On the Existence of a Point Subset with a Specified Number of Interior Points

David AVIS*

School of Computer Science, McGill University,
3480 University, Montreal, Quebec, Canada, H3A 2A7

Kiyoshi HOSONO†

and

Masatsugu URABE‡

Department of Mathematics, Tokai University,
3-20-1 Orido, Shimizu, Shizuoka, 424-8610 Japan

February 1, 2000

Abstract

An *interior* point of a finite point set is a point of the set that is not on the boundary of the convex hull of the set. For any integer $k \geq 1$, let $g(k)$ be the smallest integer such that every set of points in the plane, no three collinear, containing at least $g(k)$ interior points has a subset of points containing exactly k interior points. We prove that $g(1) = 1$, $g(2) = 4$, $g(3) \geq 8$, and $g(k) \geq k + 2, k \geq 4$. We also give some related results.

1 Introduction

Throughout the paper we consider only planar point sets in which no three points are collinear. For such a point set P we distinguish its *vertices*, which lie on the boundary of its convex hull, from the remaining *interior* points. In 1935, Erdős and Szekeres [1] proved that for every $t \geq 3$ there is a number

**e-mail address*: avis@mutt.cs.mcgill.ca

†*e-mail address*: hosono@scc.u-tokai.ac.jp

‡*e-mail address*: qzg00130@scc.u-tokai.ac.jp

$f(t)$ such that every set P of at least $f(t)$ points, no three collinear, contains a subset of points whose convex hull contains precisely t *vertices*. The convex set may contain additional interior points. Somewhat surprisingly, Horton [2] has shown that it is not possible in general to specify both the number of hull vertices and number of interior points: there exist point sets with arbitrary size and no subset with t vertices and zero interior points, for each $t \geq 7$. In this paper we ask whether it is possible to specify the required number of interior points, leaving the number of vertices unspecified. Specifically, we investigate the question of when P contains a subset of points whose convex hull contains precisely k interior points of P. For any integer $k \geq 1$ let $g(k)$ be the smallest integer such that every set of points in the plane, no three collinear, containing at least $g(k)$ interior points has a subset whose convex hull contains exactly k interior points. We will show that $g(1) = 1$ and $g(2) = 4$. We do not know if $g(k)$ is finite for $k \geq 3$.

Let $\{v_1, v_2, \ldots, v_m\}$ be the vertices and $\{p_1, p_2, \ldots, p_s\}$ be the interior points of a point set P. When indexing a set of t points, we identify indices modulo t. We define $f(m, k)$ to be the smallest integer such that every point set with m vertices and at least $f(m, k)$ interior points has a subset of points whose convex hull contains precisely k interior points of P. Clearly

$$g(k) = \sup_m f(m, k).$$

To investigate $f(m, k)$ it is useful to consider *deficient* point sets $P = P(m, s, k)$, which are sets with m vertices and s interior points which do not contain a subset of points with precisely k interior points. It is trivial to construct deficient point sets when $s < k$, so we assume $s \geq k \geq 1$ throughout the paper. The existence of a deficient point set $P(m, s, k)$ implies that $f(m, k) \geq s + 1$ and hence $g(k) \geq s + 1$. The following theorem gives some bounds on $f(m, k)$.

Theorem 1.
(a) $f(m, k) \geq k + 2$, *for* $m \geq k + 1 \geq 3$.
(b) $f(m - t, k) \leq f(m, k) \leq f(m - t, k) + f(t + 2, k) - 1$, *for* $m - 3 \geq t \geq 1$, $k \geq 1$.
(c) $f(3, k) \leq f(m, k) \leq (m - 2)f(3, k) - m + 3$, *for* $k \geq 1$.
(d) *If P contains a subset with $k \geq 2$ interior points, then it contains a convex t-gon with exactly k interior points and $t \leq 2k$.*

From the definitions of f and g, Theorem 1(a) implies that $g(k) \geq k + 2$ for each $k \geq 2$. Theorem 1(c) motivates the study of the case $m = 3$, since it implies that $f(m, k)$ is finite for $m \geq 4$ if and only if $f(3, k)$ is finite. Theorem 1(d) implies that if $f(m, k)$ is finite, the desired convex subset can be found with at most $2k$ vertices. The next two theorems give exact values of $g(k)$ when $k = 1$ and 2 respectively.

Theorem 2. *Let p be an interior point of a planar point set P. There exists a triangle in P containing only p, that is $g(1) = 1$.*

Theorem 3. *Every planar point set with $m \geq 3$ vertices and at least 4 interior points contains a triangle or a convex quadrilateral with 2 interior points, that is $g(2) = 4$. In particular, when $m = 3$, there exists a triangle with 2 interior points.*

Figure 1 shows that for $m = 4$, a quadrilateral is sometimes necessary. There are deficient point sets $P(3, s, 3)$ for $4 \leq s \leq 7$, so $g(3) \geq f(3, 3) \geq 8$. Figures 2 (a) and (b) show $P(3, 4, 3)$ and $P(3, 7, 3)$ respectively. Although we do not know if $g(k)$ is finite for $k \geq 3$, we can prove the following weaker results.

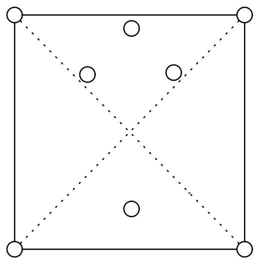
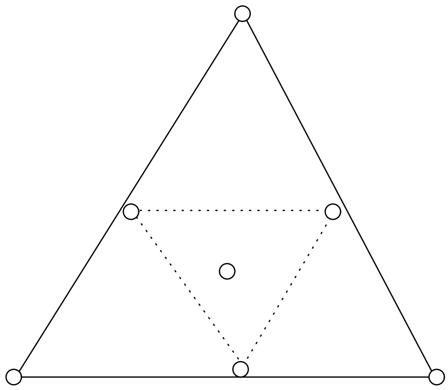
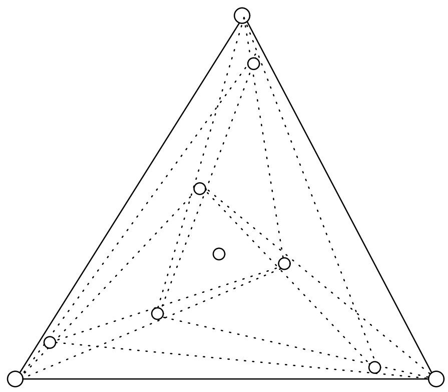


Figure 1: No triangle contains 2 interior points.



(a) $P(3, 4, 3)$



(b) $P(3, 7, 3)$

Figure 2:

Theorem 4. *Every planar point set with at least k interior points contains a convex subset with between k and $\lfloor 3k/2 \rfloor$ interior points.*

Corollary 5. *Every planar point set with at least 3 interior points contains a convex subset with either 3 or 4 interior points.*

It might be expected that this corollary would be useful in determining $g(3)$, but its usefulness seems to be limited by the existence of deficient point sets $P(m,4,3)$ for $m \geq 3$ (see Section 2).

2 Proofs

To prove the results we will make use of the following lemmas.

Extension Lemma. *Every deficient point set $P(m,s,k)$ can be extended to a deficient point set $P(m+1,s,k)$.*

(*Proof*) Let $P = P(m,s,k)$ and let v be any vertex. Place a new convex hull vertex u near v such that the remaining points of P are in the same angular order about u as they are about v creating the point set Q. If Q has a convex subset with k interior points, this subset must contain vertex u. By the order condition, u can be replaced by v creating a subset of P with k interior points, a contradiction. Hence Q is a deficient point set $P(m+1,s,k)$. □

Using the Extension Lemma, we can extend the examples of the previous section to see that there are deficient point sets $P(m,s,3)$ for each choice of $m \geq 3$ and $4 \leq s \leq 7$.

Reduction Lemma. *Let P be a planar point set with $m \geq 3$ vertices and $s \geq 0$ interior points. There exists a point v_i on the convex hull of P, s.t. $P' = P \backslash \{v_i\}$ has m' vertices, s' interior points and either:*
(a) $m' = m - 1$ and $s' = s$ or
(b) $m' = m + t$ and $s' = s - t - 1$, where $t = 0, 1, ..., \lfloor s/m \rfloor$.

(*Proof*) If there is a triangle $\triangle v_{i-1}v_iv_{i+1}$ of P containing no interior points, we can delete vertex v_i and case (a) applies. We assume therefore that each of these triangles contains an interior point. Take the closest point to each of the m edges of the boundary of the convex hull of P. Suppose first that one point u is closest to two consecutive edges $v_{i-1}v_i$ and v_iv_{i+1}. Since $\triangle v_{i-1}v_iv_{i+1}$ is non empty, u must lie in this triangle. If we delete v_i, u becomes a new vertex and (b) holds with $t = 0$. Otherwise we have m distinct closest points. Look at the triangles formed at each vertex with its two neighbouring closest edge points. An example for $m = 5$ is given in Figure 3. By the pigeon hole principle, some triangle T has at most $\lfloor (s-m)/m \rfloor$ interior points. Let v_i be the vertex of P defining this triangle, and let $P' = P \backslash \{v_i\}$. Since $\triangle v_{i-1}v_iv_{i+1}$ is non empty, at least one point of triangle T (interior point or vertex) must be on the convex hull of P'. On the other hand, the only new points that can appear on the convex hull of P' are the vertices or interior points of T, which number at most $2 + \lfloor (s-m)/m \rfloor$. Therefore the net increase in convex hull vertices is at most $1 + \lfloor (s-m)/m \rfloor = \lfloor s/m \rfloor$. □

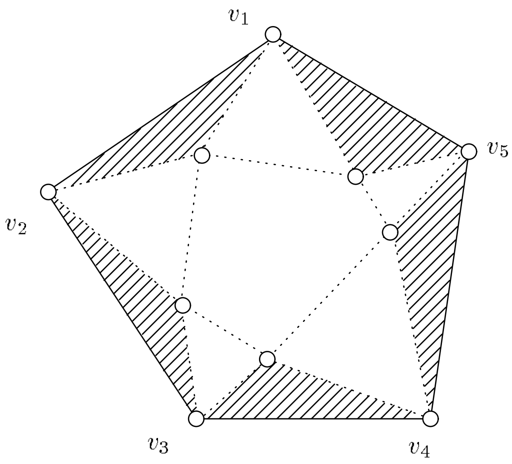


Figure 3: Shaded area are empty.

Theorem 1.
(a) $f(m,k) \geq k+2$, *for* $m \geq k+1 \geq 3$.
(b) $f(m-t,k) \leq f(m,k) \leq f(m-t,k)+f(t+2,k)-1$, *for* $m-3 \geq t \geq 1$, $k \geq 1$.
(c) $f(3,k) \leq f(m,k) \leq (m-2)f(3,k)-m+3$, *for* $k \geq 1$.
(d)*If P contains a subset with* $k \geq 2$ *interior points, then it contains a convex* t*-gon with exactly* k *interior points and* $t \leq 2k$.

(*Proof*) (a) We first exhibit deficient point sets $P(k+1,k+1,k)$ for each $k \geq 2$. Let P be a set of $2k+2$ points constructed from an equilateral $(k+1)$-gon by placing an interior point near to the midpoint of each of its edges. If we delete any vertex v of P, we are left with a convex $(k+2)$-gon, $P \backslash \{v\}$ containing $k-1$ interior points. Therefore we cannot find a subset of P with k interior points. Hence P is a deficient point set $P(k+1,k+1,k)$. By repeated application of the Extension Lemma, we can extend this example to a deficient point set $P(m,k+1,k)$ for any $m \geq k+1$.

(b) To show the upper bound, consider the $(m-t)$-gon $v_1, v_2, ..., v_{m-t}$ and the $(t+2)$-gon $v_1, v_{m-t}, v_{m-t+1}, ..., v_m$. The required convex subset will be found if either the $(m-t)$-gon contains at least $f(m-t,k)$ interior points, or the $(t+2)$-gon contains at least $f(t+2,k)$ interior points. One of these two cases must occur if P contains $f(m-t,k)+f(t+2,k)-1$ interior points.

For the lower bound, by applying the Extension Lemma t times, a deficient point set $P(m-t,s,k)$ can be extended to a deficient point set $P(m,s,k)$.

(c) To show the upper bound, create a star triangulation of P by joining hull vertex v_1 to each of the other hull vertices, creating $m-2$ triangles. The required convex subset will be found if any of the triangles contains at least $f(3,k)$ interior points. This will occur if P has at least $(m-2)(f(3,k)-1)+1$ interior points.

The lower bound is just part (b) with $t = m - 3$.

(d) Choose a convex subset with the minimum number t of vertices, $u_1, ..., u_t$, that has precisely k interior points, and assume, by contradiction, that $t \geq 2k+1 \geq 5$. Consider the t triangles $\triangle u_{i-1}u_iu_{i+1}$. No three of these triangles can have an interior point in common. Furthermore, if for some i, $\triangle u_{i-1}u_iu_{i+1}$ does not contain an interior point, we could delete u_i, reducing t by one. Since each interior point lies in at most two triangles, $t \leq 2k$ contradicting our assumption. □

Theorem 2. *Let p be an interior point of a planar point set P. There exists a triangle in P containing only p, that is $g(1) = 1$.*

(*Proof*) Let p be an interior point of P. Consider a minimum area triangle $T = \triangle abc$, with vertices from P, that contains p. We claim that T contains no other interior point. For otherwise, if q is another interior point of T, extend the half line qp until it intersects an edge, say ab, of T. Now $\triangle abq$ contains p, and has smaller area than T, a contradiction. □

Before proving Theorem 3, we first give the following lemma. An interior point of a triangle T is called a (x, y, z)-*splitter of* T if it partitions T into triangles with $x \geq y \geq z$ interior points, respectively. In case $z = 0$, we abbreviate this to (x, y)-*splitter*. Note that this will happen if the splitter is the closest point to one of the edges of T.

Lemma 1. $f(3, 2) \leq 4$.

(*Proof*) Let P be a point set with three points v_1, v_2, v_3 on the convex hull and $s \geq 4$ interior points, $p_1, p_2, ..., p_s$. We consider several cases, and in each show that a triangle can be found containing precisely two interior points.

(*Case 1*) $s = 4$:

If some interior point is a $(2, 1)$-splitter we are done. Otherwise, since the closest point to each edge of the convex hull is a $(3, 0)$-splitter, there must exist at least two distinct $(3, 0)$-splitters, say p_1 and p_2. Let L be the line through p_1 and p_2. Label the points so that L intersects edges v_1v_2 and v_1v_3, and so that p_1 is closer to v_1v_2 than p_2 is, as in Figure 4. The remaining interior points p_3 and p_4 must lie in both $\triangle v_1p_1v_3$ and $\triangle v_1p_2v_2$. If they both lie on the same side of the line as v_1, then $\triangle v_1p_1p_2$ contains precisely these two points. If they both lie on the same side of the line as v_2, then $\triangle v_2p_1p_2$ contains the two points, since $\triangle v_2p_2v_3$ is empty. Now suppose p_3 lies on the same side as v_1 and p_4 lies on the same side as v_2. Either $\triangle v_1v_2p_4$ contains p_1 and p_3, or $\triangle v_1v_3p_4$ contains p_3 and p_2.

(*Case 2*) $s = 5$:

If some interior point of P is a $(2, 1, 1)$- or $(2, 2)$-splitter, we are done immediately, and if it is a $(4, 0)$-splitter we are done by Case 1. We can therefore assume that every interior point of P is a $(3, 1)$-splitter. There is one interior point, say p_1, such that there are two other interior points on each side of the

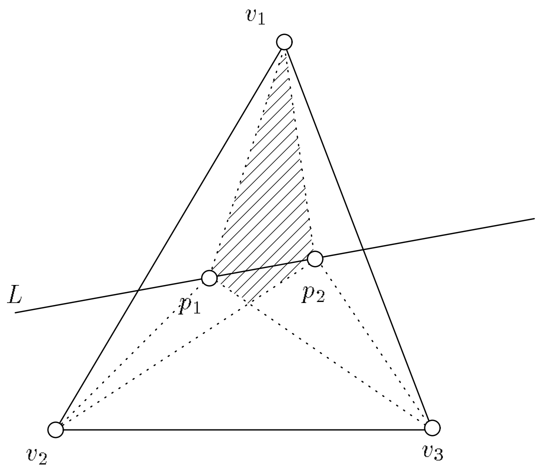


Figure 4: p_3, p_4 lie inside shaded area.

line v_1p_1. Since p_1 is a $(3,1)$-splitter, $\triangle p_1v_2v_3$ contains three interior points. We may assume without loss of generality that two of them, say p_2 and p_3 lie on the same side of the line v_1p_1 as v_2, and the other one, say p_4 lies on the other side. The remaining point, p_5 lies in $\triangle v_1p_1v_3$. If p_4 is a $(2,0)$-splitter of $\triangle p_1v_2v_3$ we are done, so it must be a (1,1)-splitter (see Figure 5). Let p_2 be the point in $\triangle p_1p_4v_2$. Now if p_5 lies on the same side of the line v_1p_4 as v_3, then $\triangle v_1p_4v_2$ contains the two points p_1 and p_2. Otherwise $\triangle v_1p_4p_2$ contains the two points p_1 and p_5.

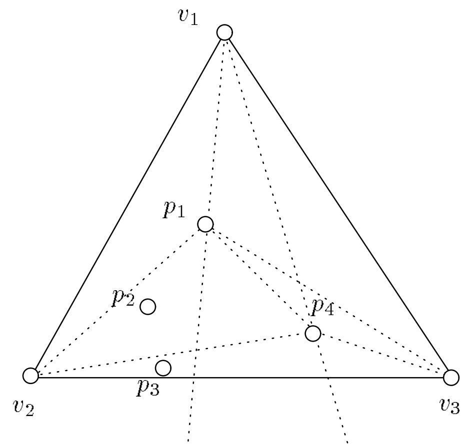


Figure 5: p_5 lies in $\triangle v_1p_1v_3$.

(*Case 3*) $s = 6$:

Take any zero splitter. If it is a $(3,2)$-splitter we are done immediately, otherwise it is a $(4,1)$-splitter or $(5,0)$-splitter reducing to Case 1 or 2 respectively.

(*Case 4*) $s = 7$:

If some interior point is a (4,2)- or (3,2,1)- or (2,2,2)-splitter we are done immediately. Moreover, if it is a (4,1,1)- (5,1)- or (6,0)-splitter we may reduce the problem to Case 1, 2 or 3 respectively. So we may assume every interior point is a $(3,3)$-splitter. It follows that for some edge of the triangle, say v_1v_2, there exist at least two interior points p_1, p_2 such that $\triangle p_1v_1v_2$ and $\triangle p_2v_1v_2$ are empty. We may label the two interior points so that p_1 lies in $\triangle v_1p_2v_3$. Since p_1 is a $(3,3)$-splitter, $\triangle p_1v_1v_3$ contains three interior points, and so $\triangle v_1p_2v_3$ contains at least four interior points, contradicting the assumption that p_2 is a (3,3)-splitter.

(*Case 5*) $s \geq 8$:

Choose a (x,y)-splitter with $x + y = s - 1,\ x \geq y$. Then

$$s - 1 \geq x \geq \left\lceil \frac{s-1}{2} \right\rceil \geq 4,$$

so we reduce to a smaller case. □

Theorem 3. *Every planar point set with $m \geq 3$ vertices and at least 4 interior points contains a triangle or a convex quadrilateral with 2 interior points, that is $g(2) = 4$. In particular, when $m = 3$, there exists a triangle with 2 interior points.*

(*Proof*) The inequality $f(m,2) \geq 4$ follows from Theorem 1(a), and the inequality $f(3,2) \leq 4$ holds by Lemma 1. We prove the upper bound for $m \geq 4$ by induction on m.

Consider $\triangle v_1v_2v_3$. If this is empty we delete v_2 obtaining a point set with $m-1$ vertices and the same number of interior points. If the triangle has two interior points we are done immediately. If it has at least four we are done by Lemma 1. If it has three interior points, consider the closest point u to edge v_1v_2. If u is a (2,0)-splitter we are done, else u is a (1,1)-splitter. In this case we delete v_2 and the point in $\triangle uv_2v_3$, and relabel u to be v_2. It now follows that $\triangle v_1v_2v_3$ contains exactly one interior point. Now we find the first non empty triangle $T = \triangle v_1v_iv_{i+1}$, with $i \geq 3$. This must exist since we started with at least four interior points. If T has either two, or more than four points, we are done as before. If it has one point, then the convex polygon $v_1v_2..v_{i+1}$ has two interior points. Finally if T has three points, we take the closest interior point w to edge v_1v_{i+1}. Like u, it must be a (1,1)-splitter or we are done. Now $v_1v_2...v_iw$ is convex with two interior points. □

Theorem 4. *Every planar point set with at least k interior points contains a convex subset with between k and $\lfloor 3k/2 \rfloor$ interior points.*

(*Proof*) Let Q be the smallest cardinality subset of P with at least k interior points. Suppose Q has m vertices and s interior points. If $s > \lfloor 3k/2 \rfloor$ we use the Reduction Lemma to get a contradiction. Indeed this is immediate if (a) holds in the lemma. So suppose (b) holds. Since $m \geq 3$, and $s \geq \lfloor 3k/2 \rfloor + 1$, we find a point set P' with at least

$$s - \lfloor s/m \rfloor - 1 \geq s - \lfloor s/3 \rfloor - 1 = \lceil 2s/3 \rceil - 1 \geq \left\lceil \frac{2\lfloor 3k/2 \rfloor + 2}{3} \right\rceil - 1 \geq k$$

interior points. This is a contradiction to the choice of Q. $\square$

3 Acknowledgements

We would like to thank János Pach for pointing out Theorem 4 to us, strengthening a result in an earlier draft of the paper.

References

[1] P. Erdős and G. Szekeres, A Combinatorial Problem in Geometry, *Compositio Mathematica* 2, 463–470 (1935).

[2] J. Horton, Sets with No Empty 7-gons, *Canad. Math. Bull.* 26, 482–484 (1983).