Polyhedral and semidefinite approaches to classical and quantum Bell inequalities

David Avis[1] * Tsuyoshi Ito[2] †

[1] *School of Computer Science, McGill University, 3480 University, Montreal, Quebec, Canada H3A 2A7.*
[2] *Department of Computer Science, The University of Tokyo, 7-3-1 Hongo, Bunkyo-ku, Tokyo 113-0033, Japan.*

Abstract. In this paper we explore further the connections between convex bodies related to quantum correlation experiments with dichotomic variables and related bodies studied in combinatorial optimization, especially cut polyhedra. Such a relationship was established in Avis, Imai, Ito and Sasaki (*J. Phys. A: Math. Gen.* 38 10971–10987, 2005) with respect to Bell inequalities. We show that several well known bodies related to cut polyhedra are equivalent to bodies such as those defined by Tsirelson (*Hadronic J. S.* 8 329–345, 1993) to represent hidden deterministic behaviors, quantum behaviors, and no-signaling behaviors. Among other things, our results allow a unique representation of these bodies, give a necessary condition for vertices of the no-signaling polytope, and give a method for bounding the quantum violation of Bell inequalities by means of a body that contains the set of quantum behaviors. Optimization over this latter body may be performed efficiently by semidefinite programming.

Keywords: Bell inequalities, quantum behaviors, the cut polytope, the no-signaling polytope, semidefinite programming

1 Introduction

Classical and quantum Bell inequalities play an important role in understanding the existence and the limit of quantum nonlocality [12]. In [12], Tsirelson compares and investigates the convex sets $X_{\mathrm{HDB}} \subseteq X_{\mathrm{QB}} \subseteq X_{\mathrm{B}}$: the polytope X_{HDB} consisting of local deterministic behaviors, the set X_{QB} of quantum behaviors, and the polytope X_{B} consisting of all behaviors satisfying the nonnegativity, the normalization, and the no-signaling conditions. We focus on behaviors over the behavior scheme $(\overbrace{2,\dots,2}^{m};\overbrace{2,\dots,2}^{n})$, where two parties have m (≥ 2) and n (≥ 2) ± 1-valued measurements. We let $X_{\mathrm{HDB}} = X_{\mathrm{HDB}}(m,n)$, $X_{\mathrm{QB}} = X_{\mathrm{QB}}(m,n)$, $X_{\mathrm{B}} = X_{\mathrm{B}}(m,n)$. In this case, a behavior is specified by $4mn$ nonnegative reals $q_{ab|ij}$ which represent the probability with which A_i measures to a and B_j to b simultaneously for $1 \leq i \leq m$, $1 \leq j \leq n$, $a,b \in \{\pm 1\}$, which satisfy the normalization condition $\sum_{a,b\in\{\pm1\}} q_{ab|ij} = 1$ and the no-signaling conditions $\sum_{b\in\{\pm1\}} q_{ab|ij} = \sum_{b\in\{\pm1\}} q_{ab|ij'}$ for any a, i, i' and $\sum_{a\in\{\pm1\}} q_{ab|ij} = \sum_{a\in\{\pm1\}} q_{ab|i'j}$ for any a, j, j'.

A (classical) Bell inequality can be seen as a linear inequality valid for $X_{\mathrm{HDB}}(m,n)$, whereas a quantum Bell inequality can be seen as a linear inequality valid for $X_{\mathrm{QB}}(m,n)$. However, classical and quantum Bell inequalities in this form are cumbersome for certain purposes because, as is pointed out by Froissart [8], adding any linear combination of the normalization and no-signaling conditions to an inequality gives apparently different representations of essentially the same inequality.

The cut polytope is a convex polytope studied intensively in combinatorial optimization. A book [7] by Deza and Laurent is a definite reference on the cut polytope, and readers are referred to it for its definition and properties. Avis, Imai, Ito, Sasaki [2] point out that the polytope $X_{\mathrm{HDB}}(m,n)$ is isomorphic to the cut polytope $\mathrm{Cut}(\nabla \mathrm{K}_{m,n})$ of a certain graph $\nabla \mathrm{K}_{m,n}$ via an affine mapping. In this paper, we apply the same mapping to $X_{\mathrm{QB}}(m,n)$ and $X_{\mathrm{B}}(m,n)$, and show their relationships to the rooted semimetric polytope $\mathrm{RMet}(\nabla \mathrm{K}_{m,n})$ and the elliptope $\mathcal{E}(\nabla \mathrm{K}_{m,n})$, which arise as linear and semidefinite relaxations of the cut polytope, respectively.

The result presented in this extended abstract will be described in more detail in an upcoming paper [3].

*avis@cs.mcgill.ca
†tsuyoshi@is.s.u-tokyo.ac.jp

2 Affine isomorphism

We describe the affine isomorphism from $X_{\mathrm{HDB}}(m,n)$ to $\mathrm{Cut}(\nabla \mathrm{K}_{m,n})$. Here $\mathrm{K}_{m,n} = (V_{m,n}, E_{m,n})$ denotes the complete bipartite graph with the vertices $V_{m,n} = \{\mathrm{A}_1,\dots,\mathrm{A}_m,\mathrm{B}_1,\dots,\mathrm{B}_n\}$ and the edges $E_{m,n} = \{\mathrm{A}_i\mathrm{B}_j \mid 1 \leq i \leq m,\ 1 \leq j \leq n\}$, and $\nabla \mathrm{K}_{m,n} = (\nabla V_{m,n}, \nabla E_{m,n})$ denotes its suspension graph obtained from $\mathrm{K}_{m,n}$ by adding a new vertex X and $m+n$ edges XA_i and XB_j. The affine isomorphism from $X_{\mathrm{HDB}}(m,n)$ to $\mathrm{Cut}(\nabla \mathrm{K}_{m,n})$ maps $\boldsymbol{q} \in \mathbb{R}^{4mn}$ to a vector $\boldsymbol{x} \in \mathbb{R}^{\nabla E_{m,n}}$ defined by $x_{\mathrm{XA}_i} = \sum_{a,b\in\{\pm1\}} a q_{+1,b|ij}$,[1] $x_{\mathrm{XB}_j} = \sum_{a,b\in\{\pm1\}} a q_{a,+1|ij}$, $x_{\mathrm{A}_i\mathrm{B}_j} = \sum_{a,b\in\{\pm1\}} ab q_{+1,+1|ij}$. We denote the image of $X_{\mathrm{QB}}(m,n)$ under this isomorphism by $\mathcal{Q}_{\mathrm{Cut}}(m,n)$.

3 Correlation functions and Tsirelson's theorem

The *correlation function* $x_{\mathrm{A}_i\mathrm{B}_j}$ is the expected value of the product ab of the outcomes of measurements by two parties. Considering only the mn correlation functions is equivalent to considering the projection of the vector $\boldsymbol{x}$ from $\mathbb{R}^{\nabla E_{m,n}}$ to $\mathbb{R}^{E_{m,n}}$ via the standard projection π.

[1] The right hand side of this equation does not depend on j due to the no-signaling condition. The same holds for the next equation and i.

Tsirelson's theorem (Theorem 1 in [5]) can be stated in terms of the elliptope. The *elliptope* $\mathcal{E}(G)$ of a graph $G = (V, E)$ with $n = |V|$ nodes is the convex body consisting of vectors $\boldsymbol{x} \in \mathbb{R}^E$ such that there exist a unit vector $\boldsymbol{u}_i$ in $\mathbb{R}^n$ for each node $i \in V$ satisfying $x_{ij} = \boldsymbol{u}_i \cdot \boldsymbol{u}_j$, see Section 28.4 of [7]. By using the notion of the elliptope, Tsirelson's theorem can be stated as $\pi(\mathcal{Q}_{\mathrm{Cut}}(m,n)) = \mathcal{E}(\mathrm{K}_{m,n})$.

It is well-known that the elliptope can also be characterized by using nonnegative definite matrices called Gram matrices (see e.g. Section 28.4.1 of [7]). This allows one to maximize an arbitrary linear function over the elliptope efficiently by using the interior-point method, and therefore to compute the maximum quantum violation of Bell inequalities involving only the correlation functions, as is pointed out by [6].

Since $\pi(\mathrm{Cut}(\nabla \mathrm{K}_{m,n})) = \mathrm{Cut}(\mathrm{K}_{m,n})$, a vector $\boldsymbol{y} \in \mathbb{R}^{E_{m,n}}$ represents the correlation functions in some hidden deterministic behavior if and only if $\boldsymbol{y} \in \mathrm{Cut}(\mathrm{K}_{m,n})$.

As is pointed out by Tsirelson [12], Grothendieck [9] proves that for a vector $\boldsymbol{y} \in \mathbb{R}^{E_{m,n}}$ to belong to $\mathcal{E}(\mathrm{K}_{m,n})$, it is necessary that the vector $\boldsymbol{z} \in \mathbb{R}^{E_{m,n}}$ defined by $z_{\mathrm{A}_i\mathrm{B}_j} = (2/\pi) \arcsin y_{\mathrm{A}_i\mathrm{B}_j}$ belongs to $\mathrm{Cut}(\mathrm{K}_{m,n})$. Tsirelson asks in [12] whether this is also sufficient for $m = n = 2$. This condition is known under the name *cut condition* [7, Section 31.3.1] in combinatorial optimization, and by Laurent's result [10], the cut condition for $\mathcal{E}(\mathrm{K}_{m,n})$ is sufficient if and only if $\min\{m, n\} \le 2$. This answers Tsirelson's question affirmatively.

4 The no-signaling polytope and the rooted semimetric polytope

The image of X_{B} under the isomorphism stated in Section 1 is identical to the polytope referred to as the rooted semimetric polytope $\mathrm{RMet}(\nabla \mathrm{K}_{m,n})$. It follows immediately from [11] that the coordinates of the vertices of $X_{\mathrm{B}}(m,n)$ are in $\{0, 1/2, 1\}$.

If we only specify the mn correlation functions $x_{\mathrm{A}_i\mathrm{B}_j} \in [-1, 1]$, there always exists $\boldsymbol{x} \in \mathrm{RMet}(\nabla \mathrm{K}_{m,n})$. The implication of this is that all correlations between observables A_i, B_j are possible under the no-signaling condition alone.

In a related work [4], Barrett, Linden, Massar, Pironio, Popescu and Roberts investigate the vertices of the no-signaling polytope with two k-outcome observables per party.

5 Implication of Tsirelson's theorem on $\mathcal{Q}_{\mathrm{Cut}}(m,n)$

As stated earlier, Tsirelson's theorem can be stated as $\pi(\mathcal{Q}_{\mathrm{Cut}}(m,n)) = \mathcal{E}(\mathrm{K}_{m,n})$. A question arises what $\mathcal{Q}_{\mathrm{Cut}}(m,n)$ looks like. The following theorem partially answers this question.

Theorem 1 $\mathcal{Q}_{\mathrm{Cut}}(m,n) \subseteq \mathcal{E}(\nabla \mathrm{K}_{m,n}) \cap \mathrm{RMet}(\nabla \mathrm{K}_{m,n})$.

Since linear functions can be optimized efficiently over $\mathcal{E}(\nabla \mathrm{K}_{m,n})$ by the interior-point method, Theorem 1 can be used to give an upper bound of the maximum quantum violation of any Bell inequality.

Like Tsirelson's theorem, Theorem 1 can be used to test quantum mechanics itself. As Froissart pointed out [8], finding violation of Bell inequalities does not prove or disprove quantum mechanics. Theorem 1 provides a stronger test than Tsirelson's theorem, since there is $\boldsymbol{x} \in \mathrm{RMet}(\nabla \mathrm{K}_{m,n}) \setminus \mathcal{E}(\nabla \mathrm{K}_{m,n})$ such that $\pi(\boldsymbol{x}) \in \mathcal{E}(\mathrm{K}_{m,n})$ for all $m, n \ge 2$ [1].

Acknowledgment

The authors would like to thank Nicolas Gisin for discussions at the ERATO Conference on Quantum Information Science (EQIS2005), Aug. 26–30, 2005, Tokyo, where he suggested we turn our attention to correlation inequalities. We also thank Hiroshi Imai for helpful discussions and comments. Research of the first author is supported by N.S.E.R.C. and F.Q.R.N.T., and the second author is grateful for a support by the Grant-in-Aid for JSPS Fellows.

References

[1] D. Avis and T. Ito. In preparation.

[2] D. Avis, H. Imai, T. Ito, and Y. Sasaki. Two-party Bell inequalities derived from combinatorics via triangular elimination. *J. Phys. A: Math. General*, 38 (50):10971–10987, 2005.

[3] D. Avis, H. Imai, and T. Ito. On the relationship between convex bodies related to correlation experiments with dichotomic observables. arXiv:quant-ph/0605148, 2006. To appear in *J. Phys. A: Math. General.*

[4] J. Barrett, N. Linden, S. Massar, S. Pironio, S. Popescu, and D. Roberts. Nonlocal correlations as an information-theoretic resource. *Phys. Rev. A*, 71(022101), 2005.

[5] B. S. Cirel'son. Quantum generalizations of Bell's inequality. *Lett. Math. Phys.*, 4(2):93–100, 1980.

[6] R. Cleve, P. Høyer, B. Toner, and J. Watrous. Consequences and limits of nonlocal strategies. In *Proc. of 19th IEEE Conference on Computational Complexity*, pages 236–249, 2004.

[7] M. M. Deza and M. Laurent. *Geometry of Cuts and Metrics*, vol. 15 of *Algorithms and Combinatorics*. Springer, 1997.

[8] M. Froissart. Constructive generalization of Bell's inequalities. *Nuovo cimento*, 64B(2):241–251, 1981.

[9] A. Grothendieck. Résumé de la théorie métrique des produits tensoriels topologiques. *Boletim da Sociedade de Matemática de São Paulo*, 8:1–79, 1953.

[10] M. Laurent. The real positive semidefinite completion problem for series-parallel graphs. *Linear Algebra Appl.*, 252(1–3):347–366, 1997.

[11] M. Padberg. The Boolean quadric polytope: Some characteristics, facets and relatives. *Math. Program.*, 45(1):139–172, 1989.

[12] B. S. Tsirelson. Some results and problems on quantum Bell-type inequalities. *Hadronic J. Supplement*, 8(4):329–345, 1993.