

$$q_{(1,1),(1,1)} - q_{(1,1),(2,2)} - q_{(2,2),(2,1)} - q_{(2,1),(1,1)} \leq 0,$$

²We exclude the trivial case where the restriction of φ to Q is injective.

³This is because any correlation table in $\mathcal{B}^\square(n+1, m, v)$ satisfies the no-signaling condition.

which supports a facet of $\mathcal{B}^\square(2, 2, 2)$. The lifting of the CHSH inequality by φ is the inequality

$$\begin{aligned} & (q_{(1,1),(1,1),(1,1)} + q_{(1,1),(1,1),(1,2)}) - (q_{(1,1),(2,2),(1,1)} + q_{(1,1),(2,2),(1,2)}) \\ & - (q_{(2,2),(2,1),(1,1)} + q_{(2,2),(2,1),(1,2)}) - (q_{(2,1),(1,1),(1,1)} + q_{(2,1),(1,1),(1,2)}) \leq 0. \end{aligned} \quad (36)$$

By the fact described above, the inequality (36) is valid for $\mathcal{B}^\square(3, 2, 2)$. However, the inequality (36) does not support a facet of $\mathcal{B}^\square(3, 2, 2)$ by the following theorem.

Theorem 7.1. *The inequality (35) never supports a facet of $\mathcal{B}^\square(n+1, m, v)$.*

Outline of proof. By normalization condition (1), we can assume $a_0 = 0$ without loss of generality.

Let F be the face of $\mathcal{B}^\square(n, m, v)$ supported by (34), and F' be the face of $\mathcal{B}^\square(n+1, m, v)$ supported by (35). For $\mathbf{c} \in \{1, \dots, v\}^m$, define $\mathbf{a}^{\mathbf{c}} \in \mathbb{R}^{(mv)^{n+1}}$ by

$$a_{(j_1, k_1), \dots, (j_n, k_n), (j_{n+1}, k_{n+1})}^{\mathbf{c}} = \begin{cases} a_{(j_1, k_1), \dots, (j_n, k_n)} & \text{if } k_{n+1} = c_{j_{n+1}}, \\ 0 & \text{otherwise.} \end{cases}$$

Then $(\mathbf{a}^{\mathbf{c}})^\top \mathbf{q}' = 0$ for any vertex \mathbf{q}' of F' and any $\mathbf{c} \in \{1, \dots, v\}^m$. This means that the face F' of $\mathcal{B}^\square(n+1, m, v)$ lies in the intersection of the hyperplanes $\mathbf{a}^{\mathbf{c}} \mathbf{q}' = 0$, and therefore F' is not a facet of $\mathcal{B}^\square(n+1, m, v)$. \square

For example, the CHSH inequality is not a facet-supporting inequality for $\mathcal{B}^\square(n, 2, 2)$ for any $n > 2$.

7.2 Projection from $\mathcal{B}^\square(n, m+1, v)$ to $\mathcal{B}^\square(n, m, v)$

By using a linear mapping $\varphi: \mathbb{R}^{((m+1)v)^n} \rightarrow \mathbb{R}^{(mv)^n}$ which maps every point $\mathbf{q}' \in \mathbb{R}^{((m+1)v)^n}$ to a point $\mathbf{q} \in \mathbb{R}^{(mv)^n}$ defined by

$$q_{(j_1, k_1), \dots, (j_n, k_n)} = q'_{(k_1, j_1), \dots, (k_n, j_n)},$$

we have $\varphi(\mathcal{B}^\square(n, m+1, v)) = \mathcal{B}^\square(n, m, v)$. Therefore, if an inequality

$$\mathbf{a}^\top \mathbf{q} \leq a_0 \quad (37)$$

is valid for $\mathcal{B}^\square(n, m, v)$, its lifting by φ

$$\mathbf{a}^\top \varphi(\mathbf{q}') \leq a_0 \quad (38)$$

is valid for $\mathcal{B}^\square(n, m+1, v)$. In contrast to the previous case of lifting to $\mathcal{B}^\square(n+1, m, v)$, this time the following theorem holds.

Theorem 7.2. *In case of $n = v = 2$, if the inequality (37) supports a facet of $\mathcal{B}^\square(n, m, v)$, its lifting (38) supports a facet of $\mathcal{B}^\square(n, m+1, v)$.*

Outline of proof. The proof is immediate from the affine isomorphism of Bell polytopes and cut polytopes:

$$\begin{aligned} \mathcal{B}^\square(2, m, 2) & \cong_{\text{aff}} \text{CUT}^\square(\mathbf{K}_{1, m, m}), \\ \mathcal{B}^\square(2, m+1, 2) & \cong_{\text{aff}} \text{CUT}^\square(\mathbf{K}_{1, m+1, m+1}), \end{aligned}$$

and the 0-lifting theorem of cut polytope of graphs [6]. \square

For example, the CHSH inequality supports a facet of $\mathcal{B}^\square(2, m, 2)$ for any $m \geq 2$.⁴

7.3 Projection from $\mathcal{B}^\square(n, m, v+1)$ to $\mathcal{B}^\square(n, m, v)$

We define a mapping $\psi: \{1, \dots, v+1\} \rightarrow \{1, \dots, v\}$ by

$$\psi(k') = \begin{cases} k' & \text{if } 1 \leq k' \leq v-1, \\ v & \text{if } k' \in \{v, v+1\}. \end{cases}$$

⁴As one direction of research, we can study the characteristics of facets of $\mathcal{B}^\square(2, m+1, 2)$ which do not appear in $\mathcal{B}^\square(2, m, 2)$.

By using a linear mapping $\varphi: \mathbb{R}^{(m(v+1))^n} \rightarrow \mathbb{R}^{(mv)^n}$ which maps every point $\mathbf{q}' \in \mathbb{R}^{(m(v+1))^n}$ to the point $\mathbf{q} \in \mathbb{R}^{(mv)^n}$ defined by

$$q_{(j_1, k_1), \dots, (j_n, k_n)} = \sum_{\substack{\mathbf{k}' \in \{1, \dots, v+1\}^n \\ \psi(k'_i) = k_i \ (1 \leq i \leq n)}} q'_{(j_1, k'_1), \dots, (j_n, k'_n)},$$

we see that $\varphi(\mathcal{B}^\square(n, m, v+1)) = \mathcal{B}^\square(n, m, v)$.

We have not yet determined whether this operation is facet-preserving.

7.4 Projection from correlation tables to correlation functions

For every $\mathbf{q} \in \mathbb{R}^{(mv)^n}$, consider a point $\mathbf{s} \in \mathbb{R}^{m^n v}$ defined by

$$s_{j_1 \dots j_n k} = \sum_{\substack{\mathbf{k} \in \{1, \dots, v\}^n \\ k_1 + \dots + k_n \equiv k \pmod{v}}} q_{(j_1, k_1), \dots, (j_n, k_n)},$$

and denote this point \mathbf{s} by $\varphi(\mathbf{q})$. This defines a linear mapping $\varphi: \mathbb{R}^{(mv)^n} \rightarrow \mathbb{R}^{m^n v}$.

For every correlation table $\mathbf{q} \in \mathcal{B}^\square(n, m, v)$ induced by a classical (n, m, v) -system, the point $\varphi(\mathbf{q})$ is called the *full correlation function* (in [20] in $(n, 2, 2)$ case) or *correlation function* (in [5] in $(2, 2, v)$ case) defined by \mathbf{q} .

Let $\mathcal{W}(n, m, v) = \varphi(\mathcal{B}^\square(n, m, v))$. Werner and Wolf [20] show that $\mathcal{W}(n, 2, 2)$ is affinely isomorphic to the 2^n -dimensional crosspolytope. In $(2, 2, 2)$ case, the facets of the crosspolytope correspond to trivial and CHSH inequalities of $\mathcal{B}^\square(2, 2, 2)$, and the facets of the crosspolytope Werner and Wolf consider can be seen as generalization of these inequalities. By Theorem 7.1, none of them are lifted to facets of $\mathcal{B}^\square(n, 2, 2)$. Collins, Gisin, Linden, Massar and Popescu [5] give a valid inequality for $\mathcal{W}(2, 2, v)$, which is later called the CGLMP inequality, and Masanes [13] shows that the lifting of the CGLMP inequality by φ is a facet of $\mathcal{B}^\square(2, 2, v)$. It is not known whether the lifting of a facet of $\mathcal{W}(n, m, v)$ always supports a facet of $\mathcal{B}^\square(n, m, v)$.

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