On the Existence of a Point Subset with 4 or 5 Interior Points

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1 Abstract

An *interior* point of a finite planar point set is a point of the set that is not on the boundary of the convex hull of the set. For any integer $k \geq 1$, let $h(k)$ be the smallest integer such that every set of points in the plane, no three collinear, containing at least $h(k)$ interior points has a subset of points containing k or $k+1$ interior points. We proved that $h(3) = 3$ in an earlier paper. In this paper we prove that $h(4) = 7$.

2 Introduction

Throughout the paper we consider only planar point sets in which no three points are collinear. For such a point set P we distinguish its *vertices* which lie on the boundary of its convex hull, from the remaining *interior* points. In 1935, Erdős and Szekeres [2] proved that for every $t \geq 3$ there is a number

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$f(t)$ such that every set P of at least $f(t)$ points, no three collinear, contains a subset of points whose convex hull contains precisely t *vertices*. Horton [3] has shown that it is not possible in general to specify both the number of hull vertices and number of interior points: there exist point sets with arbitrary size and no subset with t vertices and zero interior points, for each $t \geq 7$. In [1], we investigated the question of when P contains a subset of points whose convex hull contains precisely k *interior* points of P. For any integer $k \geq 1$ let $g(k)$ be the smallest integer such that every set of points in the plane, no three collinear, containing at least $g(k)$ interior points has a subset whose convex hull contains exactly k interior points. Then we showed that $g(1) = 1$ and $g(2) = 4$, but were unable to determine if $g(k)$ is finite for $k \geq 3$.

Let $\{v_1, v_2, \ldots, v_m\}$ be the vertices and $\{p_1, p_2, \ldots, p_s\}$ be the interior points of a point set P. Let S be a subset of P. By $CH(S)$ we refer to the convex hull of S. The *interior points of* S are the interior points of P that are also interior points of $CH(S)$. If S has three points, say x, y, z, then we use the notation $\triangle xyz$ for $CH(S)$. Note that if S has interior point(s) and a vertex x, then $CH(S \setminus x)$ has at least as many vertices as $CH(S)$. When indexing a set of t points, we identify indices modulo t. For any integer $k \geq 1$, let $h(k)$ be the smallest integer such that every planar points set, no three collinear, containing at least $h(k)$ interior points has a convex subset with k or $k+1$ interior points. In [1] we showed that $h(3) = 3$. In this paper, we show the following result.

Theorem *$h(4)$=7, that is, every planar point set P with at least 7 interior points contains a convex subset with 4 or 5 interior points.*

For $k \geq 5$, we do not know if $h(k)$ is finite.

To investigate lower bounds on $h(k)$ we consider *deficient* point sets $P = P(m, s, k)$, which are sets with m vertices and s interior points which do not contain a subset of points with k or $k+1$ interior points. It is trivial to construct deficient point sets when $s < k$, so we assume $s \geq k \geq 1$ throughout the paper. The existence of a deficient point set $P(m, s, k)$ implies that $h(k) \geq s + 1$. In Figure 1 we give a deficient point set $P(3, 8, 5)$, showing that $h(5) \geq 9$. For $k \geq 6$ we have $h(k) \geq \lceil \frac{3k+3}{2} \rceil$, as shown by deficient point sets $P(3, \lceil \frac{3k+1}{2} \rceil, k)$ given in Figure 2 (a), (b) for the case of k even and k odd respectively. These examples can be extended to deficient point sets $P(m, \lceil \frac{3k+1}{2} \rceil, k)$ for all $m \geq 3$ by the following lemma, which is analogous to a similar lemma in [1] in a slightly different setting.

Extension Lemma *Every deficient point set $P(m, s, k)$ can be extended to a deficient point set $P(m+1, s, k)$.*

(*Proof*) Let $P = P(m, s, k)$ and let v be any vertex. Place a new convex hull vertex u near v such that the remaining points of P are in the same angular order about u as they are about v creating the point set Q. Suppose Q has a convex subset T with k or $k+1$ interior points. Clearly none of these interior points could be either u or v. It is then easy to verify that $T \cup \{v\} \setminus \{u\}$ is

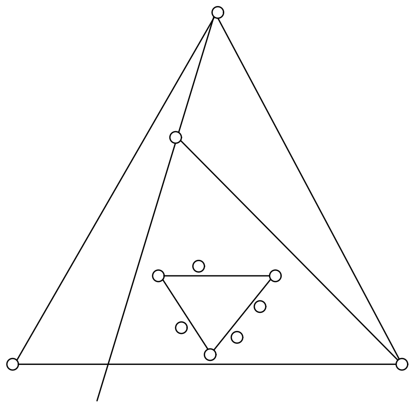
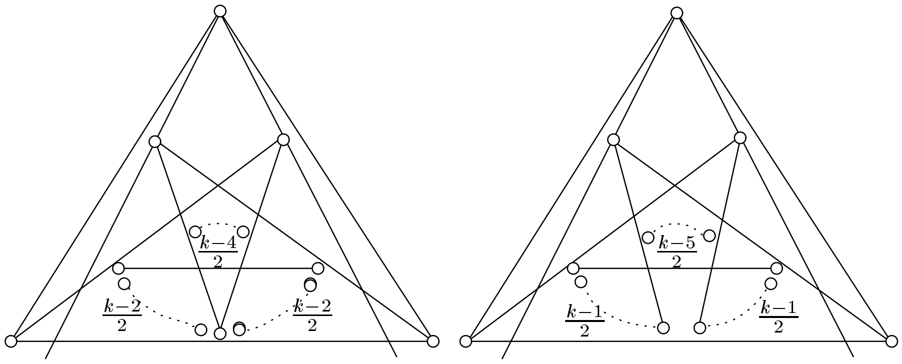


Figure 1: Example of $P(3, 8, 5)$



(a) (b)

Figure 2: Example of $P(3, \lceil \frac{3k+1}{2} \rceil, k)$, (a) k even (b) k odd

a subset of P with the same number of interior points as T, a contradiction. Hence Q is a deficient point set $P(m + 1, s, k)$. $\square$

We also will make use of the following lemma proved in [1].

Reduction Lemma *Let P be a planar point set with $m \geq 3$ vertices and $s \geq 0$ interior points. Then there exists a vertex v_i of P, such that $P' = P \backslash \{v_i\}$ has m' vertices, s' interior points and either:*
(a) $m' = m - 1$ and $s' = s$ or
(b) $m' = m + t$ and $s' = s - t - 1$, where $t = 0, 1, ..., \lfloor s/m \rfloor$.

3 Proof of Theorem

An interior point of a triangle T is called a *x, y, z-splitter of T* if it partitions T into three triangles with $x \geq y \geq z$ interior points, respectively. In case $z = 0$, we abbreviate this to *x, y-splitter*. Note that this will happen if the splitter is the closest point to one of the edges of T. For $i = 1, 2, 3$ we use the notation $v_{i,i+1}$ to refer to the closest interior point to the edge $v_i v_{i+1}$ in $\triangle v_1 v_2 v_3$.

Proof of $h(4) \geq 7$

Figure 3 shows that there exists a planar point set with 6 interior points which does not contain a convex subset with 4 or 5 interior points, that is $h(4) \geq 7$. This configuration, $P(3, 6, 4)$, also plays an important role below in proving the upper bound on $h(4)$. Using the Extension Lemma, we can extend it to give deficient point sets $P(m, 6, 4)$ for each choice of $m \geq 3$.

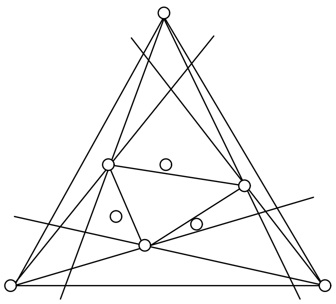


Figure 3: Monster: $P(3, 6, 4)$

Proof of $h(4) \leq 7$

The proof is constructed from a set of base cases, and then completed by induction using the Reduction Lemma. First of all, we investigate triangles with 6 interior points that do not contain a subset with 4 or 5 interior points. Consider such a triangle T with vertices v_1, v_2, v_3. Then it is easy to verify the following properties, for each $i = 1, 2, 3$:

(i) edge $v_i v_{i+1}$ has a unique closest point $v_{i,i+1}$ which is a $3, 2$-splitter,

(ii) $\triangle v_i v_{i-1,i} v_{i,i+1}$ contains exactly one interior point,

(iii) $CH(T \backslash v_i)$ is a convex pentagon containing exactly three interior points,

(iv) the interior points of T form a convex hexagon.

We call a triangle with these properties a *monster*, and an example is shown in Figure 3. Henceforce, we may assume that every triangle containing 6 interior points in P is a monster. Now we distinguish several cases.

(Case 1) $m = 3, s = 7$:

We may assume that every x, y-splitter is a 6,0- or 3,3-splitter. First, consider the case that there exists a 6,0-splitter, say u_1, where $\triangle u_1 v_2 v_3$ contains 6 points. Then $T = \triangle u_1 v_2 v_3$ is a monster and we may assume that $v_{2,3}$ is on the same side of the extended line $\overline{v_1u_1}$ as v_3. Let $u_{1,2}$ and $u_{3,1}$ be the closest interior points in T, respectively, of edges u_1v_2 and v_3u_1. Properties (ii) and (iii) applied to $\triangle u_1v_2v_3$ imply that $Q = CH(u_1, u_{3,1}, v_{2,3}, v_2)$ is a convex quadrilateral containing 3 interior points. If $v_{2,3}$ is a $3,3$-splitter, it follows that $v_1u_{3,1}v_{2,3}v_2$ is a convex quadrilateral containing Q, and hence containing 4 interior points, see Figure 4(a). Otherwise, if $v_{2,3}$ is a $6,0$-splitter, $\triangle v_1v_2v_{2,3}$ is also a monster. Now since $\triangle u_1v_2v_3$ is a monster, $\triangle u_1u_{1,2}u_{3,1}$ contains a point w by property (ii). But then $\{u_1, u_{1,2}, u_{3,1}, w\}$ is a non-convex interior point set of monster $\triangle v_1v_2v_{2,3}$, contradicting property (iv), see Figure 4(b).

Next, we may assume that each closest point $v_{i,i+1}$ is unique and is a 3,3-splitter. This in turn implies that every quadrilateral $v_{i-1}v_{i-1,i}v_{i,i+1}v_{i+1}$ is convex. By the pigeon-hole principle, one of the triangles $v_iv_{i-1,i}v_{i,i+1}$, contains 0 or 1 interior points. This implies that its neighbouring convex quadrilateral $v_{i-1}v_{i-1,i}v_{i,i+1}v_{i+1}$ contains 5 or 4 points, respectively, as required.

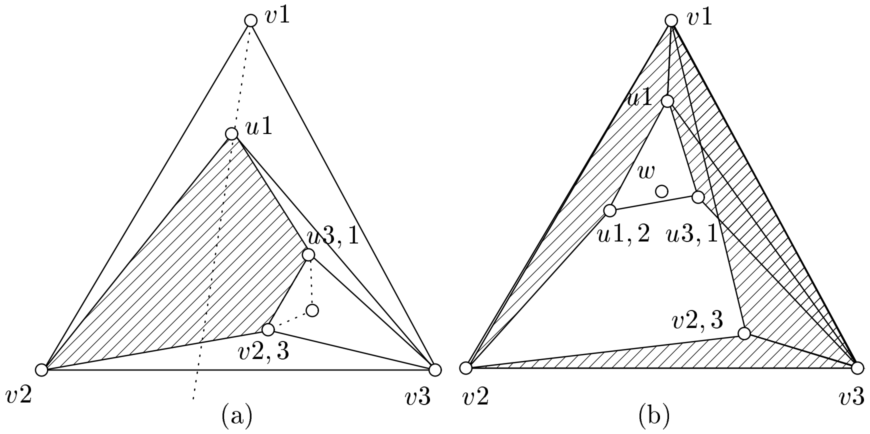


Figure 4: $m = 3, s = 7$: (a) Shaded region contains remaining 3 interior points. (b) Shaded region is empty.

(Case 2) $m \geq 4, s = 7$:

To show this case, create a star triangulation of P by joining hull vertex v_1 to each of the other hull vertices, creating $m-2$ triangles. If there exists a triangle which contains 4 or 5 or 7 points, we are done. If each triangle contains at most 3 points, it is easy to verify that we may find a convex subset with 4 or 5 interior points by concatenating a set of adjacent triangles. Thus we have only to consider the case where one triangle, say $\triangle v_1v_iv_{i+1}$, has 6 interior points, one triangle, say $\triangle v_1v_jv_{j+1}$, has one interior point and the others have no interior points. We may assume that $j \geq i+1$, the case $j \leq i$ being symmetric. Consider the subset $Q = CH(v_1, v_i, v_{i+1}, v_j, v_{j+1})$, which has 7 interior points, and is either a convex pentagon (see Figure 5), or quadrilateral if $j = i+1$. $\triangle v_1v_iv_{i+1}$ is a monster, so by property (iii) if we delete v_i from it we have a convex pentagon with 3 interior points. Therefore $CH(Q \backslash v_i)$ has 4 interior points.

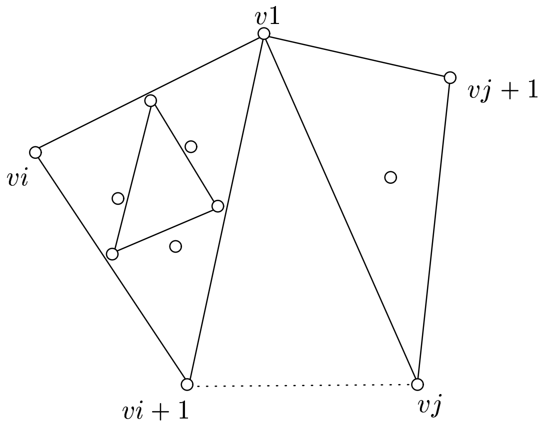


Figure 5: $m \geq 4, s = 7$: Convex pentagon Q

(Case 3) $m = 3, s = 8$:

We may assume that every x, y-splitter is a 6,1-splitter. Since $v_{1,2}$ is a 6,1-splitter, we may assume that $\triangle v_1 v_3 v_{1,2}$ contains just one point $u_{3,1}$ where $u_{3,1}$ is also a 6,1-splitter. If $\triangle v_1 v_2 u_{3,1}$ contains 6 points, then $\triangle v_2 v_{1,2} u_{3,1}$ contains 5 points. Thus we may assume that the region $\triangle v_2 v_{1,2} v_3 \cap \triangle v_2 u_{3,1} v_3$ contains all 6 remaining points, see Figure 6. Let w be the closest point in $\triangle v_2 v_{1,2} v_3$ to the edge $v_2 v_{1,2}$. By property (i), w is a 3,2-splitter. If w is on the same side of the line $\overline{v_1 v_{1,2}}$ as v_2, then $\triangle v_1 v_2 w$ is empty. Therefore w is a 6,1-splitter, contradicting the assumption that w is a 3,2-splitter in a monster. Otherwise, by properties (ii) and (iii), $CH(v_{1,2}, w, v_{2,3}, v_3)$ is a convex quadrilateral with 3 interior points. It extends by adding v_1 into a convex pentagon containing 4 interior points: the original 3 plus $u_{3,1}$.

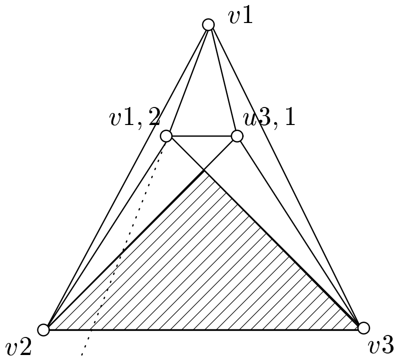


Figure 6: $m = 3, s = 8$: Shaded region contains all 6 remaining interior points.

(Case 4) $m \geq 4, s = 8$:

As in Case 2, we form a star triangulation of $CH(P)$ from v_1. We are done unless some triangle, say $\triangle v_1 v_i v_{i+1}$, has 6 interior points. If some other triangle has one interior point, we have a set with 7 interior points, reducing to Case 2. Otherwise some triangle, say $\triangle v_1 v_j v_{j+1}$, has the remaining two interior points. We define Q as in Case 2, and conclude by an analogous argument that if $j \geq i+1$ then $CH(Q \backslash v_i)$ has 5 interior points.

(Case 5) $m = 3, s = 9$:

We may assume that every x,y-splitter is a 6,2-splitter. Since $v_{1,2}$ is a 6,2-splitter, we may assume that $\triangle v_1 v_3 v_{1,2}$ contains two points. Let $u_{3,1}$ be the closest point in $\triangle v_1 v_3 v_{1,2}$ to the edge $v_1 v_3$. If $\triangle v_1 v_{1,2} u_{3,1}$ is empty, then the convex quadrilateral $CH(v_2, v_3, u_{3,1}, v_{1,2})$ has 7 interior points, and the problem reduces to Case 2. Therefore $\triangle v_1 v_{1,2} u_{3,1}$ contains one point. If $\triangle v_1 v_2 u_{3,1}$ contains 6 points, then $\triangle v_2 v_{1,2} u_{3,1}$ contains 4 points. Therefore the region $\triangle v_2 v_{1,2} v_3 \cap \triangle v_2 u_{3,1} v_3$ contains 6 points where both $\triangle v_2 v_{1,2} v_3$ and $\triangle v_2 u_{3,1} v_3$ are monsters, see Figure 7. Notice that at least one of the quadrilaterals $v_1 v_2 v_{2,3} u_{3,1}$ and $v_1 v_{1,2} v_{2,3} v_3$ is convex. If the quadrilateral $v_1 v_2 v_{2,3} u_{3,1}$ is convex, it contains 4 or 5 points since $v_{2,3}$ is a 2,3-splitter in $\triangle u_{3,1} v_2 v_3$. We can argue the same way for the convex quadrilateral $v_1 v_{1,2} v_{2,3} v_3$.

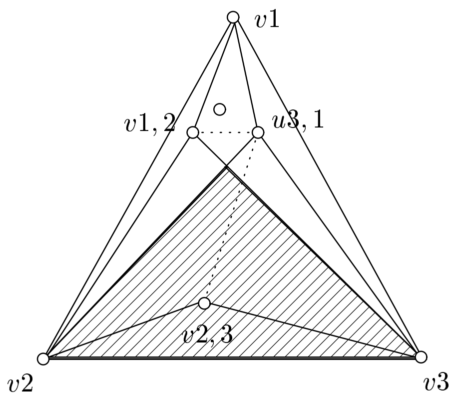


Figure 7: $m = 3, s = 9$: Shaded region contains 6 interior points.

(Case 6) $m = 4, s = 9$:

Decompose the convex quadrilateral $v_1 v_2 v_3 v_4$ into two triangles by adding the diagonal $v_1 v_3$. If either of the two triangles formed have 0,4,5,7,8 or 9 points we reduce to a previous case and are done. We may therefore assume that $\triangle v_1 v_2 v_3$ contains 6 interior points, and is a monster. The other triangle contains 3 interior points. Let $v_{3,1}$ be the closest point in $\triangle v_1 v_2 v_3$ to the edge $v_3 v_1$. Since $\triangle v_1 v_2 v_3$ is a monster, $\triangle v_1 v_{1,2} v_{3,1}$ contains one interior point w. Property (iii) implies that $\triangle v_1 w v_3$ contains 1 or 2 interior points. Therefore the convex quadrilateral $v_1 w v_3 v_4$ contains 4 or 5 interior points.

(Case 7) $m = 3, s = 10$:

This case is similar to Case 5. We may assume that every x, y-splitter is a 6,3-splitter. Since $v_{1,2}$ is a 6,3-splitter, we may assume that $\triangle v_1 v_3 v_{1,2}$ contains 3 points. Let $u_{3,1}$ be the closest point in $\triangle v_1 v_3 v_{1,2}$ to the edge $v_1 v_3$ and let $Q = CH(v_2, v_3, u_{3,1}, v_{1,2})$. Q is a quadrilateral since $v_{1,2}$ cannot lie in $\triangle v_2 v_3 u_{3,1}$ because $u_{3,1}$ is also a 6,3-splitter of $\triangle v_1 v_2 v_3$. We can assume that $\triangle v_1 v_{1,2} u_{3,1}$ contains 2 points for otherwise the convex quadrilateral Q has 7 or 8 interior points, and we reduce to Case 2 or Case 4, respectively.

If $\triangle v_1 v_2 u_{3,1}$ contains 6 interior points, then both $\triangle v_{1,2} v_2 u_{3,1}$ and $\triangle v_2 u_{3,1} v_3$ contain 3 interior points each, see Figure 8. Now $CH(\triangle v_2 v_{1,2} u_{3,1} \backslash v_{1,2})$ is not a convex pentagon since $\triangle v_1 v_2 u_{3,1}$ is a monster, and this would violate property (iv). Therefore $CH(Q \backslash v_{1,2})$ has 4 or 5 interior points.

Otherwise, $\triangle v_2 v_3 u_{3,1}$ contains 6 points. Then both $\triangle v_2 v_{1,2} v_3$ and $\triangle v_2 u_{3,1} v_3$ are monsters. This situation was essentially treated in Case 5 (compare Figure 7), since interior points of $\triangle v_1 v_{1,2} u_{3,1}$ do not come into play.

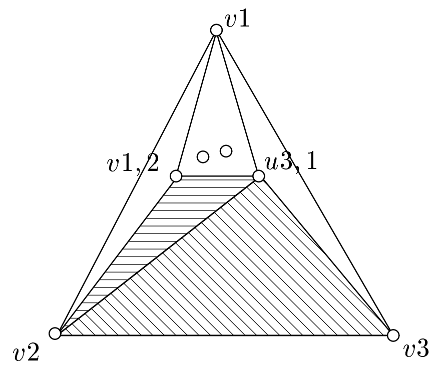


Figure 8: $m = 3, s = 10$: Both shaded triangles have 3 interior points.

We now complete the proof. Cases 1 and 2 settle $s = 7$, and Cases 3 and 4 settle $s = 8$. For $s \geq 9$ we apply the Reduction Lemma and induction. By Case 5 and 6, we may assume $m \geq 5$ for the case $s = 9$. Then the Reduction Lemma shows the existence of a proper subset to which we can apply induction: if (a) of the lemma holds we have $s' = 8$, and if (b) holds we have a proper subset of P with $s' = 9 - \lfloor 9/m \rfloor - 1 \geq 7$ interior points, for which we may again use induction. In the case $s = 10$, we may assume that $m \geq 4$ by Case 7. We apply the Reduction Lemma as before, and note that if (b) holds in the lemma, we have a proper subset with $s' = 10 - \lfloor 10/m \rfloor - 1 \geq 7$ interior points. Finally, if $m \geq 3$, $s \geq 11$, we have $s - \lfloor s/m \rfloor - 1 \geq 7$, and so the result follows by the Reduction Lemma and induction. $\square$

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